Chapter 6

MAGNETOHYDRODYNAMICS IN GENERALIZED COORDINATES

We have developed until now a wave propagation method with constrained transport for magnetohydrodynamics only on strictly Cartesian grids. However, for the applications we would like to pursue in this work, it is necessary to generalize this method to magnetohydrodynamic flows on general curved manifolds. In this chapter we modify the wave propagation method and the constrained transport technique to solve ideal MHD and SMHD in generalized coordinates.

6.1 Generalized coordinates

We describe in this section the additional mathematics required to consider hyperbolic conservation laws on general curved manifolds.

6.1.1 Contravariant vector notation

In our notation, all vector components are in contravariant form. Because of this simplification, we will use a slightly different version of the tensor notation found in Misner, Thorne, and Wheeler [65]. Vector components are denoted by a subscript (i.e., $\boldsymbol{U} = (U_1, U_2, U_3)$). Superscripts are not used in this context since they can be confused with exponentiation. The components of the metric for a given coordinate basis is denoted by $\boldsymbol{h}_{mn}$ and the components of the inverse metric by $\boldsymbol{h}^{-1}_{mn}$. Since we do not have lowered and raised indices in our notation, summation is implied when an index is repeated:

$$U_k V_k = \sum_k U_k V_k \,. \tag{6.1}$$

Our notation is summarized and compared to the notation of Misner, Thorne, and Wheeler [65] in Table 6.1.

6.1.2 Some concepts from differential geometry

In this work, we consider generalized coordinates as the coordinates (ξ, η, ζ) that describe an arbitrary manifold $\mathcal{M}$. We say that these are related to the standard Cartesian coordinates (x, y, z) through the transformations

$$x = X(\xi, \eta, \zeta) \tag{6.2}$$
$$y = Y(\xi, \eta, \zeta) \tag{6.3}$$
$$z = Z(\xi, \eta, \zeta)\,. \tag{6.4}$$

Table 6.1: Vector notation used in this work compared to that of Misner, Thorne, and Wheeler [65].

VECTOR NOTATION		
	Our notation	**Misner, Thorne, Wheeler** [65]
contravariant	$\boldsymbol{U} = (U_1, U_2, U_3)$	$\boldsymbol{U} = (U^1, U^2, U^3)$
covariant	Not used	$\boldsymbol{U} = (U_1, U_2, U_3)$
metric in basis	$\boldsymbol{h}_{mn}$	$\boldsymbol{h}_{mn}$
metric in dual	$\boldsymbol{h}_{mn}^{-1}$	$\boldsymbol{h}^{mn}$
summation	$U_k U_k = \sum_k U_k U_k$	$U^k U_k = \sum_k U^k U_k$

We will refer to (ξ, η, ζ) as the *computational coordinates* in which conservation laws will be solved and (x, y, z) as the *physical coordinates.*

Once the coordinate transformations have been chosen, it can easily be shown that vectors in computational space (V_1, V_2, V_3) transform to a vectors in physical space (V_x, V_y, V_z) through the Jacobian J in the following way:

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = J \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \partial_\xi X & \partial_\eta X & \partial_\zeta X \\ \partial_\xi Y & \partial_\eta Y & \partial_\zeta Y \\ \partial_\xi Z & \partial_\eta Z & \partial_\zeta Z \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} . \tag{6.5}$$

Therefore, the coordinate transformations directly give us a natural basis in which to represent vectors in that coordinate system. We will refer to such a basis as a *coordinate basis.*

Furthermore, associated with each coordinate system is a symmetric matrix called the *metric tensor* which we denote by $\boldsymbol{h}$. The metric relates *true distances* as measured in $\mathbb{R}^3$ to the coordinate distances measured in the computational coordinate system. In particular, the line element $ds^2 = dx^2 + dy^2 + dz^2$ in $\mathbb{R}^3$ is related to $d\xi$, $d\eta$, and $d\zeta$ through

$$ds^2 = \boldsymbol{h}_{mn}\, d\xi_m\, d\xi_n \,, \tag{6.6}$$

where $(\xi_1, \xi_2, \xi_3) = (\xi, \eta, \zeta)$. The distance along a curve $C(\lambda)$ parameterized by λ from $C(a)$ to $C(b)$ is given by

$$L = \int_a^b \left| \boldsymbol{h}_{mn} \left(\frac{d\xi_m}{d\lambda} \right) \left(\frac{d\xi_n}{d\lambda} \right) \right|^{1/2} d\lambda \,. \tag{6.7}$$

Using the Jacobian matrix discussed above, the computational metric can be constructed as follows:

$$\boldsymbol{h} = J^t J \,. \tag{6.8}$$

Table 6.2: Examples of metric tensors for some standard coordinate systems.

STANDARD METRIC TENSORS			
Name	**Coordinates**	**Transformations**	**Metric Tensor**
Cartesian	$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$\boldsymbol{h} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Cylindrical	$\begin{pmatrix} x \\ r \\ \theta \end{pmatrix} = \begin{pmatrix} \text{length} \\ \text{radius} \\ \text{azimuth} \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$	$\boldsymbol{h} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix}$
Spherical	$\begin{pmatrix} \phi \\ \lambda \\ r \end{pmatrix} = \begin{pmatrix} \text{longitude} \\ \text{latitude} \\ \text{radius} \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\lambda)\cos(\phi) \\ r\cos(\lambda)\sin(\phi) \\ r\sin(\lambda) \end{pmatrix}$	$\boldsymbol{h} = \begin{bmatrix} r^2\cos^2(\lambda) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Volume in $\mathbb{R}^3$ is related to volume in computational space through the square root of the determinant of the metric tensor,

$$\kappa(\xi, \eta, \zeta) \equiv \sqrt{\det(\boldsymbol{h})}\,, \tag{6.9}$$

such that

$$\int_{\Omega(x,y,z)} q\, dx\, dy\, dz = \int_{\Omega(\xi,\eta,\zeta)} \kappa\, q\, d\xi\, d\eta\, d\zeta\,. \tag{6.10}$$

The metric tensors for some standard coordinate systems are shown in Table 6.2.

