Properties of Relations and Infinite Domains

For our final topic, we will cover a few points that are not fully covered in *LPL*, but that are important in two ways. First, they are important for understanding the logic of arguments involving binary predicates. Second, they help to bring into sharper focus some of the theoretical limitations of using Tarski's World counterexamples to show FOL arguments to be invalid.

Although most of this topic is omitted in *LPL*, there is a useful discussion of the properties of binary relations in §15.5 (pp. 422-424). I suggest that you read these pages now, and then return to this point and resume reading.

Binary relations

A **binary relation** is what gets expressed by a binary (2-place) predicate. For example, Larger expresses the *larger than* relation, FrontOf expresses the relation of *being in front of*, and = expresses the identity relation.

The true atomic sentence Older(ringo, paul) expresses the fact that Ringo stands in the *older than* relation to Paul.

Properties of binary relations

Binary relations may themselves have properties. For example, if a relation *R* is such that everything stands in the relation *R* to itself, *R* is said to be **reflexive**. Some relations, such as *being the same size as* and *being in the same column as*, are reflexive. Others, such as *being in front of* or *being larger than* are not.

We can express the fact that a relation is reflexive as follows: a relation, *R*, is reflexive iff it satisfies the condition that $\forall x\, R(x, x)$.

LPL has a brief discussion of these properties of relations, and provides a list of some of the most important ones, on p. 422. We summarize them here.

Reflexivity:	$\forall x\, R(x, x)$
Irreflexivity:	$\forall x\, \neg R(x, x)$
Transitivity:	$\forall x \forall y \forall z\, [(R(x, y) \land R(y, z)) \rightarrow R(x, z)]$
Symmetry:	$\forall x \forall y\, (R(x, y) \rightarrow R(y, x))$
Asymmetry:	$\forall x \forall y\, (R(x, y) \rightarrow \neg R(y, x))$
Antisymmetry:	$\forall x \forall y\, [(R(x, y) \land R(y, x)) \rightarrow x = y]$

It would probably be useful for you at this point to think of the various relations expressed by the binary predicates of the blocks language and figure out, for each of those relations, which of the properties above it has. (A shorter version of this project appears in *LPL* as Exercise 15.36.)

OK, do you have your list? I'll run through each of the properties, give a brief explanation of it, and give some examples of relations that have that property.

Reflexivity

A reflexive relation is one that everything bears to itself. The blocks language predicates that express reflexive relations are: SameSize, SameShape, SameCol, SameRow, and =. Other reflexive relations include *lives in the same city as*, *is (biologically) related to*.

Irreflexivity

An irreflexive relation is one that nothing bears to itself. The blocks language predicates that express reflexive relations are: Adjoins, Larger, Smaller, LeftOf, RightOf, FrontOf, and BackOf. Other irreflexive relations include *is different from*, *occurred earlier than*.

Transitivity

The property of transitivity is probably more clearly and efficiently expressed by its FOL formula than by trying to state it in English. One might try to put it like this: a transitive relation is one such that if one thing bears it to a second, and the second bears it to a third, then the first thing bears it to the third. Do you see what I mean? The FOL version is simpler and more straightforward.

The blocks language predicates that express transitive relations are: Larger, Smaller, LeftOf, RightOf, FrontOf, BackOf, SameSize, SameShape, SameCol, SameRow and =. Wow, that's quite a list! In fact, it includes every blocks language binary predicate except for one: Adjoins. If it is not clear to you why *adjoins* is not transitive, you might want to check it out using Tarski's World. Other transitive relations include *older than*, *occurred earlier than*, *lives in the same city as*, *ancestor of*.

Symmetry

A symmetric relation is one that is always reciprocated. That is, if one thing bears it to a second, the second also bears it to the first. (The only thing wrong with putting it this way is that the "first thing" and the "second thing" don't have to be two different things!)

The blocks language predicates that express symmetric relations are: Adjoins, SameSize, SameShape, SameCol, SameRow and =. Other symmetric relations include *lives near*, *is a sibling of*.

Asymmetry

An asymmetric relation is one that is never reciprocated. That is, if one thing bears it to a second, the second does not bear it to the first.

The blocks language predicates that express asymmetric relations are: Larger, Smaller, LeftOf, RightOf, FrontOf, and BackOf. Other asymmetric relations include *older than*, *daughter of*.

Antisymmetry

Looking at the definition of antisymmetry above, you may have a hard time putting it into English. You might try this: an antisymmetric relation is one such that if two things bear it to one another, then they are identical. But that doesn't sound quite right, does it? You'll come up with a better way of putting it if you review the discussion of "at most one" in the study guide section on §14.1.

