Chapter 11: Multiple Quantifiers

§ 11.1 Multiple uses of a single quantifier

We begin by considering sentences in which there is more than one quantifier of the same "quantity"—i.e., sentences with two or more existential quantifiers, and sentences with two or more universal quantifiers. Only later will we consider the more difficult cases of "mixed" quantifiers.

Avoid prenex form

The examples on pp. 289-90 are instructive. In both cases, we see that there are two equivalent FOL sentences that adequately translate the same English sentence:

Some cube is left of a tetrahedron.

$$\exists x\ \exists y\ [\text{Cube}(x) \land \text{Tet}(y) \land \text{LeftOf}(x, y)]$$

$$\exists x\ [\text{Cube}(x) \land \exists y\ (\text{Tet}(y) \land \text{LeftOf}(x, y))]$$

Every cube is left of every tetrahedron.

$$\forall x\ \forall y\ [(\text{Cube}(x) \land \text{Tet}(y)) \rightarrow \text{LeftOf}(x, y)]$$

$$\forall x\ [\text{Cube}(x) \rightarrow \forall y\ (\text{Tet}(y) \rightarrow \text{LeftOf}(x, y))]$$

The first FOL sentence in each case has all the quantifiers out in front—in "prenex" form, as logicians say. But there is an advantage to using the second FOL sentence, with one of the quantifiers embedded. For this way of translating English into FOL makes clearer the overall "Aristotelian" structure of the sentence, and hence such an FOL translation will be easier to come by in a systematic way.

The overall Aristotelian structure becomes clear if we treat each of the phrases *left of a tetrahedron* and *left of every tetrahedron* as a single, indissoluble, unit:

left-of-a-tetrahedron *left-of-every-tetrahedron*

and represent each as a single FOL predicate, say G and H, respectively. In that case, we can think of our original sentences as:

Some cube is G. *Every cube is H.*

And it is easy to translate these into FOL as, respectively:

$\exists x\ [\text{Cube}(x) \land G(x)]$ $\forall x\ [\text{Cube}(x) \rightarrow H(x)]$

Our next task is to replace the temporary wffs G(x) and H(x) with proper FOL wffs. Since G(x) represents *x is left of a tetrahedron* and H(x) represents *x is left of every tetrahedron*, we must translate these into wffs of FOL. In so doing, we must be sure that in each case our translation contains a free occurrence of *x*, and hence is not a sentence. (Remember, a wff with a free occurrence of a variable is not a sentence.) But if we ignore the fact that these wffs are not sentences, we will recognize their forms as familiar Aristotelian ones.

G(x):
- *x* is left of a tetrahedron
- Some tetrahedron has *x* to its left
- ∃*y* (*y* is a tetrahedron and *x* is to the left of *y*)
- $\exists y\ (\text{Tet}(y) \land \text{LeftOf}(x, y))$

H(x) { *x* is left of every tetrahedron
Every tetrahedron has *x* to its left
∀*y* (if *y* is a tetrahedron, then *x* is to the left of *y*)
∀y (Tet(y) → LeftOf(x, y)) }

Notice that in both cases, we chose a new variable, *y*, for our new quantifier. (We did this to keep our occurrences of *x* free.) Now we simply replace G(x) with ∃y (Tet(y) ∧ LeftOf(x, y)) and H(x) with ∀y (Tet(y) → LeftOf(x, y)), and our FOL translations are complete:

∃x [Cube(x) ∧ ∃y (Tet(y) ∧ LeftOf(x, y))]

∀x [Cube(x) → ∀y (Tet(y) → LeftOf(x, y))]

The moral of this story is to translate complex sentences into FOL by first figuring out their overall structure (usually an Aristotelian form) and then replacing the embedded wffs with more complex wffs containing quantifiers. If you do this, you will find that you **seldom** produce an FOL sentence in prenex form.

Multiple quantifiers don't guarantee multiple objects

It is tempting to read ∃x ∃y as saying *there are two objects, x and y* …. But this would be a mistake, for the variables x and y may pick out the same object. To see why this is so, open Tarski's World and write ∃x ∃y (Cube(x) ∧ Cube(y)) in a new sentence file.

Next, create a new world with a single cube in it. Then try playing the game committed to false. Do you see why you can't win? Tarski will name the one cube n_1 and will pick it as the value for both x and y. You will end up committed to the falsity of Cube(n_1) ∧ Cube(n_1), which is a losing position.

