Chapter 4

THE SHALLOW WATER MAGNETOHYDRODYNAMIC EQUATIONS

The shallow water magnetohydrodynamic or SMHD equations can be written as

$$\partial_t \begin{bmatrix} h \\ h\mathbf{u} \\ h\mathbf{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} h\mathbf{u} \\ h\mathbf{u} \otimes \mathbf{u} - h\mathbf{B} \otimes \mathbf{B} + \frac{1}{2} g h^2 \mathcal{I} \\ h\mathbf{u} \otimes \mathbf{B} - h\mathbf{B} \otimes \mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -gh\,\nabla b \\ 0 \end{bmatrix} \tag{4.1}$$

$$\nabla \cdot (h\mathbf{B}) = 0\,, \tag{4.2}$$

where $\nabla = (\partial_x, \partial_y)^t$, h is the height, $\mathbf{u} = (u_1, u_2)^t$ is the velocity field, $\mathbf{B} = (B_1, B_2)^t$ is the magnetic field, g is the gravitational constant, and b is the bottom topography. These equations were first introduced by Gilman [30] in order to obtain a simplified set of equations from the full MHD equations. The SMHD equations model the dynamics of a constant density, shallow, and electrically conducting fluid that is hydrostatically balanced in the vertical direction. This system of equations is a combination of the of shallow water equations [53] and Maxwell's equations [41]. Because of the constant density assumption, all sound waves are filtered out of the SMHD equations and the resulting eigenstructure is less complicated than MHD. Since both the SMHD and MHD equations require special treatment of a divergence-free constraint, it is instructive to consider the SMHD system as an important first step in solving the ideal MHD system. This system can also be used as a mathematical model in certain applications, such as in the dynamics of the solar tachocline [30]. In this chapter we derive the SMHD system from ideal magnetohydrodynamics, explore its hyperbolic structure, and develop an approximate Riemann solver.

4.1 Derivation of the SMHD equations

In this section we derive the SMHD equations using the ideal MHD equations as a starting point. Our derivation is equivalent to the one first presented by Gilman [30], although our derivation is much more detailed. We begin with the ideal MHD equations under the influence of a gravitational force,

$$\partial_t \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \mathcal{E} \\ \tilde{\mathbf{B}} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u} \otimes \mathbf{u} + \tilde{p}\mathcal{I} - \tilde{\mathbf{B}} \otimes \tilde{\mathbf{B}} \\ \mathbf{u}\big(\mathcal{E} + \tilde{p}\big) - \tilde{\mathbf{B}}(\mathbf{u} \cdot \tilde{\mathbf{B}}) \\ \mathbf{u} \otimes \tilde{\mathbf{B}} - \tilde{\mathbf{B}} \otimes \mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho g\,\hat{\mathbf{k}} \\ 0 \\ 0 \end{bmatrix} \tag{4.3}$$

$$\nabla \cdot \tilde{\mathbf{B}} = 0\,. \tag{4.4}$$

In this notation g is the gravitational constant and $\hat{\mathbf{k}}$ is the unit vector in the vertical direction. From the MHD equations we would now like to derive a set of simplified equations

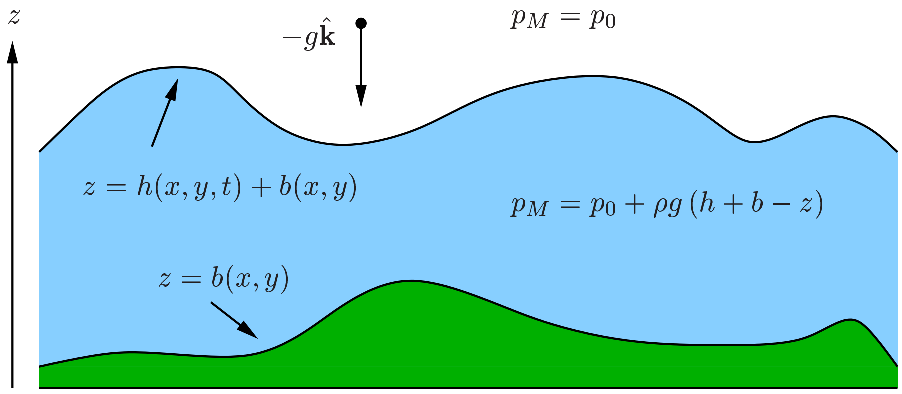

Figure 4.1: The SMHD equations model the dynamics of a conducting fluid that has a free surface at $z = h(x, y, t)$, is homogeneous in density, and is magnetohydrostatically balanced in the vertical direction.

that govern the dynamics of a free surface fluid that has homogeneous density and is in magnetohydrostatic balance in the vertical direction. Such a fluid is depicted in Figure 4.1.