Did that help? An antisymmetric relation is one that no **two** things ever bear to one another. The blocks language predicates that express antisymmetric relations are: Larger, Smaller, LeftOf, RightOf, FrontOf, BackOf, and =.

It might at first seem odd that *larger than*, for example, is antisymmetric. But nevertheless

$$(\text{Larger}(x, y) \land \text{Larger}(y, x)) \rightarrow x = y$$

is true for all values of x and y. This is one of those "vacuously true" universal generalizations whose antecedent is always false. And since Larger(x, y) ∧ Larger(y, x) is **never** true, the number of things that are both larger than each other is zero. Since the number of such things is zero, it follows trivially that the number is not more than one. That is, no **two** things are both larger than each other.

The (equivalent) contrapositive form of the antisymmetry condition is perhaps easier to understand:

$$\forall x \forall y\ [x \neq y \rightarrow \neg(R(x, y) \land R(y, x))]$$

This is in turn equivalent to:

$$\forall x \forall y\ [x \neq y \rightarrow (R(x, y) \rightarrow \neg R(y, x))]$$

And this says, once again, that if something stands in the R relation to something **else**, then the relation is not reciprocated. Clearly, asymmetry implies antisymmetry, although the converse does not hold.

Notice that neither *same column* nor *same row* is antisymmetric (two different blocks can be in the same column, and two different blocks can be in the same row). But now consider the **logical product** of these two relations, that is, the relation *same column and same row*. This relation is antisymmetric. For if x and y are in both the same column and the same row as one another, then $x = y$. No **two** blocks can be in both the same column and the same row. That's because in a Tarski world, you cannot fit more than one block into a single square. For practical purposes, the relation *same column and same row* just **is** the identity relation in Tarski's World.

Arguments involving binary relations

When a relation has one of these properties, that means that a certain kind of argument involving atomic sentences is valid. Thus, for example, it is because the *larger than* relation is transitive that the following argument is valid.

| Larger(a, b)
| Larger(b, c)
|—
| Larger(a, c)

And it is because the *adjoins* relation is symmetric that the following argument is valid.

| Adjoins(b, c)
|—
| Adjoins(c, b)

A good way to become familiar with these properties of relations is to do exercises 15.30 – 15.36.

Notice that every relation expressed by a binary atomic predicate in the blocks language (SameSize, Larger, Adjoins, etc.) is either reflexive or irreflexive, and either symmetric or asymmetric. This is because in the language of the blocks world, all the binary predicates (except for =) stand for **spatial relations**.

But when you consider relations more broadly, you will find some that are neither reflexive nor irreflexive; some are neither symmetric nor asymmetric. Examples: *loves*, *hates*, *shaves*, *respects*. Test these out for yourself: $\forall x$ Hates(x, x) is not true, but neither is $\forall x \neg$Hates(x, x). (Not everyone hates himself, but surely at least some self-hatred exists.) And
$\forall x \forall y$ (Shaves(x, y) $\to$ Shaves(y, x)) is not true, but neither is
$\forall x \forall y$ (Shaves(x, y) $\to \neg$Shaves(y, x)). (People do not typically shave one another, but there is probably at least one pair of people who engage in this strange behavior.)

Relations such as *loves*, *hates*, *admires*, etc. have this interesting feature: they have **none** of the properties we have been discussing. Takes *loves*, for example. It is not reflexive (not everyone loves himself or herself), not irreflexive (at least one person does love himself), not transitive (one does not always love the ones one's beloved loves), not symmetric (love is not always reciprocated), not asymmetric (love is sometimes reciprocated), and not antisymmetric (sometimes two people really do love one another). I guess this shows there's just not much that's logical about love!

Equivalence relations

When a relation is transitive, symmetric, and reflexive, it is called an **equivalence** relation. *Being the same size as* is an equivalence relation; so are *being in the same row as* and *having the same parents as*. The most familiar (and important) example of an equivalence relation is **identity**.

Unlike *same size* and *same row*, however, *identity* has the additional property of being antisymmetric. (That is, no **two** things bear the *identity* relation to one another.) This makes it special among equivalence relations. In fact, you can think of identity as just a very special case of equivalence.

Confirm to your own satisfaction (if you are not already clear about this) that identity is transitive, symmetric, reflexive, and antisymmetric.