In other words, just as the truth of Cube(a) ∧ Cube(b) does not guarantee that there is more than one cube, neither does the truth of the quantified sentence ∃x ∃y (Cube(x) ∧ Cube(y)) guarantee this. For just as a and b may name the same object, so too may the quantifiers ∃x and ∃y pick out the same object. In fact, the FOL sentence ∃x ∃y x = y is a logical truth! In every (non-empty) world, there is sure to be some object satisfying the condition ∃y x = y (that is, the condition of *being identical to something*), since we can always pick the same object as the value for both x and y. Some object is identical to *something*, since some object is identical to itself. That is, ∃x x = x logically implies ∃x ∃y x = y.

§ 11.2 Mixed quantifiers

We now consider sentences with multiple quantifiers in which the quantifiers are "mixed"—some universal and some existential.

A simple Aristotelian form

Consider a slight variation on an example we looked at above:

Every cube is left of a tetrahedron.

This clearly has an Aristotelian form, ∀x (P(x) → Q(x)), where P(x) means *x is a cube* and Q(x) means *x is left of a tetrahedron*. Earlier, we saw that we could translate the wff *x is left of a tetrahedron* as ∃y (Tet(y) ∧ LeftOf(x, y)), so we just plug that in here for Q(x). The result is this FOL sentence:

∀x [Cube(x) → ∃y (Tet(y) ∧ LeftOf(x, y))]

Note, by the way, that the embedded wff $\exists y\ (\text{Tet}(y) \wedge \text{LeftOf}(x, y)$ is itself of the Aristotelian form **I**: *Some tetrahedron has x to its left*. So our translation has the overall structure of an Aristotelian **A** sentence with an **I** wff embedded inside it as the consequent of the conditional.

Order of quantifiers

When quantifiers in the same sentence are of the same quantity (all universal or all existential), the order in which they occur does not matter. But when they are mixed, the order in which they occur becomes crucial. Consider these examples:

$$\forall x\ \forall y\ \text{Likes}(x, y) \quad \Leftrightarrow \quad \forall y\ \forall x\ \text{Likes}(x, y)$$
$$\exists x\ \exists y\ \text{Likes}(x, y) \quad \Leftrightarrow \quad \exists y\ \exists x\ \text{Likes}(x, y)$$

These are clearly equivalent pairs. The first pair contains two different ways of saying *everyone likes everyone*. The second contains two different ways of saying *someone likes someone*.

Now consider this mixed quantifier case:

$$\forall x\ \exists y\ \text{Likes}(x, y) \quad \not\Leftrightarrow \quad \exists y\ \forall x\ \text{Likes}(x, y)$$

Clearly these are not equivalent sentences. The one on the left says (very plausibly) that everyone likes someone (or other), but allows for the possibility that different people have different likes—I like Edgar Martinez, you like Ken Griffey, Jr., Madonna likes herself, etc. The one on the right, however, says something much stronger—it says that there is at least one person so well liked that *everyone* likes him or her. (It's very unlikely that there is such a person, and so very unlikely that the sentence on the right is true.)

Notice that the stronger sentence (on the right) logically implies the weaker one (on the left). In general, an $\exists\forall$ sentence logically implies its $\forall\exists$ counterpart. (We will return to these "stronger – weaker" pairs later in this chapter.)

For a more dramatic contrast, consider this pair of sentences:

$$\forall x\ \exists y\ x = y \quad \not\Leftrightarrow \quad \exists y\ \forall x\ x = y$$

Again, these are not equivalent. The one on the left is a logical truth; it says *everything is identical to something*. The one on the right says *there is something such that everything is identical to that thing*, and this comes very close to being logically false. (It is not logically false, because there are at least some worlds in which it is true. Can you think of one? You should be able to. If you can't, try constructing a Tarski World in which it comes out true.)

To cement your understanding of mixed quantifier sentences, do the **You try it** on p. 295.

§ 11.3 The step-by-step method of translation

We have already encountered the step-by-step method of translation in our discussion of the advisability of avoiding prenex form in §11.1. The trick is to start with the outer or "gross" structure of the sentence, and then move inward. (For this reason, the step-by-step method is sometimes called *paraphrasing inward*.)

Let's try our hand at a fairly simple example:

Some cube that adjoins a dodecahedron is larger than every tetrahedron.

The step-by-step procedure is outlined in the file Step-by-step1.sen, on the Supplementary Exercises page. Here's a short description of the procedure:

- First, find the gross structure of the sentence. In this case, it's one of the Aristotelian forms, **I**: *Some P's are Q's*, or ∃x (P(x) ∧ Q(x)). This gives us the overall form:

  ∃*x* (*x* is a cube that adjoins a dodecahedron ∧ *x* is larger than every tetrahedron).

- Then isolate the embedded wffs:

  x is a cube that adjoins a dodecahedron
  x is larger than every tetrahedron

  and translate those into FOL wffs with free x.