4.1.1 The magnetohydrostatic approximation

The first approximation to be made, and perhaps the most limiting, is to assume that the fluid has constant density:

$$\rho(x, y, z, t) = \text{constant}\,. \tag{4.5}$$

Let us also re-scale the magnetic field so that $\mathbf{B} = \rho^{-1/2}\,\tilde{\mathbf{B}}$. With this assumption, the conservation of vertical momentum can be written as

$$\partial_t u_3 + \mathbf{u}\cdot\nabla_3 u_3 - \mathbf{B}\cdot\nabla_3 B_3 + \rho^{-1}\,\partial_z\left(p + \frac{\rho}{2}\,|\mathbf{B}|^2\right) = -g\,. \tag{4.6}$$

Since our main interest lies in large-scale flows for which the horizontal scale is much larger than the vertical scale, we can safely say that the vertical velocity and the vertical component of the magnetic field are negligible in comparison to the vertical pressure gradient and the gravitational force,

$$u_3 \approx 0 \quad \text{and} \quad B_3 \approx 0\,. \tag{4.7}$$

Because the vertical scale is small, z derivatives will be large; and therefore, one might argue that all terms with z derivatives should remain in the equation. Note however that these terms are

$$u_3\,\partial_z u_3,\;\; B_3\,\partial_z B_3, \text{ and }\;\partial_z\left(p + \frac{\rho}{2}\,|\mathbf{B}|^2\right)\,, \tag{4.8}$$

and that the first two terms are negligible because their scaling involves the *square* of small quantities: u_3^2 and B_3^2, respectively. Eliminating the appropriate terms according to (4.7) leads to the so-called *magnetohydrostatic balance* equation:

$$\partial_z\left(p + \frac{\rho}{2}\,|\mathbf{B}|^2\right) = -\rho g\,. \tag{4.9}$$

4.1.2 Boundary conditions at the bottom and the free surface

Our goal is to eliminate all vertical dependence in the fluid equations by explicitly integrating the MHD equations in the vertical direction. To accomplish this we will need to know the vertical components of the velocity field and the magnetic field at the bottom boundary and the free surface. For the velocity, we require that the vertical component at the free surface ($z = h(x, y, t) + b(x, y)$) is equal to the rate of displacement of the free surface Dh/Dt. The same boundary condition must also be applied at the bottom ($z = b(x, y)$), although here the boundary stays rigid (i.e., $\partial_t b = 0$). Mathematically this results in the following expression:

$$u_3\Big|_{z=b} = \frac{Db}{Dt} = u_1\,\partial_x b + u_2\,\partial_y b \tag{4.10}$$

$$u_3\Big|_{z=h+b} = \frac{D}{Dt}\big(h+b\big) = \partial_t h + u_1\,\partial_x\,(h+b) + u_2\,\partial_y\,(h+b)\ . \tag{4.11}$$

Furthermore, we need to impose that the magnetic field, both at the free surface and at the bottom, remains locally parallel to these surfaces. Mathematically this requires that

$$B_3\Big|_{z=b} = \mathbf{B}\cdot\nabla b = B_1\,\partial_x b + B_2\,\partial_y b \tag{4.12}$$

$$B_3\Big|_{z=h+b} = \mathbf{B}\cdot\nabla\big(h+b\big) = B_1\,\partial_x\,(h+b) + B_2\,\partial_y\,(h+b)\ . \tag{4.13}$$

In the next section we will apply the above boundary conditions as well as the assumptions made on the density and vertical momentum equation to obtain a simplified version of the MHD system.