The final concept from differential geometry which we need to introduce is the *Christoffel symbol*. Christoffel symbols involve spatial derivatives of the metric tensor $\boldsymbol{h}$ and characterize the spatial variation of the underlying basis. In particular, in a coordinate basis the Christoffel symbols are given by (see Misner, Thorne, and Wheeler [65]):

$$\Gamma^k_{mn} \equiv \frac{1}{2}\boldsymbol{h}^{-1}_{\alpha k}\left(\partial_n \boldsymbol{h}_{\alpha m} + \partial_m \boldsymbol{h}_{\alpha n} - \partial_\alpha \boldsymbol{h}_{mn}\right)\,, \tag{6.11}$$

where there is an implied sum on the repeated index α.

In Table 6.3 we summarize some standard vector operations in generalized coordinates. These operations are valid for a general three-dimensional metric for which

$$\boldsymbol{h}_{mn} = \boldsymbol{h}_{mn}(\xi, \eta, \zeta)\,. \tag{6.12}$$

For a detailed derivation of these vector operations see Heinbockel [38].

Table 6.3: Vector operations in 3D generalized coordinates. All results are given in terms of contravariant components and all input vectors are in contravariant form (i.e., U_m and V_m are all contravariant components).

3D GENERALIZED COORDINATE VECTOR OPERATIONS	
metric	$\boldsymbol{h} = \begin{bmatrix} \boldsymbol{h}_{11} & \boldsymbol{h}_{12} & \boldsymbol{h}_{13} \\ \boldsymbol{h}_{21} & \boldsymbol{h}_{22} & \boldsymbol{h}_{23} \\ \boldsymbol{h}_{31} & \boldsymbol{h}_{32} & \boldsymbol{h}_{33} \end{bmatrix}$
Jacobian	$\kappa(\xi, \eta, \zeta) = \sqrt{\det(\boldsymbol{h})}$
inner product	$\boldsymbol{U} \cdot \boldsymbol{V} = U_m V_n \boldsymbol{h}_{mn}$
cross product	$\boldsymbol{U} \times \boldsymbol{V} = \frac{1}{\kappa} \begin{bmatrix} (\boldsymbol{h}_{2m}\boldsymbol{h}_{3n} - \boldsymbol{h}_{3m}\boldsymbol{h}_{2n})\, U_m V_n \\ (\boldsymbol{h}_{3m}\boldsymbol{h}_{1n} - \boldsymbol{h}_{1m}\boldsymbol{h}_{3n})\, U_m V_n \\ (\boldsymbol{h}_{1m}\boldsymbol{h}_{2n} - \boldsymbol{h}_{2m}\boldsymbol{h}_{1n})\, U_m V_n \end{bmatrix}$
gradient	$\nabla\Theta = \begin{bmatrix} \boldsymbol{h}_{11} & \boldsymbol{h}_{12} & \boldsymbol{h}_{13} \\ \boldsymbol{h}_{21} & \boldsymbol{h}_{22} & \boldsymbol{h}_{23} \\ \boldsymbol{h}_{31} & \boldsymbol{h}_{32} & \boldsymbol{h}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \partial_\xi\Theta \\ \partial_\eta\Theta \\ \partial_\zeta\Theta \end{bmatrix}$
divergence	$\nabla \cdot \boldsymbol{U} = \frac{1}{\kappa}\partial_\xi(\kappa U_1) + \frac{1}{\kappa}\partial_\eta(\kappa U_2) + \frac{1}{\kappa}\partial_\zeta(\kappa U_3)$
curl	$\nabla \times \boldsymbol{U} = \frac{1}{\kappa} \begin{bmatrix} \boldsymbol{h}_{3k}\, \partial_\eta U_k - \boldsymbol{h}_{2k}\, \partial_\zeta U_k \\ \boldsymbol{h}_{1k}\, \partial_\zeta U_k - \boldsymbol{h}_{3k}\, \partial_\xi U_k \\ \boldsymbol{h}_{2k}\, \partial_\xi U_k - \boldsymbol{h}_{1k}\, \partial_\eta U_k \end{bmatrix}$

6.1.3 2D conservation laws in generalized coordinates

As was the case in Cartesian coordinates, we restrict ourselves to wave propagation in two space dimensions. Let us consider a prototypical vector of conserved quantities q such that

$$q = \begin{bmatrix} \Theta \\ V_1 \\ V_2 \end{bmatrix} . \tag{6.13}$$

Here Θ represents some scalar quantity (i.e., density or energy) and $\boldsymbol{V} = (V_1, V_2)^t$ a vector quantity (i.e., momentum or magnetic field). Furthermore, let the flux tensor associated with q be

$$F = \begin{bmatrix} S_1 & S_2 \\ T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} . \tag{6.14}$$

We now consider the conservation law

$$\partial_t q + \nabla \cdot F = 0 \tag{6.15}$$

on a general curved manifold $\mathcal{M}$ with metric $\boldsymbol{h}(\xi, \eta)$. Following [6], equation (6.15) takes the following form in the coordinate basis associated with $\boldsymbol{h}(\xi, \eta)$:

$$\partial_t \Theta + \frac{1}{\kappa}\, \partial_\xi \big(\kappa\, S_1\big) + \frac{1}{\kappa}\, \partial_\eta \big(\kappa\, S_2\big) = 0 \tag{6.16}$$

$$\partial_t V_k + \frac{1}{\kappa}\, \partial_\xi \big(\kappa\, T_{k1}\big) + \frac{1}{\kappa}\, \partial_\eta \big(\kappa\, T_{k2}\big) = -\Gamma^k_{mn} T_{mn} \,, \tag{6.17}$$

where summation is implied on the m and n indices in the term $\Gamma^k_{mn} T_{mn}$. Note the presence of the Christoffel symbols and the square root of the determinant of the metric. We refer to (6.16) and (6.17) as the *coordinate* form of the equations.