Proofs about relations

There are some interesting generalizations that can be proved about the properties of relations. For example, if a relation is transitive and irreflexive,[1] it must also be asymmetric. That is to say, the following argument is valid.

$$\forall x \forall y \forall z\, [(R(x, y) \wedge R(y, z)) \to R(x, z)]$$
$$\forall x\, \neg R(x, x)$$
$$\overline{\quad\quad}$$
$$\forall x \forall y\, (R(x, y) \to \neg R(y, x))$$

To prove that this is so, go to the Supplementary Exercises page and open the file Asymmetry.prf. You should be able to deduce the conclusion from the premises fairly easily. Use generalized conditional proof strategy, assuming R(a, b) with boxed constants a and b. Then deduce $\neg$R(b, a) and apply $\forall$ **Intro**. The trick in this proof is to choose the right substitutions of constants for variables in applying $\forall$ **Elim** to the first premise. To see my version of the proof, look at Proof Asymmetry.prf. But don't look until you've worked out your own proof.

Here's another exercise to try. We noted above that asymmetry implies antisymmetry. Prove that this is so by completing the proof in Antisymmetry.prf. Then compare your proof with my version (only six steps!) in Proof Antisymmetry.prf

[1] Mathematically, a relation that is transitive and irreflexive is known as a *strict partial ordering*.

Counter-examples to generalizations about relations

When a generalization about a relation is false, you should be able to establish this by means of a counter-example. For example, we can show that not every symmetric relation is transitive by producing a counter-example to this inference:

$$\forall x \forall y\, (R(x, y) \rightarrow R(y, x))$$

$$\forall x \forall y \forall z\, [(R(x, y) \wedge R(y, z)) \rightarrow R(x, z)]$$

An obvious counter-example here would be to take *R* to be Adjoins. A little reflection will show that

∀x∀y (Adjoins(x, y) → Adjoins(y, x))

is a logical truth (although not, of course, an FO validity). That is, the *adjoins* relation is symmetric. But this relation is certainly **not** transitive. That is, we can easily construct a Tarski World in which the premise of the argument below is true and the conclusion is false:

1. ∀x∀y (Adjoins(x, y) → Adjoins(y, x))
2. ∀x∀y∀z [(Adjoins(x, y) ∧ Adjoins(y, z)) → Adjoins(x, z)]

Go to the Supplementary Exercises page and open Symmetry.sen. Then construct a world in which sentence (1) is true and sentence (2) is false. You will find this very easy. In fact, as long as there are two adjoining blocks somewhere in your world, (2) will be false. (Since (1) is a logical truth, it will be true in every Tarski world.) This shows that *adjoins* is not a transitive relation. Therefore, we have provided a counter-example to the generalization that every symmetric relation is transitive.

> [If you do not see why (2) can come out false in a two-block world, try this experiment: create a world with exactly two adjoining blocks. Now play the game against Tarski on (2) and commit to its truth. You will soon discover why Tarski will always win. Remember, three different variables do not require three different objects.]

Seriality and infinite domains

In our last example, we were able to falsify a generalization about a relation by constructing a counter-example with only two objects in its domain. (Our Tarski's World invalidating the argument needed only two blocks.) But some false generalizations about relations cannot be shown to be false in Tarski's World, or in any world containing only a finite number of objects in its domain.

To put the point another way, there are some invalid FOL arguments that cannot be shown to be invalid by means of any counter-example containing only finitely many objects. Sometimes, an **infinite domain** is required.

Examples of this kind of argument frequently involve relations that have an important property that we have not yet considered—the property of **seriality**. (It is sometimes called "totality," as it is on p. 427 where it is defined.) Here is the definition:

Seriality: $\forall x \exists y\, R(x, y)$

That is, a relation *R* is serial just in case for each object in the domain there is something to which it stands in the relation *R*.

Here, then, is a famous example of an invalid argument that has no counter-example in any finite domain. Notice that it has the same premises as Asymmetry.prf, but a different conclusion:

$$\begin{array}{|l} \forall x \forall y \forall z\, [(R(x, y) \wedge R(y, z)) \rightarrow R(x, z)] \\ \forall x\, \neg R(x, x) \\ \hline \neg \forall x \exists y\, R(x, y) \end{array}$$