- This yields these wffs:

  Cube(x) ∧ ∃y (Dodec(y) ∧ Adjoins(x, y))
  ∀y (Tet(y) → Larger(x, y))

- Finally, plug these wffs into our overall **I** form ∃x (P(x) ∧ Q(x)) in place of the two conjuncts P(x) and Q(x) This yields our completed translation:

  ∃x [Cube(x) ∧ ∃y (Dodec(y) ∧ Adjoins(x, y)) ∧
  ∀y (Tet(y) → Larger(x, y))]

To check that this translation is correct, open the file Step-by-step1.wld. The sentence we've written should come out true in this world. Try making some changes to the world and confirm that the resulting evaluation of our sentence is appropriate.

For example, move the dodecahedron away from the cube—the sentence should become false. Next, put the dodecahedron back where it was, but make one of the tetrahedra larger—the sentence should become false. Finally, make the tetrahedron small again, but shrink the cube—the sentence should become false. If you do not get these results, your translation is incorrect.

Now let's attempt the difficult example mentioned on p. 298:

No cube to the right of a tetrahedron is to the left of a larger dodecahedron.

We can begin by determining the gross structure of the sentence. Is it an Aristotelian form? If so, which? Clearly, it is an **E** sentence. Let us use our hyphenation technique to make this evident:

No cube-to-the-right-of-a-tetrahedron is to-the-left-of-a-larger-dodecahedron.

We then treat the hyphenated phrases as if they were simple predicates, and put the sentence into its "gross" Aristotelian form:

∀*x* (*x* is a cube-to-the-right-of-a-tetrahedron → ¬ *x* is to-the-left-of-a-larger-dodecahedron)

Our next task is to translate the two embedded wffs. First, we tackle the antecedent, proceeding in a step-by-step way:

x is a cube-to-the-right-of-a-tetrahedron

x is a cube ∧ *x* is to the right of a tetrahedron

x is a cube ∧ some tetrahedron has *x* to its right

x is a cube ∧ ∃*y* (*y* is a tetrahedron ∧ *x* is right of *y*)

$Cube(x) \land \exists y\ (Tet(y) \land RightOf(x, y))$

Next, the consequent:

x is to-the-left-of-a-larger-dodecahedron

Before we can begin to put this wff into FOL, we must decide what the dodecahedron is being said to be larger than. There seem to be two possibilities: (1) a dodecahedron larger than *x*, and (2) a dodecahedron larger than the tetrahedron mentioned in the antecedent. The sentence seems genuinely ambiguous between these possibilities, although (1) seems more likely to my ears, so we will go with that reading.

x is to the left of a dodecahedron that is larger than *x*

There is a dodecahedron that *x* is to the left of and that is larger than *x*

There is a dodecahedron such that *x* is to the left of it and it is larger than *x*

$\exists y\ (Dodec(y) \land LeftOf(x, y) \land Larger(y, x))$

We now have our outer framework (the **E** sentence):

$\forall x\ (P(x) \rightarrow \neg Q(x))$

and the two wffs that will become its embedded antecedent and consequent. All that remains is to assemble the pieces—we substitute our two wffs for $P(x)$ and $Q(x)$, respectively:

$\forall x\ ((Cube(x) \land \exists y\ (Tet(y) \land RightOf(x, y))) \rightarrow \neg\exists y\ (Dodec(y) \land LeftOf(x, y) \land Larger(y, x)))$

And that's how the step-by-step method of translation works.

§ 11.4 Paraphrasing English

There are times when the step-by-step method cannot be applied directly. This happens frequently in cases in which the quantifier word *something* is used with **universal** force. Example:

If something is a cube, it is not a tetrahedron.

The tip-off that the *something* here is a universal quantifier is the occurrence of the pronoun *it* in the consequent. This *it* functions in English as a variable, so it must be bound by a quantifier. But the only quantifier around is the one in the antecedent. If we make it existential and include the variable *it* in its scope, we would get:

There is something such that, if is a cube, it is not a tetrahedron.

$\exists x\ (Cube(x) \rightarrow \neg Tet(x))$

But this sentence is too weak, as we've already seen, to say what the English sentence says. (The existence of a single non-cube, for example, makes it true.) But if we restrict the scope of $\exists x$ to the antecedent, we get:

$\exists x\ Cube(x) \rightarrow \neg Tet(x)$

and this wff is not a sentence (the x in $Tet(x)$ is free). The step-by-step method seems to have failed us.

What we must do, instead, is to **paraphrase** the original sentence in a way that gives the quantifier large scope. When we do this, we see that the quantifier is actually universal:

If anything is a cube, it is not a tetrahedron.

For anything you like, if it is a cube, it is not a tetrahedron.

No cube is a tetrahedron.