If we *orthonormalize* equations (6.16) and (6.17) at a given point, we obtain

$$\partial_t \Theta + \partial_{\hat{\xi}} S_1 + \partial_{\hat{\eta}} S_2 = 0 \tag{6.18}$$

$$\partial_t \hat{V}_k + \partial_{\hat{\xi}} \hat{T}_{k1} + \partial_{\hat{\eta}} \hat{T}_{k2} = 0 \,, \tag{6.19}$$

where the $\hat{\cdot}$ symbol refers to a quantity in the orthonormal frame. It is important to note that (6.18) and (6.19) look like conservation laws in Cartesian coordinates but are only exactly equivalent to (6.16) and (6.17) at a single point. In a small neighborhood around this point, the orthonormal equations approximate the coordinate equations [65]. These equations are important because solving Riemann problems is simpler in orthonormal coordinates than in the coordinate basis. In the next section we will develop a wave propagation method for solving (6.16) and (6.17) using the Riemann solutions to (6.18) and (6.19).

6.2 The wave propagation method in generalized coordinates

The wave propagation method as derived in the previous sections is known as a *Cartesian grid method*, since it is based on propagating waves in the orthonormal x and y Cartesian coordinates. In many physical applications, however, it is necessary to consider more sophisticated geometries. We address this issue by generalizing the scheme developed thus far to apply to hyperbolic conservation laws on a general manifold $\mathcal{M}$. This manifold can either be flat or curved and the choice of coordinate system we lay down on this manifold is arbitrary. The numerical solutions are computed on *logically Cartesian grids* with appropriate generalizations of the first order upwind, high resolution, and transverse corrections. We follow the approach and philosophy outlined in [3, 4, 6], which makes use of orthonormal Riemann solvers to propagate waves and a local treatment of geometric source terms based on parallel transport. The goal of this chapter is to extend the method of [3, 4, 6] to magnetohydrodynamics by coupling it to the constrained transport method developed in the previous chapter.

6.2.1 An overview of the method

In this work we consider a generalization of the wave propagation method which was proposed and extensively studied in the Ph.D. thesis of Derek S. Bale [3]. The underlying assumptions of this approach are that two important pieces of information are known. First, the metric tensor associated with the curved manifold must be given. Second, normal and transverse Riemann solvers using the flux difference approach outlined in Chapter 2 must exist for the orthonormal (same as Cartesian) form the equations. The orthonormal Riemann solvers are devoid of metric information and are in general much easier to solve than the full coordinate Riemann solvers. The approach of [3, 4, 6] is to thus update the coordinate form of the equations by patching together orthonormal Riemann problems that are only locally valid.

Under the above assumptions, the solution procedure for the wave propagation method on curved manifolds is as follows:

- **Step 1: parallel transport**
  $\longrightarrow$ transform the flux and solution vectors on either side of the cell interface into the same coordinate frame
- **Step 2: orthonormalize**
  $\longrightarrow$ transform the flux and solution vectors into a locally valid orthonormal frame at the cell interface
- **Step 3: solve orthonormal Riemann problems**
  $\longrightarrow$ solve the simpler Cartesian Riemann problems
- **Step 4: de-orthonormalize**
  $\longrightarrow$ transform results from the Riemann problem back to original coordinate basis
- **Step 5: construct high-resolution corrections**
- **Step 6: solve transverse Riemann problems**
  $\longrightarrow$ no parallel transport required on transverse corrections for second order accuracy
- **Step 7: update solution**.

Bale [3] has shown that this method is second order accurate in smooth regions of the flow. The stability requirements of this method are that the Courant number be less than one. The Courant number is computed in by treating (6.16)-(6.17) as a balance law with a capacity function $\kappa(\xi, \eta) = \sqrt{\boldsymbol{h}(\xi, \eta)}$ (see Section 2.6).

In the next several sections we will discuss the details of the parallel transport and orthonormalization operations. We will first describe these operations from the differential geometry point of view and then discuss how these operations affect the numerical update for equations (6.16)-(6.17).

6.2.2 Parallel transport in 2D

Parallel transport is an operation that takes a vector $\boldsymbol{U}$ from point A to point B along a curve $c(\lambda)$ such that

$$\frac{d}{d\lambda}\boldsymbol{U} = 0\,, \tag{6.20}$$

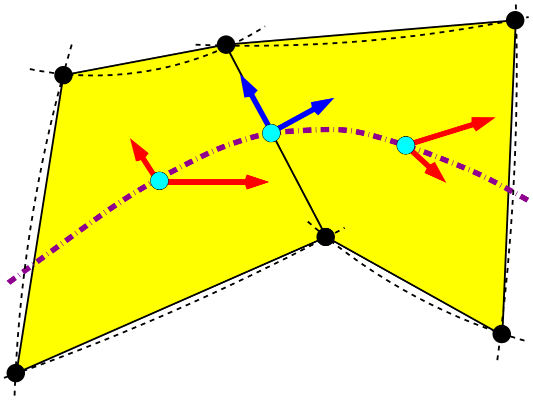

Figure 6.1: Parallel transport and orthonormalization along a coordinate line.

