Chapter 3: The Boolean Connectives

These are truth-functional connectives: the truth value (truth or falsity) of a compound sentence formed with such a connective is a function of (i.e., is completely determined by) the truth value of its components.

§ 3.1 Negation symbol: ¬

The negation of a true sentence is false; the negation of a false sentence is true. This information is recorded in the "truth-table" on p. 69. (Here and in other such tables we will abbreviate *true* by **T** and *false* by **F**.)

P	¬P
T	**F**
F	**T**

This table tells us that the negation of a sentence has the opposite truth value.

Some **terminology**: If P is atomic, then both P and ¬P are called "literals." Thus, Cube(a) and ¬Cube(a) are literals, but ¬¬Cube(a) is not a literal.

§ 3.2 Conjunction symbol: ∧

Writing conjunctions in FOL and in English

In English, conjunction is expressed by *and*, *moreover*, and *but*.

George is wealthy and John is not wealthy

George is wealthy but John is not wealthy

are both translated in FOL as Wealthy(george) ∧ ¬Wealthy(john)

Note that we read this FOL sentence as: "Wealthy George and not wealthy John." This way of reading FOL sentences will make it much easier later when we come to write them using Tarski's World.

∧ vs. 'and'

In English, 'and' often conveys a temporal meaning: 'and then' or 'and next'. Thus, these aren't equivalent:

Max went home and Claire went to sleep

Claire went to sleep and Max went home

The first suggests that Claire retired after Max left; the second suggests that Max didn't leave until after Claire retired.

But in FOL, the following sentences are equivalent:

WentHome(max) ∧ WentToSleep(claire)

WentToSleep(claire) ∧ WentHome(max)

That is, ∧ requires nothing more than joint truth, not temporal order.

The semantics of ∧

See the truth table for ∧ on p. 72.

P	Q	P ∧ Q
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

This table shows that a **conjunction** P ∧ Q is true in just one case: the case in which P is true and Q is true.

To see how this works, try playing the game in Tarski's World. Do the **You try it** on p. 72.

∧ in FOL where there's no corresponding English connective

How do we translate *d is a large cube* into FOL? Although the English sentence has no connective, we treat it as if it had an *and* in it: *d is a cube and d is large*. The advantage of this is that it makes translation easy—our FOL translation looks like this:

Cube(d) ∧ Large(d).

We can do this because in Tarski's World, we treat the size of an object as being entirely independent of its shape. Whether an object is a cube or a tetrahedron has no effect on whether it is counted as large, medium, or small.

This approach to translation into FOL keeps things simple, but it does not always give satisfactory results. Suppose we try putting *Dumbo is a small elephant* into FOL as:

Elephant(dumbo) ∧ Small(dumbo)

But small elephants are still large objects, so one might plausibly assert: *Although Dumbo is a small elephant, Dumbo is large*. If we put this into FOL using the scheme above, we get:

Elephant(dumbo) ∧ Small(dumbo) ∧ Large(dumbo)

This translation, however, is problematic. For one thing, this FOL sentence never comes out true, since nothing can be simultaneously, and without qualification, both small and large.

> To confirm this, try the following experiment: open Fitch and start a new proof with no premises. Add a new line, and enter the sentence ¬(Elephant(dumbo) ∧ Small(dumbo) ∧ Large(dumbo)). Now justify the line using **Ana Con** and click on **`Check Step`**. You will see that it checks out, which means that it is always true. Hence the sentence it negates, Elephant(dumbo) ∧ Small(dumbo) ∧ Large(dumbo), is always false.

For another thing, it is unclear what this FOL sentence is supposed to mean. Since the order of the conjuncts in an FOL conjunction has no effect on its meaning, we could translate it equally well in either of the following ways:

Although Dumbo is a small elephant, Dumbo is large.

Although Dumbo is a large elephant, Dumbo is small.

These English sentences are certainly not equivalent, so they cannot both correspond to the same FOL sentence. In English, *Dumbo is a small elephant* really means that Dumbo is small **for** an elephant. But there is no way to express *Dumbo is small for an elephant* in FOL using only the predicates Small and Elephant and the truth-functional connectives.