$\forall x\ (Cube(x) \rightarrow \neg Tet(x))$

Donkey sentences

The classic example of a so-called “donkey sentence” is this:

Every farmer who owns a donkey beats it.

The difficulty with such sentences is that they resemble ones in which the phrase *a donkey* is properly treated as an existential quantifier. For example:

Every farmer who owns a donkey buys hay.

This goes into FOL straightforwardly as:

$\forall x\ ((Farmer(x) \wedge \exists y\ (Donkey(y) \wedge Owns(x, y)) \rightarrow BuysHay(x)))$

Note that the scope of the existential quantifier stops at the end of the antecedent. If we try to translate the classic donkey sentence this way, we get:

$\forall x\ ((Farmer(x) \wedge \exists y\ (Donkey(y) \wedge Owns(x, y)) \rightarrow Beats(x, y)))$

and this wff is not a sentence, since the y in the consequent is free. We can see this by translating the wff back into English:

Every farmer who owns a donkey beats y.

In order to have a sentence (a wff with no free variables) we must make sure that the y variable in $Beats(x, y)$ is bound by the quantifier (“a donkey”) in the antecedent. This means we must paraphrase the original English sentence, perhaps in one of the following ways:

Any farmer who owns any donkey beats it.

Every farmer is such that any donkey he owns is beaten by him.

Every farmer beats every donkey he owns.

This makes clear that the original sentence contains two universal quantifiers:

$\forall x\ (Farmer(x) \rightarrow \forall y\ ((Donkey(y) \wedge Owns(x, y)) \rightarrow Beats(x, y)))$

In *LPL* (p. 301), a slightly different (but equivalent) translation was obtained:

$\forall x\ (Donkey\ (x) \rightarrow \forall y\ ((Farmer(y) \wedge Owns(y, x)) \rightarrow Beats(y, x)))$

§ 11.5 Ambiguity and context sensitivity

Sentences containing both universal and existential quantifiers can be ambiguous, depending on the scope the quantifiers receive. Here’s an example:

Some man has been calling Becky every hour.

When the existential quantifier is given wide scope, we get what is called the “strong” reading:

$\exists x\ (Man(x) \wedge \forall y\ (Hour(y) \rightarrow Calls(x, becky, y)))$

This FOL sentence suggests that Becky is being harassed by a single persistent (and unwanted) caller. On the other hand, if we take the English sentence to mean merely that Becky is popular, and has been receiving calls from many different interested gentlemen, the right way to put it would be this (the “weak” reading):

$\forall y\ (Hour(y) \rightarrow \exists x\ (Man(x) \wedge Calls(x, becky, y)))$

The weak reading is a logical consequence of the strong reading, but not conversely.

In other cases, the context makes the weak reading obviously the intended one. Consider the following sentence (attributed to the showman P. T. Barnum):

There's a sucker born every minute.

The strong reading here is obviously inappropriate:

$\exists x\ (Sucker(x) \wedge \forall y\ (Minute(y) \rightarrow BornAt(x, y)))$

The trouble with this FOL translation is that it says that some unfortunate individual has the property of being born (again, and again) at each and every minute. What the original sentence obviously intended was the weaker claim, that no matter what minute you pick, some sucker is being born then (a different sucker at each succeeding minute, of course, since each of us is born only once). Here's the FOL version of the intended (weak) reading:

$\forall y\ (Minute(y) \rightarrow \exists x\ (Sucker(x) \wedge BornAt(x, y)))$

The Doris Day principle

In our next example, there are multiple sources of ambiguity—not just the scope of the quantifiers, but their quantity. Here's the example:

Everybody loves a lover.

Only four words, but a mare's nest of ambiguity! First, there is the **order** of the quantifiers: does *everybody* have wide scope, or does *a lover* have wide scope? Second, there is the **quantity** of the quantifiers: is *a lover* an existential quantifier ("some lover") or universal ("every lover")? We'll begin with those two questions, but as we'll see later, there's yet a further possible source of ambiguity.

Quantity

Does *a lover* here mean *some lover* or *every lover*? Without a context, it's hard to tell, so we'll have to keep both options open.

Order

Which of the two quantifiers has wide scope? Again, it seems we'll have to keep both options open. This would seem to give us, at least in the abstract, four possibilities. We can represent them (temporarily) in the following slightly unorthodox way:

1. ∃ lover *y* ∀ person *x* : *x* loves *y*
2. ∀ person *x* ∃ lover *y*: *x* loves *y*
3. ∀ lover *y* ∀ person *x* : *x* loves *y*
4. ∀ person *x* ∀ lover *y*: *x* loves *y*

Since (3) and (4) do not involve mixed quantifiers, they are clearly equivalent. (3) says that every lover is loved by every person, and (4) says that every person loves every lover. So we only need to consider one of them—we'll drop (4) from consideration. But the other three are still in the running.