where λ is some parameterization of this curve. For our purposes, we need to be able to transport vectors along coordinate lines. In this case, (6.20) becomes

$$\partial_n \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \Gamma^1_{1n} & \Gamma^1_{2n} \\ \Gamma^2_{1n} & \Gamma^2_{2n} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \,, \tag{6.21}$$

where $\partial_1 = \partial_\xi$ and $\partial_2 = \partial_\eta$. The action of parallel transport along a coordinate line is depicted in Figure 6.1. Equation 6.21 is a system of two coupled ordinary differential equations. Instead of computing the solution to this system exactly for each grid cell, we approximate the solution with a Taylor series:

$$\begin{aligned} \boldsymbol{V}(x_n + \Delta x_n) &= \boldsymbol{V}(x_n) + \Delta x_n \, \partial_n \boldsymbol{V}(x_n) + \mathcal{O}(\Delta x_n^2) \\ &= \boldsymbol{V}(x_n) - \Delta x_n \begin{bmatrix} \Gamma^1_{1n} & \Gamma^1_{2n} \\ \Gamma^2_{1n} & \Gamma^2_{2n} \end{bmatrix} \boldsymbol{V}(x_n) + \mathcal{O}(\Delta x_n^2) \,. \end{aligned} \tag{6.22}$$

From this Taylor series, we can now define a parallel transport matrix that transforms the vector $\boldsymbol{V}$ from the coordinate basis at x_n to the coordinate basis at $x_n + \Delta x_n$:

$$\begin{bmatrix} \Theta \\ V_1 \\ V_2 \end{bmatrix}_{x_n + \Delta x_n} \approx \mathcal{P}^{n\pm}_{ij} \begin{bmatrix} \Theta \\ V_1 \\ V_2 \end{bmatrix}_{x_n} \,, \tag{6.23}$$

where

$$\mathcal{P}^{n\pm}_{ij} = \mathcal{I} \mp \Delta x_n \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1n} & \Gamma^1_{2n} \\ 0 & \Gamma^2_{1n} & \Gamma^2_{2n} \end{bmatrix}_{ij} \,. \tag{6.24}$$

In the above expressions, ij is the grid point location, the n superscript denotes the coordinate direction along which the propagation will occur ($n = 1, 2$), and the $\pm$ determines if the transport is in the positive (+) or negative (−) coordinate direction. To invert the parallel transport operation one can simply invert matrix (6.24) exactly. In order to solve a Riemann problem we will have to parallel transport not only Q values but also the corresponding fluxes. The transported values can be written as

$$[f_1]_r \equiv \mathcal{P}^{n-}_{ij}\,[f_1]_{ij} \tag{6.25}$$

$$[f_1]_l \equiv \mathcal{P}^{n+}_{i-1,j}\,[f_1]_{i-1,j} \tag{6.26}$$

$$Q_r \equiv \mathcal{P}^{n-}_{ij}\,Q_{ij} \tag{6.27}$$

$$Q_l \equiv \mathcal{P}^{n+}_{i-1,j}\,Q_{i-1,j}\,. \tag{6.28}$$

Having applied parallel transport, the quantities $[f_1]_r$, $[f_1]_l$, Q_r, and Q_l are all in approximately the same reference frame.

6.2.3 Parallel transport and the geometric source term

From the geometric point view, parallel transport is a natural process in which two vectors that exist at different spatial locations can be expressed in the same reference frame. In this section we investigate what effect parallel transport has from the point of view of numerically solving the partial differential equations (6.16)-(6.17).

Let us consider the Riemann problem between the cells centered at (ξ_i, η_j) and (ξ_{i-1}, η_j). The fluxes in each cell are given by $[f_1]_{ij}$ and $[f_1]_{i-1,j}$, respectively. In order to compute a flux difference $[\Delta f_1]_{i-1/2,j}$ we apply the parallel transport matrices:

$$[\Delta f_1]_{i-1/2,j} = \mathcal{P}^{1-}_{ij}\,[f_1]_{ij} - \mathcal{P}^{1+}_{i-1,j}\,[f_1]_{i-1,j} \tag{6.29}$$

$$= \left(\mathcal{I} + \frac{\Delta\xi}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{11} & \Gamma^1_{21} \\ 0 & \Gamma^2_{11} & \Gamma^2_{21}\end{bmatrix}_{ij}\right)[f_1]_{ij} - \left(\mathcal{I} - \frac{\Delta\xi}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{11} & \Gamma^1_{21} \\ 0 & \Gamma^2_{11} & \Gamma^2_{21}\end{bmatrix}_{i-1,j}\right)[f_1]_{i-1,j}\,. \tag{6.30}$$

Expanding the above expression out, we obtain the following:

$$[\Delta f_1]_{i-1/2,j} = [f_1]_{ij} - [f_1]_{i-1,j} - \Delta\xi\,[\psi_1]_{i-1/2,j}\,, \tag{6.31}$$

where

$$[f_1]_{ij} = \begin{bmatrix} S \\ T_{11} \\ T_{21}\end{bmatrix}_{ij} \tag{6.32}$$

and

$$[\psi_1]_{i-1/2,j} \equiv -\frac{1}{2}\left(\begin{bmatrix} 0 \\ \Gamma^1_{11}T_{11} + \Gamma^1_{21}T_{21} \\ \Gamma^2_{11}T_{11} + \Gamma^2_{21}T_{21}\end{bmatrix}_{ij} + \begin{bmatrix} 0 \\ \Gamma^1_{11}T_{11} + \Gamma^1_{21}T_{21} \\ \Gamma^2_{11}T_{11} + \Gamma^2_{21}T_{21}\end{bmatrix}_{i-1,j}\right). \tag{6.33}$$

