Some Useful Quantifier Equivalences

DeMorgan laws for quantifiers

$$\neg\forall x\ P(x) \quad\Leftrightarrow\quad \exists x\ \neg P(x)$$

$$\neg\exists x\ P(x) \quad\Leftrightarrow\quad \forall x\ \neg P(x)$$

¬A is equivalent to O

$$\neg\forall x\ (P(x) \rightarrow Q(x)) \quad\Leftrightarrow\quad \exists x\ (P(x) \wedge \neg Q(x))$$

¬I is equivalent to E

$$\neg\exists x\ (P(x) \wedge Q(x)) \quad\Leftrightarrow\quad \forall x\ (P(x) \rightarrow \neg Q(x))$$

Distributing ∀ through ∧

$$\forall x\ (P(x) \wedge Q(x)) \quad\Leftrightarrow\quad \forall x\ P(x) \wedge \forall x\ Q(x)$$

Distributing ∃ through ∨

$$\exists x\ (P(x) \vee Q(x)) \quad\Leftrightarrow\quad \exists x\ P(x) \vee \exists x\ Q(x)$$

Null quantification

In the following examples, P represents any wff in which x does not occur free.

$$\forall x\ P \quad\Leftrightarrow\quad P$$

$$\exists x\ P \quad\Leftrightarrow\quad P$$

$$\forall x\ (P \vee Q(x)) \quad\Leftrightarrow\quad P \vee \forall x\ Q(x)$$

$$\exists x\ (P \wedge Q(x)) \quad\Leftrightarrow\quad P \wedge \exists x\ Q(x)$$

Null quantification over →

$$\forall x\ (P \rightarrow Q(x)) \quad\Leftrightarrow\quad P \rightarrow \forall x\ Q(x)$$

$$\exists x\ (P \rightarrow Q(x)) \quad\Leftrightarrow\quad P \rightarrow \exists x\ Q(x)$$

$$\forall x\ (Q(x) \rightarrow P) \quad\Leftrightarrow\quad \exists x\ Q(x) \rightarrow P$$

$$\exists x\ (Q(x) \rightarrow P) \quad\Leftrightarrow\quad \forall x\ Q(x) \rightarrow P$$

Replacing bound variables

$$\forall x\, P(x) \iff \forall y\, P(y)$$

$$\exists x\, P(x) \iff \exists y\, P(y)$$

Some non-equivalences to beware of

The following pairs of sentences may appear to be equivalent, but they are **not**. Please beware of them.

$\forall x\, (P(x) \lor Q(x))$ and $\forall x\, P(x) \lor \forall x\, Q(x)$

$\exists x\, (P(x) \land Q(x))$ and $\exists x\, P(x) \land \exists x\, Q(x)$

$\forall x\, (Q(x) \rightarrow P)$ and $\forall x\, Q(x) \rightarrow P$

$\exists x\, (Q(x) \rightarrow P)$ and $\exists x\, Q(x) \rightarrow P$