(1) says that there is some lover, *y*, such that everyone loves *y*. (This might have been true back in the early days of motion pictures—Rudolph Valentino was a lover, and everybody loved him.)

(2) says that for each person, *x*, there is a lover, *y*, such that *x* loves *y*. (This leaves open the possibility, which (1) does not, that different people might love different lovers—e.g., Julia Roberts is a lover, and Brad Pitt is a lover, and I love Julia (but I don't love Brad), and you love Brad (but not Julia), etc.

(3) says that every lover is loved by everyone. This seems to have been the original intention of the poet Ralph Waldo Emerson when he wrote "Here's to the happy man: All the world loves a lover." That is, no matter who you are, all you have to do is to be a lover, and everyone will love you.

So (3) seems to be the favored reading of this potentially ambiguous sentence. It certainly is the correct reading in the context in which I first ran across it, which was in a song that Doris Day made popular. (It rose to #6 on the charts in 1958, and got a Grammy nomination.) The song begins:

> *Everybody loves a lover, I'm a lover, everybody loves me. …And I love everybody, since I fell in love with you.*

Doris seems to be advancing an argument here, with two conclusions:

| Everybody loves a lover
| Doris is a lover.
|—
| Everybody loves Doris.
| Doris loves everybody.

Charity demands that we interpret the argument as valid, if we can. And this argument is valid only if we interpret the ambiguous first premise as meaning (3). So that is its likely meaning in this context. (Exercise: can you explain why the argument would be invalid if the first premise is interpreted as (1) or (2)?)

Hence, our preferred reading of our ambiguous sentence is:

$$\forall \text{ person } x \ \forall \text{ lover } y\text{: } x \text{ loves } y$$

This, of course, is not an FOL sentence. But it is easy to see how to put it into FOL. For it says that no matter which objects *x* and *y* we take, if *x* is a person and *y* is a lover, then *x* loves *y*. That is:

$$\forall x\ \forall y\ ((x \text{ is a person} \land y \text{ is a lover}) \rightarrow x \text{ loves } y)$$

If we take the domain of discourse to be restricted to *persons* we can simply drop the conjunct "*x* is a person." So we can put this into FOL as:

$$\forall x\ \forall y\ (\mathrm{Lover}(y) \rightarrow \mathrm{Loves}(x, y))$$

We must now consider one final potential source of ambiguity: the predicate *is a lover*. What, exactly, does this mean? It seems clear that we should be able to express the meaning of the **unary** predicate Lover(y) in terms of the **binary** predicate Loves(y, z). But how should we do this? The following seems to me to be correct:

$$\mathrm{Lover}(y) =_{df} \exists z\ \mathrm{Loves}(y, z)$$

But I can imagine a case being made for one of the following, among others:

$$\text{Lover}(y) =_{df} \forall z\ \text{Loves}(y, z)$$

$$\text{Lover}(y) =_{df} \exists z\ (\text{Loves}(y, z) \wedge \text{Loves}(z, y))$$

$$\text{Lover}(y) =_{df} \exists z\ \text{Loves}(y, z) \wedge \exists z\ \text{Loves}(z, y)$$

The first option seems best: to be a lover is simply to love someone. In its favor is that it, alone, passes Grice's cancellability test. (Roughly: you can be a lover without either loving everyone, or being loved by someone you love; or even being loved by anyone at all, but you cannot be a lover without loving someone.)

So we can replace Lover(y) with ∃z Loves(y, z), and come up with our FOL version of what I'll call the **Doris Day principle**:

$$\forall x\ \forall y\ (\exists z\ \text{Loves}(y, z) \rightarrow \text{Loves}(x, y))$$

So much for the translation issue. We will revisit the Doris Day principle in Chapter 13, where it will figure in some proofs.

§ 11.7 Prenex Form

When we started doing translations involving multiple quantifiers (§ 11.1), I warned you that when doing translations, it is best to avoid "prenex form," i.e., placing all quantifiers at the beginning of the FOL sentence. The reason was that attempting to produce such sentences was likely to lead to translation errors.

But it turns out that for many purposes, it is advantageous to have an FOL sentence in prenex form. Furthermore, every FOL sentence has an equivalent sentence (in fact, many equivalent sentences) in prenex form. In this section, we discuss methods for putting sentences into prenex form. But first, let's refresh ourselves on why trying to put sentences directly into prenex form is likely to lead to error.