§ 3.3 Disjunction symbol: ∨

Writing disjunctions in FOL and in English

In English, disjunction is expressed by *or*.

George is wealthy or John is wealthy

Either George or John is wealthy

are both translated in FOL as Wealthy(george) ∨ Wealthy(john)

We read this FOL sentence as: "Wealthy George or wealthy John."

∨ vs. 'or'

In English, *or* is sometimes used in an "exclusive" sense, meaning *one or the other but not both*. But it will be our practice to use it in the (more common) "inclusive" sense, in which it means *one or the other or both*. (This is sometimes called "and/or.")

Thus, in our example above, the ∨ sentence comes out true in the event that both George and John are wealthy. If we need to say that **exactly** one of the two is wealthy (either George or John but not both), we can always write in FOL:

(Wealthy(george) ∨ Wealthy(john)) ∧ ¬(Wealthy(george)∧ Wealthy(john))

The semantics of ∨

See the truth table for ∨ on p. 75.

P	Q	P ∨ Q
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

This table shows that a **disjunction** P ∨ Q is true in three cases: P true and Q true, P true and Q false, and P false and Q true. That is, it is false in just one case: the case in which P is false and Q is false.

To see how this works, try playing the game in Tarski's World. Do the **You try it** on p. 76.

Some connectives that are not truth-functional

Lots of English connective words are not truth-functional. That is, if you use one of these words as the main connective in a compound sentence, the truth-value of the resulting sentence does not depend in all cases solely on the truth-values of the component sentences. An easy way to see that a connective is not truth-functional is to try to construct a truth-table for a compound in which it is the main connective. You will notice that you cannot complete all the rows.

Claire fed Scruffy while Max slept.

Fed(claire, scruffy)	Slept(max)	Fed(claire, scruffy) **while** Slept(max)
T	T	**?**
T	F	**F**
F	T	**F**
F	F	**F**

In this case, we know that if either component is false, the whole compound must be false. For example, if Claire did not feed Scruffy, it is false that she fed Scruffy while Max slept. The problem occurs when both components are true. It may be true that Claire fed Scruffy and true that Max slept, and nothing follows about whether the feeding and sleeping took place at the same time or not. The truth of both component sentences is compatible with either the truth or the falsity of the entire compound sentence.

Claire went home because she found Max boring

WentHome(claire)	Bored(max, claire)	WentHome(claire) **because** Bored(max, claire)
T	T	**?**
T	F	**F**
F	T	**F**
F	F	**F**

Once again, we know that if either component is false, the whole compound must be false. For example, if Claire did not find Max boring, it is false that she went home for that reason. Again, the problem occurs when both components are true. It may be true that Claire went home and true that Max bored her, and nothing follows about whether or not his boring her was the reason she went home. The truth of both component sentences is compatible with either the truth or the falsity of the entire compound sentence.

§ 3.4 Remarks about the game

The game rules for ¬, ∧, and ∨ are summarized on p. 78. There is no need to memorize them, though, as Tarski's World will always tell you what your commitments are (after you choose your initial commitment), and will tell you when it is your turn to move.

To play the game and be sure of winning, you will need to know not only **that** a sentence has the truth value you say it has (your commitment), but also **why** it does. This means, for example, that if you know that a disjunction is true, you will need to know **which** disjunct is true in order to be sure of winning. Similarly, if you know that a conjunction is false, you will need to know **which** conjunct is false in order to be sure of winning.

Sometimes, however, you may know the truth value of an entire compound sentence without knowing the truth values of its components. Suppose you have the sentence Cube(d) ∨ ¬Cube(d). You know that this is true even though you don't know which disjunct is the true one. If *d* is a cube, the left disjunct is true; otherwise, it's the right disjunct that's true. But you may not be able to **see** *d*; perhaps it is small, and hidden behind a larger object. Try exercise 3.11 to see how this works.