This same analysis applied to a Riemann problem between the cells centered at (ξ_i, η_j) and (ξ_i, η_{j-1}) yields

$$[\Delta f_2]_{i,j-1/2} = [f_2]_{ij} - [f_2]_{i,j-1} - \Delta\eta\,[\psi_2]_{i,j-1/2}\,, \tag{6.34}$$

where

$$[\psi_2]_{i,j-1/2} \equiv -\frac{1}{2}\left(\begin{bmatrix} 0 \\ \Gamma^1_{12}T_{12} + \Gamma^1_{22}T_{22} \\ \Gamma^2_{12}T_{12} + \Gamma^2_{22}T_{22} \end{bmatrix}_{ij} + \begin{bmatrix} 0 \\ \Gamma^1_{12}T_{12} + \Gamma^1_{22}T_{22} \\ \Gamma^2_{12}T_{12} + \Gamma^2_{22}T_{22} \end{bmatrix}_{i,j-1}\right). \tag{6.35}$$

We showed in Section 2.7.2 of Chapter 2 that appropriately modifying the flux differences which are sent to the Riemann solver yields a numerical method for solving balance laws. In particular, the modifications introduced into the flux differences by parallel transport yield a numerical update for a balance law with the following source term:

$$\psi_1 + \psi_2 = -\begin{bmatrix} 0 \\ \Gamma^1_{11}T_{11} + \Gamma^1_{21}T_{21} + \Gamma^1_{12}T_{12} + \Gamma^1_{22}T_{22} \\ \Gamma^2_{11}T_{11} + \Gamma^2_{21}T_{21} + \Gamma^2_{12}T_{12} + \Gamma^2_{22}T_{22} \end{bmatrix} = -\begin{bmatrix} 0 \\ \Gamma^1_{mn}T_{mn} \\ \Gamma^2_{mn}T_{mn} \end{bmatrix}. \tag{6.36}$$

This is precisely the source term found on the right hand side of the balance laws (6.16)-(6.17). We conclude from this analysis that the application of parallel transport to the flux functions produces a natural numerical method for dealing with the geometric source term in (6.17). To obtain a formally second order update we can again introduce a correction term of the form (2.150) [3].

6.2.4 Orthonormalization in 2D

For some hyperbolic partial differential equations, the Riemann problem in a coordinate system whose metric tensor is a dense matrix may be significantly more difficult to solve than the Riemann problem for the same set of equations in an orthonormal frame. Furthermore, we would like make use of the Cartesian Riemann solvers already developed in the previous chapters. In order to accomplish this, we will apply *orthonormalization matrices* to the previously parallel transport quantities. This same idea was used in the context of special versus general relativistic Riemann solvers in the paper by Pons, Font, Ibáñez, Marti, and Miralles [70]. Bale [3] applied this idea to solving hyperbolic equations on curved manifolds.

If we again consider the model system of equations (6.16)-(6.17), then the orthonormalization matrices in the 1- and 2-directions are given by

$$\mathrm{O}^1 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \kappa/\sqrt{\boldsymbol{h}_{22}} & \boldsymbol{h}_{21}/\sqrt{\boldsymbol{h}_{22}} \\ 0 & 0 & \sqrt{\boldsymbol{h}_{22}} \end{bmatrix} \tag{6.37}$$

$$\mathrm{O}^2 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\boldsymbol{h}_{11}} & 0 \\ 0 & \boldsymbol{h}_{12}/\sqrt{\boldsymbol{h}_{11}} & \kappa/\sqrt{\boldsymbol{h}_{11}} \end{bmatrix}, \tag{6.38}$$

respectively. A full derivation of these matrices can be found in [3]. Using the above matrices, the inputs to the normal Riemann solver are as follows:

$$\textbf{Flux Difference:} \quad \Delta\hat{f}_1 = \mathrm{O}^1_{i-1/2,j}\left([f_1]_r - [f_1]_l\right) \tag{6.39}$$

$$\textbf{Left State:} \quad \hat{Q}_l = \mathrm{O}^1_{i-1/2,j} Q_l \tag{6.40}$$

$$\textbf{Right State:} \quad \hat{Q}_r = \mathrm{O}^1_{i-1/2,j} Q_r\,. \tag{6.41}$$

For nonlinear flux functions, we can linearize about a state $\hat{Q}_{i-1/2,j}$, which might be given either by arithmetic or Roe averaging. At this point we compute the wave speeds, wave strengths, and fluctuations via (2.68) -(2.73). In order obtain the correct coordinate speeds, the wave speeds in the orthonormal frame must be re-scaled as follows:

$$\textbf{Wave speed (1-direction):} \qquad s^p_{i-1/2,j} \longrightarrow \left[\kappa\sqrt{\boldsymbol{h}^{-1}_{11}}\right]_{i-1/2,j} s^p_{i-1/2,j} \tag{6.42}$$

$$\textbf{Wave speed (2-direction):} \qquad s^p_{i,j-1/2} \longrightarrow \left[\kappa\sqrt{\boldsymbol{h}^{-1}_{22}}\right]_{i,j-1/2} s^p_{i,j-1/2}\,. \tag{6.43}$$

Once the Riemann problem has been solved, the waves and fluctuations are then de-orthonormalized and inverse parallel transported to correctly update the solution. These are then used to construct high-resolution corrections.

Bale, Rossmanith, and LeVeque [6] argue that to obtain a second order accurate method, parallel transport is not needed before solving the transverse Riemann problems; and therefore, only the orthonormalization is necessary for these terms.

