

Chapter 5: Methods of Proof for Boolean Logic

§ 5.1 Valid inference steps

Conjunction elimination

Sometimes called *simplification*. From a conjunction, infer any of the conjuncts.

- From $P \wedge Q$, infer P (or infer Q).

Conjunction introduction

Sometimes called *conjunction*. From a pair of sentences, infer their conjunction.

- From P and Q , infer $P \wedge Q$.

§ 5.2 Proof by cases

This is another valid inference step (it will form the rule of *disjunction elimination* in our formal deductive system and in Fitch), but it is also a powerful proof strategy.

In a proof by cases, one begins with a disjunction (as a premise, or as an intermediate conclusion already proved). One then shows that a certain consequence may be deduced from each of the disjuncts taken separately. One concludes that that same sentence is a consequence of the entire disjunction.

- From $P \vee Q$, and from the fact that S follows from P and S also follows from Q , infer S .

The general proof strategy looks like this: if you have a disjunction, then you know that at least **one** of the disjuncts is true—you just don't know which one. So you consider the individual “cases” (i.e., disjuncts), one at a time. You assume the first disjunct, and then derive your conclusion from it. You repeat this process for each disjunct. So it doesn't matter which disjunct is true—you get the same conclusion in any case. Hence you may infer that it follows from the entire disjunction.

In practice, this method of proof requires the use of “subproofs”—we will take these up in the next chapter when we look at formal proofs.

§ 5.3 Indirect proof: proof by contradiction

Also called *indirect proof* or *reductio ad absurdum*, this is a powerful method of proof commonly used in mathematics.

In a proof by contradiction, one assumes that one's conclusion is *false*, and then tries to show that this assumption (together with the argument's premises) leads to a contradiction. This shows that the conclusion cannot be false if all the premises are true—i.e., that the conclusion *must* be true if the premises are true. That is to say, that the conclusion is a logical consequence of the premises.

We will develop this idea as a way of establishing a *negative* conclusion. Suppose you wish to establish that a conclusion of the form $\neg S$ is a logical consequence of a set of premises P_1, P_2, \dots, P_n . You assume S (equivalent to the negation of the argument's conclusion) and treat it as a premise along with P_1, P_2, \dots, P_n . You then try to deduce from these assumptions a contradiction—a pair of sentences that contradict one another, e.g., Q and $\neg Q$. You may then (no longer assuming S) conclude that $\neg S$.

For an example of indirect proof, see the proof on p. 136. In this example, we get these contradictions: $\text{Cube}(b)$ contradicts $\text{Tet}(b)$, and $\text{Cube}(b)$ contradicts $\text{Dodec}(b)$. These are not TT-contradictions (like $\text{Cube}(b) \wedge \neg\text{Cube}(b)$), but they are still logically contradictory, in that it is impossible for them both to be true.

The contradiction symbol \perp

In constructing proofs we will use the symbol \perp (an upside down “tee”) to indicate that a contradiction has been reached. Rather than struggle for a way to pronounce this symbol, we will read \perp simply as *contradiction*.

TT-contradictions vs. other types

Not all contradictions are TT-contradictions. Consider these examples:

- $\text{Cube}(b) \wedge \neg\text{Cube}(b)$ TT-contradiction
- $a \neq a$ FO-contradiction (but not a TT-contradiction)
- $\text{Cube}(b) \wedge \text{Tet}(b)$ Logical contradiction (but not a FO-contradiction)
- $\text{Large}(b) \wedge \text{Adjoins}(b, c)$ TW-contradiction (but not a logical contradiction)

The reasons for this classification are as follows:

$\text{Cube}(b) \wedge \neg\text{Cube}(b)$

A truth table shows that this sentence cannot be true. Hence, it is a TT-contradiction.

$a \neq a$

No truth table can show that this sentence cannot be true, for a truth table can assign **F** to $a = a$. So it is not a TT-contradiction. But the meaning FOL assigns to $=$ and its rules for using names like **a** make the sentence $a = a$ true in every world. That is, an identity sentence is true in any world in which the names it contains name the same object, and no name can name two different objects in the same world. So $a \neq a$ is an FO-contradiction.

$\text{Cube}(b) \wedge \text{Tet}(b)$

FOL does not assign any particular meaning to the predicates **Cube** and **Tet**. For all FOL knows, this sentence can be true. So it is not an FO-contradiction. But given the meanings of these predicates, this sentence cannot be true—it is logically impossible for something to be both a cube and a tetrahedron. Hence, this sentence is a logical contradiction.