Pitfalls of going directly to prenex

Here's a fairly simple example of a sentence not in prenex form:

1. ∀x Cube(x) → ∀y Large(y)

If we simply pull all the quantifiers to the outside, we would produce this:

2. ∀x ∀y (Cube(x) → Large(y))

(2) is in prenex form, but it is **not equivalent to (1).** If you are in any doubt about this, try evaluating the two sentences in a Tarski world. You will find it easy to get them to disagree in truth value.

Converting to prenex form

To convert (1) to prenex form, we must remember these equivalences that we learned in Chapter 10:

3. ∃x (Q(x) → P) ⇔ ∀x Q(x) → P
4. ∀x (P → Q(x)) ⇔ P → ∀x Q(x)

Remember, these equivalences require that P is either a sentence or a wff containing no free occurrence of x.

First, we apply equivalence (3) to sentence (1) and obtain:

5. ∃x (Cube(x) → ∀y Large(y))

What we did was to pull the universal quantifier off of the antecedent and change it to an existential quantifier whose scope is the entire conditional. Next, we will apply equivalence (4) to sentence (5) and obtain:

6. ∃x ∀y (Cube(x) → Large(y))

Here we simply moved the universal quantifier, ∀y, from the consequent to the entire conditional. Note that in applying (4) to (5), P is the wff Cube(x), which contains no free occurrences of y, the variable in the exported quantifier.

Comparing (6) with (2), you can see the difference: (2) begins ∀x, while (6) begins ∃x, and otherwise they are identical. It should be obvious, therefore, that they are not equivalent.

Rules for conversion to prenex

To convert an FOL sentence to prenex form, we make use of these equivalences that we learned in chapter 10:

- DeMorgan laws for quantifiers
- Distributing ∀ through ∧
- Distributing ∃ through ∨
- Null quantification
- Null quantification over →
- Replacing bound variables

In addition, we will need to use some of the handy truth-functional equivalences we learned in § 8.1, especially to get rid of biconditionals:

The "biconditional – conjunction" equivalence

P ↔ Q ⇔ (P → Q) ∧ (Q → P)

The "biconditional – disjunction" equivalence

P ↔ Q ⇔ (P ∧ Q) ∨ (¬P ∧ ¬Q)

The general strategy is to work from the inside out, moving quantifiers "outward" so that they get larger scope. Since all of our quantifiers will appear at the beginning of our ultimate sentence, we must be sure that no quantifier gets reused (e.g., we cannot have both ∀x and ∃x); each time we have a quantifier that repeats a variable, we will have to change to a new variable.

We will definitely need to get rid of biconditionals, and it is sometimes useful to get rid of conditionals, as well. The procedure is best illustrated by examples, to which we now turn.

Example #1

We'll start with a simple example:

∀x Cube(x) ∨ ¬∃x Tet(x)

The strategy will be to drive the negation sign through the quantifier ∃x, converting it to ∀x¬ (appealing to DeMorgan laws for quantifiers), then rewrite the second quantifier with a new variable, y (replacing bound variables), then pull the quantifiers to the outside (null quantification). We'll do this one step at a time.

$\forall x\ Cube(x) \lor \neg\exists x\ Tet(x)$

$\forall x\ Cube(x) \lor \forall x\ \neg Tet(x)$

$\forall x\ Cube(x) \lor \forall y\ \neg Tet(y)$

$\forall x\ (Cube(x) \lor \forall y\ \neg Tet(y))$

$\forall x\ \forall y\ (Cube(x) \lor \neg Tet(y))$

Notice that we might have performed the last two steps (pulling out the universal quantifiers) in reverse order. If we had, we would have ended up with this (equivalent) prenex form:

$\forall y\ \forall x\ (Cube(x) \lor \neg Tet(y))$

Example #2

Sentences containing biconditionals are exceptionally tricky to put into prenex form. Here's one that in its non-prenex form is very easy to understand:

$\exists x\ Cube(x) \leftrightarrow Tet(b)$

This says that a cube exists if, and only if, *b* is a tetrahedron. Now let's put it into prenex form. We'll proceed in a step-by-step fashion, applying the equivalences mentioned above.

$\exists x\ Cube(x) \leftrightarrow Tet(b)$	
$(\exists x\ Cube(x) \to Tet(b)) \land (Tet(b) \to \exists x\ Cube(x))$	bicond. – conj.
$\forall x\ (Cube(x) \to Tet(b)) \land \exists x\ (Tet(b) \to Cube(x))$	null quant. over $\to$
$\forall x\ (Cube(x) \to Tet(b)) \land \exists y\ (Tet(b) \to Cube(y))$	replace bound vbl.
$\forall x\ ((Cube(x) \to Tet(b)) \land \exists y\ (Tet(b) \to Cube(y)))$	dist. $\forall$ thru $\land$
$\forall x\ \exists y\ ((Cube(x) \to Tet(b)) \land (Tet(b) \to Cube(y)))$	dist. $\exists$ thru $\land$

Our sentence is now in prenex form. But notice the price: although the quantifiers are all out in front, the FOL sentence is hard to understand when compared to the original.