6.2.5 A conservation fix

Although the orthonormalization procedure allows us to solve much simpler Riemann problems, some care must be taken to ensure numerical conservation. Conceptually, the parallel transport step takes cell centered vectors and moves them to the cell edge. More accurately, however, parallel transport transforms vectors from the coordinate system at cell centers to the coordinate system at the cell edge. *These transformed vectors are still spatially located at cell centers.* However, orthonormalization is a strictly pointwise operation and an $\mathcal{O}(\Delta\xi)$ error is introduced when it is applied to the flux difference $f_r - f_l$.

This error in the orthonormalization has the effect of producing flux differences that when sent to the Riemann solver are not in perfect compliance with the orthonormal flux Jacobian. In terms of the numerical scheme, these errors are in general admissible since we are only solving approximate Riemann problems anyway. The only exception to this is when an eigenvalue of the orthonormal flux Jacobian is zero. In this scenario, the Rankine-Hugoniot conditions dictate that s^p and $\beta^p = \left(R^{-1}\Delta\hat{f}\right)^p$ must simultaneously vanish. As we will show in Claim 6.2.1, the inexact orthonormalization will instead result in $\beta^p \neq 0$ if $s^p = 0$, independent of how the approximate flux Jacobian is constructed. In the wave propagation method this will result in a loss of conservation since the entire flux difference cannot be reconstructed by left and right traveling fluctuations. We remedy this situation by modifying the fluctuations in the following way (the same modification as introduced in Section 2.4.2):

$$\mathcal{A}^-_1\Delta Q_{i-1/2} = \sum_{p:\, s^p<0} \mathcal{Z}^p_{i-1/2} + \frac{1}{2}\mathbb{Z}_{i-1/2} \tag{6.44}$$

$$\mathcal{A}^+_1\Delta Q_{i-1/2} = \sum_{p:\, s^p>0} \mathcal{Z}^p_{i-1/2} + \frac{1}{2}\mathbb{Z}_{i-1/2}\,, \tag{6.45}$$

where

$$\mathbb{Z}_{i-1/2} \equiv \sum_{p:s^p=0} \mathcal{Z}^p_{i-1/2}\,. \tag{6.46}$$

This approach guarantees numerical conservation because

$$\mathcal{A}_1^- \Delta Q_{i-1/2} + \mathcal{A}_1^- \Delta Q_{i-1/2} = f_1(Q_i, x_i) - f_1(Q_{i-1}, x_{i-1})\,. \tag{6.47}$$

In the case where $s^p_{i-1/2} = 0$ for $1 \leq p \leq m$, we proved (see Chapter 2) that in Cartesian coordinates $\mathbb{Z}_{i-1/2} = 0$ if $\left[\bar{A}_1\right]_{i-1/2}$ is constructed from the Roe averages. We also argued that if $\left[\bar{A}_1\right]_{i-1/2}$ is any other consistent flux Jacobian then $\mathbb{Z}_{i-1/2}$ ranges anywhere from $\mathcal{O}(\Delta\xi^2)$ to $\mathcal{O}(||Q_i - Q_{i-1}||)$ depending on the smoothness of the solution. We prove below that even in the case where $\left[\bar{A}_1\right]_{i-1/2}$ is constructed from the Roe averages, the orthonormalization process results in an $\mathcal{O}(\Delta\xi)$ error in $\mathbb{Z}_{i-1/2}$. This result implies that with the use of orthonormalization, if $s^p_{i-1/2} = 0$, then $\mathbb{Z}_{i-1/2}$ is always at least of $\mathcal{O}(\Delta\xi)$ in magnitude, independent of the averaging used in constructing the flux Jacobian and the smoothness of the solution.

Claim 6.2.1 *Let $s^p_{i-1/2} = 0$ for some $1 \leq p \leq m$ and let $\left[\bar{A}_{\mathbf{ROE}}\right]_{i-1/2}$ be the approximate flux Jacobian based on the Roe averages. If all the components of the metric tensor are smooth in $\left[\xi_{i-1/2} - \Delta\xi/2, \xi_{i-1/2} + \Delta\xi/2\right]$, then $\mathbb{Z}_{i-1/2} = \mathcal{O}(\Delta\xi)$.*

Proof. Let $\Delta f \equiv [f_1]_i - [f_1]_{i-1}$ and $\Delta Q \equiv Q_i - Q_{i-1}$. By definition, the Roe averaged flux Jacobian satisfies

$$\Delta f = \left[\bar{A}_{\mathbf{ROE}}\right]_{i-1/2} \Delta Q\,. \tag{6.48}$$

Ignoring parallel transport, we can write Δf and ΔQ as

$$\begin{aligned}
& \Delta Q = Q_i - Q_{i-1} \\
\Longrightarrow \quad & \Delta Q = [\mathrm{O}_1^{-1}]_i \hat{Q}_i - [\mathrm{O}_1^{-1}]_{i-1} \hat{Q}_{i-1} \\
\Longrightarrow \quad & \Delta Q = [\mathrm{O}_1^{-1}]_{i-1/2} \left(\hat{Q}_i - \hat{Q}_{i-1}\right) - \frac{\Delta\xi}{2} \partial_\xi [\mathrm{O}_1^{-1}]_{i-1/2} \left(\hat{Q}_i + \hat{Q}_{i-1}\right) + \mathcal{O}(\Delta\xi^2) \\
\Longrightarrow \quad & \Delta Q = [\mathrm{O}_1^{-1}]_{i-1/2} \Delta\hat{Q} + \mathcal{O}(\Delta\xi) \\
\text{Similarly:} \quad & \Delta f = [\mathrm{O}_1^{-1}]_{i-1/2} \Delta\hat{f} + \mathcal{O}(\Delta\xi)\,.
\end{aligned}$$