This argument says that if *R* is both transitive and irreflexive, it follows that it is **not** serial. To show that the argument is invalid, we must find an example of a relation that is **transitive**, **irreflexive**, and **serial**. That is, we have to produce an example of a relation that satisfies these three conditions:

$$\forall x \forall y \forall z\, [(R(x, y) \wedge R(y, z)) \rightarrow R(x, z)]$$

$$\forall x\, \neg R(x, x)$$

$$\forall x \exists y\, R(x, y)$$

Looking in vain in Tarski's World

We might try to find a counter-example in Tarski's World, but we will quickly see that we cannot succeed. Consider the binary predicates in the blocks language of Tarski's World: Adjoins, FrontOf, SameSize, Larger, =, etc. Now think of the relations expressed by these predicates. Certainly, some of these relations are transitive and serial. For example, all of the **equivalence** relations (e.g., *identity*, *same size as*, *same shape as*) are both transitive and serial. The trouble is, these are also reflexive (each block is the same size as itself, the same shape as itself, identical to itself, etc.), and we are looking for an **irreflexive** relation.

Now the blocks language does have other predicates that express relations that are both transitive and irreflexive: Larger, Smaller, FrontOf, LeftOf, etc. The trouble is that none of these relations is **serial**. That is, it is never true in a Tarski World that **every** block is in front of some block or other, or that **every** block is smaller than some block or other. This is because, in a Tarski World, the blocks in the back row are never in front of anything, and there is no block larger than a large block. There is a backmost row, a leftmost column, a smallest size, a largest size, etc.

This suggests that a successful counter-example must avoid this limitation. Whatever relation, *R*, we pick, we must be sure that there is no "*R*-most" thing—no *x* which does not bear the *R* relation to anything else. The avoidance of this limitation is what pushes us to an infinite domain.

A counter-example with an infinite domain

It turns out that an example of a relation that is transitive, irreflexive, and serial is easy to find.The *less than* relation among positive integers is one such relation. It is transitive, obviously, since if one number is less than a second and the second is less than a third, the first is less than the third. Equally clearly, it is irreflexive, since no number is less than itself. But it is also serial, in that for every number (no matter how large), there is another number that is even larger.

What makes this example work, of course, is that the supply of positive integers is infinite There is no largest integer, so that no matter what number *x* we pick, we never run out of numbers, *y*, such that *x* is less than *y*.

Proving that a domain must be infinite

It is easy to see that our counter-example above works, and that the argument against which we brought it is invalid. But it is not so easy to prove that a successful counter-example in this case **must** have an infinite domain. Still, there is a simple, intuitive way to see why the domain must be infinite. Here it is.

First, note that R is **transitive**, **irreflexive**, and **serial**. But we have already proved above that any relation that is transitive and irreflexive is also asymmetric. So we can add to our conditions that R must be **asymmetric** as well.

Now suppose we have a domain with only one object, call it n_1. Since R is serial, there is some y such that $R(n_1, y)$. But n_1 is the only object in the domain, so y must be n_1. Hence, we get $R(n_1, n_1)$, which is impossible if R is irreflexive. So our domain must have more than one object.

Next, suppose we have a domain with two objects, call them n_1 and n_2. Since R is serial, n_1 must bear R to something, and that thing cannot be n_1 (because R is irreflexive). Hence, $R(n_1, n_2)$. But n_2 must also bear R to something, and that thing cannot be n_1 (because of asymmetry) or n_2 (because of irreflexivity). So our domain must have more than two objects.

So let's add another object, n_3, and suppose that $R(n_2, n_3)$. But because R is transitive, and we have both $R(n_1, n_2)$ and $R(n_2, n_3)$, it follows that we also have $R(n_1, n_3)$. Now, which object y is such that $R(n_3, y)$? It cannot be n_1 (because of asymmetry) or n_2 (for the same reason) or n_3 (because of irreflexivity). So our domain must have more than three objects.

It is easy to see that this line of reasoning can be extended indefinitely. Each time we add an object to the domain, it is because we are forced to add a new object for its immediate predecessor to stand in the R relation to. But then the transitivity condition will come into play and guarantee that each of the previous objects in the domain bears R to the newcomer. Given the asymmetry condition, this will prevent the newcomer from bearing R to any of them, and because of the irreflexivity condition, the newcomer cannot bear R to itself. The seriality condition then forces us to introduce yet another object. So we will never be able to convert a domain that is too small into one that is large enough by adding just one object. And that is sufficient to establish that our domain will have to be infinite. For any finite domain can be constructed (however tediously) by adding objects one at a time.

To convert this intuitive sketch into a proper proof would require us to construct a proof by *mathematical induction*, which is the topic of Chapter 16 of *LPL*. So although our course ends here, if you wish to continue your study of logic, that would be an excellent place to begin.