Note that we would end up with a different, but still equivalent, prenex form if we began with the biconditional – disjunction equivalence:

$\exists x\ Cube(x) \leftrightarrow Tet(b)$	
$(\exists x\ Cube(x) \land Tet(b)) \lor (\neg\exists x\ Cube(x) \land \neg Tet(b))$	bicond. – disj.
$(\exists x\ Cube(x) \land Tet(b)) \lor (\forall x\ \neg Cube(x) \land \neg Tet(b))$	DeM quant.
$\exists x\ ((Cube(x) \land Tet(b)) \lor \forall x\ (\neg Cube(x) \land \neg Tet(b)))$	dist. $\exists$, $\forall$ thru $\land$.
$\exists x\ ((Cube(x) \land Tet(b)) \lor \forall y\ (\neg Cube(y) \land \neg Tet(b)))$	replace bound vbl.
$\exists x\ \forall y\ ((Cube(x) \land Tet(b)) \lor (\neg Cube(y) \land \neg Tet(b)))$	dist. $\forall$ thru $\lor$.

Here we have another, equivalent, prenex form of our original sentence. You can use Fitch's **FO Con** to confirm that the two prenex forms are equivalent. [Note (11/19/04): the **FO Con** in Fitch 2.2 gives the wrong evaluation of this equivalence.]

An English example: first, translate

Here we start with an English sentence, translate it into FOL, and then put it into prenex form:

No cube that adjoins a tetrahedron is back of every dodecahedron.

We make no effort to go directly to prenex form. Instead, we translate into FOL using the step-by-step method:

$\forall x$ (x is a cube-that-adjoins-a-tetrahedron $\rightarrow \neg$ x is back-of-every-dodecahedron)

$\forall x$ ((x is a cube $\wedge$ $\exists y$ (y is a tetrahedron $\wedge$ x adjoins y)) $\rightarrow \neg$ x is back-of-every-dodecahedron)

$\forall x$ ((x is a cube $\wedge$ $\exists y$ (y is a tetrahedron $\wedge$ x adjoins y)) $\rightarrow \neg$ $\forall z$ (z is a dodecahedron $\rightarrow$ x is back of z))

$\forall x\ ((\text{Cube}(x) \wedge \exists y\ (\text{Tet}(y) \wedge \text{Adjoins}(x, y))) \rightarrow \neg\forall z\ (\text{Dodec}(z) \rightarrow \text{BackOf}(x, z)))$

Next, convert to prenex

Now let's apply our conversion technique to our example.

Before applying our technique, it's handy to use (temporarily!) a more compact notation:

$$\forall x\ ((Cx \wedge \exists y\ (Ty \wedge Axy)) \rightarrow \neg\forall z\ (Dz \rightarrow Bxz))$$

This is what we do to produce the compact notation. We use single letters for the predicates; we remove the parentheses that surround their arguments; we delete the commas and spaces that separate the arguments (assuming that the arguments are all single letters, e.g., all variables). For example, we abbreviate Cube(x) as Cx, and Adjoins(x, y) as Axy, etc. When we do this, the quantifiers and connectives become more prominent, making it easier for us to see where the conversion equivalences apply.

Now we convert to prenex form. First, we drive the negation sign inside the scope of the quantifier $\forall z$:

$$\forall x\ ((Cx \wedge \exists y\ (Ty \wedge Axy)) \rightarrow \exists z\ \neg(Dz \rightarrow Bxz))$$

Next, we look at the conjunction that is the antecedent of the first conditional:

$$Cx \wedge \exists y\ (Ty \wedge Axy)$$

And we apply one of the "null quantification" equivalences:

$$\exists x\ (P \wedge Q(x)) \quad \Leftrightarrow \quad P \wedge \exists x\ Q(x)$$

This allows us to pull the existential quantifier out:

$$\exists y\ (Cx \wedge (Ty \wedge Axy))$$

Replacing this in the entire sentence yields:

$$\forall x\ (\exists y\ (Cx \wedge (Ty \wedge Axy)) \rightarrow \exists z\ \neg(Dz \rightarrow Bxz))$$

The wff in the scope of the initial universal quantifier $\forall x$ is:

$$\exists y\ (Cx \wedge (Ty \wedge Axy)) \rightarrow \exists z\ \neg(Dz \rightarrow Bxz)$$