§ 3.5 Ambiguity and parentheses

In FOL, we need to be able to avoid ambiguities that can arise in English. The form

P and *Q* or *R*

is ambiguous. Does it mean *P*, *and either Q or R*? Or does it mean *either both P and Q*, *or R*? Notice how the auxiliary words *either* and *both*, working with *or* and *and*, respectively, remove the ambiguity. (You will see these at work in exercise 3.21, problems 1, 8, and 10.)

FOL does not have such auxiliary words. We use **parentheses** to remove ambiguity:

P ∧ (Q ∨ R)   (P ∧ Q) ∨ R

The effect is the same. The parentheses remove the ambiguity by showing which is the **main** connective, and which the subsidiary. (As we will say, they show which connective has the larger "scope.")

Scope is especially important with **negation**. Compare these sentences:

¬Cube(a) ∧ Cube(b)   ¬(Cube(a) ∧ Cube(b))

The first says that *a* is not a cube, but *b* is a cube. The second does not give us such definite information about *a* and *b*. All it tells us is that that aren't **both** cubes. That is, either *a* is not a cube, or *b* is not a cube, or perhaps neither is a cube. The first is a much more informative claim.

Practice

Let's check out some sentences in a sample world. Download the files Sentences TF1 and World TF1 from the course web site—they're on the Supplementary Exercises page. Then predict the truth values of these sentences in this world, and play the game with Tarski's World.

§ 3.6 Equivalent ways of saying things

There are many different ways of saying the same thing in FOL. That is, for any given FOL sentence, we can come up with a different but **equivalent** FOL sentence. (*Equivalent* here means *comes out true or false in exactly the same cases*, or *has the same truth table*. Here are some of the more common equivalent pairs. (⇔ represents equivalence).

¬¬P ⇔ P	Double negation
¬(P ∧ Q) ⇔ (¬P ∨ ¬Q)	DeMorgan's law
¬(P ∨ Q) ⇔ (¬P ∧ ¬Q)	DeMorgan's law

Note that these can be combined to yield more equivalences:

¬(¬P ∧ ¬Q) ⇔ (P ∨ Q)	∨ defined in terms of ∧
¬(¬P ∨ ¬Q) ⇔ (P ∧ Q)	∧ defined in terms of ∨

§ 3.7 Translation

Under what conditions do we count an FOL sentence to be a correct translation of an English sentence? The only rule is that the two sentences must **agree in truth value in all possible circumstances**.

Notice that this requires more than that the two sentences both be true, or both be false. Agreement in (actual) truth value may be due to accidental circumstances that happen to obtain. The two sentences must agree even if you "change the facts."

This means that any two **equivalent** FOL sentences will be equally correct translations of any English sentence that either of them correctly translates. That is, if an FOL sentence S is a good translation of an English sentence *S*, and S is equivalent to some other FOL sentence S′, then S′ also counts as a correct translation of *S*.

A result of this policy is that some rather unnatural sounding translations will count as correct. Consider the English sentence *b is a cube and c is a tetrahedron*. The most natural translation of that into FOL is:

$$\text{Cube(b)} \land \text{Tet(c)}$$

But given the DeMorgan and Double Negation equivalences noted above, we can see that:

$$(\text{Cube(b)} \land \text{Tet(c)}) \Leftrightarrow \neg(\neg\text{Cube(b)} \lor \neg\text{Tet(c)})$$

Hence, our sentence is equally accurately translated as:

$$\neg(\neg\text{Cube(b)} \lor \neg\text{Tet(c)})$$

But even though this is (technically) correct, it is not the "best" or most natural translation, for it introduces three *nots* and an *or*, none of which were present in the English original.

Still, both Tarski's World and I will follow the policy of counting any translation that is equivalent to the "right" one as correct.

[Note that later in the term, when the sentences get more complicated, the Grade Grinder may not always be able to tell whether an answer you give is equivalent to the correct answer. If that happens, it will tell you that it "timed out"—i.e., couldn't figure out whether your answer was correct. Bring any such cases to your instructor for evaluation.]