Plugging these expressions into (6.48) yields

$$[\mathrm{O}_1^{-1}]_{i-1/2} \Delta\hat{f} = \left[\bar{A}_{\mathbf{ROE}}\right]_{i-1/2} [\mathrm{O}_1^{-1}]_{i-1/2} \Delta\hat{Q} + \mathcal{O}(\Delta\xi)\,.$$

Applying the appropriate orthonormalization matrices to this expression we obtain

$$\begin{aligned}
\Longrightarrow \quad & \Delta\hat{f} = \underbrace{[\mathrm{O}_1]_{i-1/2} \left[\bar{A}_{\mathbf{ROE}}\right]_{i-1/2} [\mathrm{O}_1^{-1}]_{i-1/2}}_{[\hat{A}_{\mathbf{ROE}}]_{i-1/2}} \Delta\hat{Q} + \mathcal{O}(\Delta\xi) \\
\therefore \quad & \Delta\hat{f} = [\hat{A}_{\mathbf{ROE}}]_{i-1/2} \Delta\hat{Q} + \mathcal{O}(\Delta\xi)\,.
\end{aligned}$$

Using the same arguments as in Claim 2.4.2 we find that

$$\beta^p_{i-1/2} = s^p_{i-1/2}\alpha_{i-1/2} + \mathcal{O}(\Delta\xi)\,.$$

Therefore, if $s^p_{i-1/2} = 0$, then $\beta^p_{i-1/2} = \mathcal{O}(\Delta\xi)$, $\mathcal{Z}^p_{i-1/2} = \mathcal{O}(\Delta\xi)$, and

$$\mathbb{Z}_{i-1/2} = \sum_{p:s^p=0} \mathcal{Z}^p_{i-1/2} = \mathcal{O}(\Delta\xi)\,.$$

It should be noted that although it was ignored in this proof, inclusion of parallel transport at any level of accuracy will not change the outcome of this proof. The $\mathcal{O}(\Delta\xi)$ error shown here arises fundamentally from the orthonormalization process. ■

6.3 *The ideal MHD equations in generalized coordinates*

We now consider the two-dimensional ideal MHD equations with a metric tensor of the form

$$\boldsymbol{h} \equiv \begin{bmatrix} \boldsymbol{h}_{11}(\xi,\eta) & \boldsymbol{h}_{12}(\xi,\eta) & 0 \\ \boldsymbol{h}_{21}(\xi,\eta) & \boldsymbol{h}_{22}(\xi,\eta) & 0 \\ 0 & 0 & \boldsymbol{h}_{33}(\xi,\eta) \end{bmatrix}. \tag{6.49}$$

This metric is the most general two-dimensional metric admitted by the ideal MHD equations. Following Bale et al. [6], the equations in the (ξ, η) coordinate basis take on the form

$$\partial_t \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \mathcal{E} \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} + \frac{1}{\kappa}\partial_\xi \left(\kappa \begin{bmatrix} \rho u_1 \\ \mathfrak{T}_{11} \\ \mathfrak{T}_{12} \\ \mathfrak{T}_{13} \\ \mathfrak{E}_1 \\ \mathfrak{X}_{11} \\ \mathfrak{X}_{12} \\ \mathfrak{X}_{13} \end{bmatrix} \right) + \frac{1}{\kappa}\partial_\eta \left(\kappa \begin{bmatrix} \rho u_2 \\ \mathfrak{T}_{21} \\ \mathfrak{T}_{22} \\ \mathfrak{T}_{23} \\ \mathfrak{E}_2 \\ \mathfrak{X}_{21} \\ \mathfrak{X}_{22} \\ \mathfrak{X}_{23} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -\Gamma^1_{mn}\mathfrak{T}_{mn} \\ -\Gamma^2_{mn}\mathfrak{T}_{mn} \\ -\Gamma^3_{mn}\mathfrak{T}_{mn} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.50}$$

$$\partial_\xi\left(\kappa B_1\right) + \partial_\eta\left(\kappa B_2\right) = 0\,, \tag{6.51}$$

where summation is implied on m and n in the source term and the components of the energy vector ($\mathfrak{E}$), the momentum tensor ($\mathfrak{T}$), and the magnetic field tensor ($\mathfrak{X}$) are given by

$$\mathfrak{E}_m \equiv u_m\left(\mathcal{E} + \tilde{p}\right) - B_m\left(\mathbf{u}\cdot\mathbf{B}\right) \tag{6.52}$$

$$\mathfrak{T}_{mn} \equiv h u_m u_n - h B_m B_n + \tilde{p}\,\boldsymbol{h}^{-1}_{mn} \tag{6.53}$$

$$\mathfrak{X}_{mn} \equiv h u_m B_n - h u_n B_m\,, \tag{6.54}$$

respectively. Note that $\mathfrak{T}_{12} = \mathfrak{T}_{21}$, $\mathfrak{X}_{11} = \mathfrak{X}_{22} = 0$, and $\mathfrak{X}_{12} = -\mathfrak{X}_{21}$. Although both the momentum and the magnetic field are vector quantities, only the momentum equation requires an extra geometric source term involving the Christoffel symbols. This is due to the fact that the magnetic field tensor ($\mathfrak{X}$) has only zeros on the diagonal and is anti-symmetric (i.e., $\mathfrak{X}_{mn} = -\mathfrak{X}_{nm}$).