And this is of the form: $\exists y\ Q(y) \rightarrow P$

which is equivalent to: $\forall y\ (Q(y) \rightarrow P)$

So we pull out the existential quantifier and change it to a universal, and embed the resulting wff inside the scope of $\forall x$:

$$\forall x\ \forall y\ ((Cx \wedge (Ty \wedge Axy)) \rightarrow \exists z\ \neg(Dz \rightarrow Bxz))$$

Finally, the existential quantifier in the consequent can be moved to the outside of the conditional (but inside the other quantifiers!), yielding:

$$\forall x\ \forall y\ \exists z\ ((Cx \wedge (Ty \wedge Axy)) \rightarrow \neg(Dz \rightarrow Bxz))$$

And that is our original sentence in prenex form. Now all we have to do is to replace our abbreviated wffs with the real ones:

$$\forall x\ \forall y\ \exists z\ ((\text{Cube}(x) \wedge (\text{Tet}(y) \wedge \text{Adjoins}(x, y))) \rightarrow \neg(\text{Dodec}(z) \rightarrow \text{BackOf}(x, z)))$$

§ 11.8 Some extra translation problems

The problems in this section make for excellent practice. 11.39 and 11.40★★ have been assigned as homework problems. But the remaining ones should be attempted too, time permitting. Here's another nice translation problem, in the form of an argument. (After we have studied Chapter 13 we will prove that it is valid.)

Dangerfield's argument

The late comedian Rodney Dangerfield was famous for the line "I don't get no respect." I always thought that this was because he himself didn't respect anyone. So when I discovered the following argument (due to the logician W. V. O. Quine) I decided to name it after Rodney. (If the argument is sound, and if my conjecture about Rodney is correct, this would explain why he had such a hard time finding a job.)

| No one respects a person who doesn't respect himself.
| No one will hire anyone (s)he doesn't respect.
|———
| Anyone who respects no one will not be hired by anyone.

Before we begin, we notice that the argument talks only about *persons*, so we will restrict our domain of discourse appropriately. This means that we will not need a predicate Person(x). The only predicates we will need, then, are Respects(x, y) and Hires(x, y).

First Premise

No one respects a person who doesn't respect himself.

Since this sentence contains two quantifiers (*no one, a person*) in the first clause, it will be best to begin with a paraphrase. If we treat the quantifier *a person* as universal, we can give it wide scope and paraphrase the sentence this way:

Any person who doesn't respect himself is respected by no one.

This sentence clearly has an Aristotelian form, which is an **A** sentence. Hence, we can proceed by the step-by-step method:

Every person-who-doesn't-respect-himself is respected-by-no-one.

∀*x* (*x* is a person who doesn't respect himself → *x* is respected by no one)

Next we attack the wffs that are embedded in this **A** sentence:

x is a person who doesn't respect himself
x doesn't respect himself
¬Respects(x, x)

x is respected by no one
no one respects *x*
∀y ¬Respects(y, x)

We then place these FOL translations of our wffs into the antecedent and consequent of our **A** sentence:

∀x (¬Respects(x, x) → ∀y ¬Respects(y, x))

Second Premise

No one will hire anyone (s)he doesn't respect.

Here again a preliminary paraphrase is helpful. The sentence clearly says that no matter what pair of persons you pick, the first will not hire the second if the first doesn't respect the second. That is:

∀*x* ∀*y* (*x* will not hire *y* if *x* does not respect *y*)
∀x ∀y (¬Respects(x, y) → ¬Hires(x, y))

Conclusion

Anyone who respects no one will not be hired by anyone.

Here we may proceed immediately with the step-by-step method, beginning with the sentence's Aristotelian form. This is clearly an **A** sentence:

∀*x* (if *x* respects no one, then *x* will not be hired by anyone)

We next translate the wffs that are embedded in this **A** sentence:

x respects no one
No one is respected by *x*
∀*y* ¬ *y* is respected by *x*
∀*y* ¬ *x* respects *y*
∀y ¬Respects(x, y)

Note that here we could have written ¬∃y instead of ∀y¬.

x will not be hired by anyone
No one will hire *x*
∀*y* ¬ *y* will hire *x*
∀y ¬Hires(y, x)

Here again we could have written ¬∃y instead of ∀y¬.

We then place these FOL translations of our wffs into the antecedent and consequent of our **A** sentence:

∀x (∀y ¬Respects(x, y) → ∀y ¬Hires(y, x))

Here, then, is the entire argument in FOL:

| ∀x (¬Respects(x, x) → ∀y ¬Respects(y, x))
| ∀x ∀y (¬Respects(x, y) → ¬Hires(x, y))
|—
| ∀x (∀y¬Respects(x, y) → ∀y ¬Hires(y, x))

We will return to Dangerfield's argument when we discuss proofs involving quantifiers in Chapter 13.