$\text{Large}(b) \wedge \text{Adjoins}(b, c)$

Given the meanings of the predicates **Large** and **Adjoins**, it should be perfectly possible for this sentence to be true. That is, we can describe a situation in which a large object adjoins another object. So this sentence is not a logical contradiction. However, there is no Tarski World in which this sentence is true. Hence, it is a TW-contradiction.

The basic rule (called \perp *introduction*) that we will use in our formal system (and in Fitch) to show that a contradiction has been reached will **require that our contradictory sentences be TT-contradictory**. This will require some extra footwork in cases in which we have other kinds of contradictions.

§ 5.4 Arguments with inconsistent premises

If a set of premises is inconsistent, any argument having those premises is valid. (If the premises are inconsistent, there is no possible circumstance in which they are all true. So no matter what the conclusion is, there is no possible circumstance in which the premises are all true and the conclusion is false.

But no such argument is sound, since a sound argument is not only valid but has true premises.

Why be interested in arguments with inconsistent premises? Well, we know that if you can derive a contradiction \perp from a set of premises, the set is inconsistent. (If it were possible for the premises all to be true, then since we have derived \perp from them, it would have to be possible for \perp to be true, and this clearly is not possible.)

We may not know, at the start, that our premises are inconsistent, but if we derive \perp from them, we have **established** that they are inconsistent. If a set of premises, or assumptions, is inconsistent, it is important to know this. And being able to deduce a contradiction from them is an excellent way of showing this. We may not be able to show, using logic alone, **which** premise is false, but we can establish that at least one of them is false.

Inconsistent premises vs. impossible sentences

If a set of premises (or any set of sentences, actually) is inconsistent, then at least one of the sentences in the set must be false. But **which** one is false depends on the world—there need not be a single sentence which is always the culprit, independent of what the facts happen to be.

To see this, open [Ch5Ex1.sen](#) and [Ch5Ex1.wld](#) on the Supplementary Exercises web page. You will see that it is impossible for all of the sentences to come out true—no matter how you change the world, at least one sentence comes out false. You can make **any three** of them true, but you can't make all four true.

Contrast this case with the case of [Ch5Ex2.sen](#) and [Ch5Ex2.wld](#). Here we have an inconsistent set of sentences where there **is** a culprit—the last sentence cannot be true (it is, in fact, TT-impossible). It is truly a “bad apple”: it will make **any** set of sentences it belongs to inconsistent.

To see the inconsistency of these sets of sentences, open [Ch5Ex1.prf](#) and [Ch5Ex2.prf](#). These two Fitch proofs contain the sets of sentences above as their premise-sets.

Notice that in both cases, the arguments are valid. That is, in both cases, \perp is a tautological consequence of the premises. (Check this out using **Taut Con**.) Notice, too, that in **Ex1**, the argument checks out **only if** all four premises are cited. But in **Ex2**, the argument checks out if (and only if) the “culprit” premise is cited.

Here we see the difference between two kinds of inconsistent sets: one (**Ex2**) contains an impossible sentence, the other (**Ex1**) does not. Each sentence in **Ex1** is possible; what is impossible is the conjunction of all four.

A connection between validity and inconsistency

When an argument is valid, its conclusion is a logical consequence of its premises. Another way to put this is to say that it would be **inconsistent** to assert the premises and deny the conclusion.

This means that for an argument to be valid is for the set of sentences consisting of all of the premises together with the **negation** of the conclusion to be inconsistent.

Examples

This set of sentences is **inconsistent**:

$$\{\text{Cube}(a) \vee \text{Cube}(b), \neg\text{Cube}(a), \neg\text{Cube}(b)\}$$

And so this argument is **valid**:

$$\begin{array}{|l} \text{Cube}(a) \vee \text{Cube}(b) \\ \neg\text{Cube}(a) \\ \hline \text{Cube}(b) \end{array}$$

This set of sentences is **consistent**:

$$\{\text{Tet}(a) \vee \text{Tet}(b), \text{Tet}(a), \text{Tet}(b)\}$$

And so this argument is **invalid**:

$$\begin{array}{|l} \text{Tet}(a) \vee \text{Tet}(b) \\ \text{Tet}(a) \\ \hline \neg\text{Tet}(b) \end{array}$$

Remember: for an argument to be valid is for its premises to be inconsistent with the negation of its conclusion.