6.4 The SMHD equations in generalized coordinates

For the SMHD equations in generalized coordinates we can assume without loss of generality that the gravitational force is everywhere orthogonal to the curved manifold on which we represent the equations. Note that any deviations from this can be handled through the topographic source term. Under this assumption, the metric tensor can be written as

$$\boldsymbol{h} \equiv \begin{bmatrix} \boldsymbol{h}_{11}(\xi,\eta) & \boldsymbol{h}_{12}(\xi,\eta) & 0 \\ \boldsymbol{h}_{21}(\xi,\eta) & \boldsymbol{h}_{22}(\xi,\eta) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{6.55}$$

Following Bale et al. [6], the equations in the (ξ,η) coordinate basis take on the form

$$\partial_t \begin{bmatrix} h \\ hu_1 \\ hu_2 \\ hB_1 \\ hB_2 \end{bmatrix} + \frac{1}{\kappa}\partial_\xi \left(\kappa \begin{bmatrix} hu_1 \\ \mathfrak{T}_{11} \\ \mathfrak{T}_{12} \\ \mathfrak{X}_{11} \\ \mathfrak{X}_{12} \end{bmatrix} \right) + \frac{1}{\kappa}\partial_\eta \left(\kappa \begin{bmatrix} hu_2 \\ \mathfrak{T}_{21} \\ \mathfrak{T}_{22} \\ \mathfrak{X}_{21} \\ \mathfrak{X}_{22} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -\Gamma^1_{mn}\mathfrak{T}_{mn} - gh\,\partial_\xi b \\ -\Gamma^2_{mn}\mathfrak{T}_{mn} - gh\,\partial_\eta b \\ 0 \\ 0 \end{bmatrix} \tag{6.56}$$

$$\partial_\xi\left(\kappa h B_1\right) + \partial_\eta\left(\kappa h B_2\right) = 0\,, \tag{6.57}$$

where summation is implied on m and n in the source term and the components of the momentum tensor ($\mathfrak{T}$) and the magnetic field tensor ($\mathfrak{X}$) are given by

$$\mathfrak{T}_{mn} \equiv hu_m u_n - hB_m B_n + \frac{1}{2} g h^2 \boldsymbol{h}^{-1}_{mn} \tag{6.58}$$

$$\mathfrak{X}_{mn} \equiv hu_m B_n - hu_n B_m\,, \tag{6.59}$$

respectively. Note that $\mathfrak{T}_{12} = \mathfrak{T}_{21}$, $\mathfrak{X}_{11} = \mathfrak{X}_{22} = 0$, and $\mathfrak{X}_{12} = -\mathfrak{X}_{21}$. Although both the momentum and the magnetic field are vector quantities, only the momentum equation requires an extra geometric source term involving the Christoffel symbols. This is due to the fact that the magnetic field tensor ($\mathfrak{X}$) has only zeros on the diagonal and is anti-symmetric (i.e., $\mathfrak{X}_{mn} = -\mathfrak{X}_{nm}$).

6.5 Constrained transport in generalized coordinates

We again restrict ourselves to two space dimensions and assume that the metric is given by

$$\boldsymbol{h} \equiv \begin{bmatrix} \boldsymbol{h}_{11}(\xi,\eta) & \boldsymbol{h}_{12}(\xi,\eta) & 0 \\ \boldsymbol{h}_{21}(\xi,\eta) & \boldsymbol{h}_{22}(\xi,\eta) & 0 \\ 0 & 0 & \boldsymbol{h}_{33}(\xi,\eta) \end{bmatrix} . \tag{6.60}$$

This form of the metric is general enough to cover all physically meaningful restricted ideal MHD and SMHD flows in which all flow variables are independent of the third coordinate ζ. The magnetic field and the magnetic potential in generalized coordinates are related in the following way:

$$\mathfrak{B} = \nabla \times \mathbb{A} = \frac{1}{\kappa}\begin{bmatrix} \partial_\eta \mathbb{A} \\ -\partial_\xi \mathbb{A} \end{bmatrix} . \tag{6.61}$$

Using Faraday's law, the magnetic potential must again satisfy the Hamilton-Jacobi equation

$$\partial_t \mathbb{A} + \Omega = 0\,. \tag{6.62}$$

Ohm's law in generalized coordinates is given by

$$\Omega = (-\mathbf{u} \times \mathfrak{B})_3 = -\kappa_s u_1 \mathfrak{B}_2 + \kappa_s u_2 \mathfrak{B}_1\,, \tag{6.63}$$

where

$$\kappa_s \equiv \frac{\kappa}{\boldsymbol{h}_{33}} = \frac{\sqrt{\boldsymbol{h}_{33}}\sqrt{\boldsymbol{h}_{11}\boldsymbol{h}_{22} - \boldsymbol{h}_{12}^2}}{\boldsymbol{h}_{33}} = \frac{\sqrt{\boldsymbol{h}_{11}\boldsymbol{h}_{22} - \boldsymbol{h}_{12}^2}}{\sqrt{\boldsymbol{h}_{33}}}\,. \tag{6.64}$$

Combining (6.61), (6.62), and (6.63) gives the following magnetic potential advection equation:

$$\partial_t\left(\kappa\mathbb{A}\right) + \kappa_s u_1 \partial_\xi \mathbb{A} + \kappa_s u_2 \partial_\eta \mathbb{A} = 0\,. \tag{6.65}$$

Although κ can formally be divided out of this expression, in the finite volume framework it is more natural to solve (6.65) since κ measures the grid cell volume and $\kappa_s u_1$ is the component of velocity orthogonal to $\xi =$ constant coordinate lines and $\kappa_s u_2$ is the component of velocity orthogonal to $\eta =$ constant coordinate lines.

We conclude from the above that the framework developed in Chapter 5 for constrained transport can be left unchanged in the generalized coordinate setting with the exception of introducing the modified Hamilton-Jacobi equation (6.65) and either the modified ideal MHD (6.50) or the modified SMHD (6.56) equations. Instead of considering numerical test problems in this chapter, we will apply the generalized coordinate MPACT scheme to a specific application problem in the next Chapter.