4.1.3 The SMHD equations

The complete list of the assumptions made on the original ideal MHD equations are given below:

1. $\rho(x, y, z, t) = \text{constant} \implies \text{let}\quad \mathbf{B} \equiv \rho^{-1/2}\,\tilde{\mathbf{B}}$
2. Boundary conditions at bottom:
   $u_3\big|_{z=b} = u_1\,\partial_x b + u_2\,\partial_y b$
   $B_3\big|_{z=b} = B_1\,\partial_x b + B_2\,\partial_y b$
3. Boundary conditions at free surface:
   $u_3\big|_{z=h+b} = \partial_t h + u_1\,\partial_x\,(h+b) + u_2\,\partial_y\,(h+b)$
   $B_3\big|_{z=h+b} = B_1\,\partial_x\,(h+b) + B_2\,\partial_y\,(h+b)$
4. Magnetohydrostatic balance: $\partial_z\left(p + \frac{\rho}{2}\,|\mathbf{B}|^2\right) = \partial_z p_M = -\rho g$
5. The magnetohydrostatic pressure at the surface is constant: $p_M(x, y, h, t) = p_0$
6. The 5 unknowns: $h(x, y, t),\ u_1(x, y, t),\ u_2(x, y, t),\ B_1(x, y, t),\ B_2(x, y, t)$.

If the magnetohydrostatic balance equation is integrated from $z = b$ to $z = h + b$ and the appropriate boundary condition is applied, the resulting expression is given by

$$p_M = p + \frac{\rho}{2}\,|\mathbf{B}|^2 = p_0 + \rho g\,(h + b - z)\ . \tag{4.14}$$

Using this last result, the MHD equations reduce to the following system of equations for the unknowns (h, u_1, u_2, B_1, B_2),

$$\partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0 \tag{4.15}$$
$$\partial_t u_1 + \partial_x \left(u_1^2 - B_1^2 + g\,h + g\,b\right) + \partial_y \left(u_1 u_2 - B_1 B_2\right) + \partial_z \left(u_1 u_3 - B_1 B_3\right) = 0 \tag{4.16}$$
$$\partial_t u_2 + \partial_x \left(u_1 u_2 - B_1 B_2\right) + \partial_y \left(u_2^2 - B_2^2 + g\,h + g\,b\right) + \partial_z \left(u_2 u_3 - B_2 B_3\right) = 0 \tag{4.17}$$
$$\partial_t B_1 + \partial_y \left(u_2 B_1 - u_1 B_2\right) + \partial_z \left(u_3 B_1 - u_1 B_3\right) = 0 \tag{4.18}$$
$$\partial_t B_2 + \partial_x \left(u_1 B_2 - u_2 B_1\right) + \partial_z \left(u_3 B_2 - u_2 B_1\right) = 0 \tag{4.19}$$
$$\partial_x B_1 + \partial_y B_2 + \partial_z B_3 = 0\,. \tag{4.20}$$

Note that the equation for u_3, B_3, and $\mathcal{E}$ are not listed above since these equations are identically satisfied under the above mentioned assumptions. The z-dependence can be completely eliminated if (4.15)-(4.20) are vertically integrated from the bottom to the free surface,

$$\int_b^{h+b} \cdot \; dz\,. \tag{4.21}$$

Integrating equation (4.15) yields

$$\begin{aligned} & h\partial_x u_1 + h\partial_y u_2 + u_3\big|_{z=h+b} - u_3\big|_{z=b} = 0 \\ \Longrightarrow\; & h\partial_x u_1 + h\partial_y u_2 + \partial_t h + u_1\,\partial_x h + u_2\,\partial_y h = 0 \\ \Longrightarrow\; & \partial_t h + \partial_x \left(h u_1\right) + \partial_y \left(h u_2\right) = 0\,. \end{aligned}$$

Integrating equation (4.16) yields

$$\begin{aligned} & h\partial_t u_1 + h\partial_x \left(u_1^2 - B_1^2 + g\,h\right) + h\partial_y \left(u_1 u_2 - B_1 B_2\right) \\ & \qquad + u_1 \left(u_3\big|_{z=h+b} - u_3\big|_{z=b}\right) + B_1 \left(B_3\big|_{z=h+b} - B_3\big|_{z=b}\right) = -gh\,\partial_x b \\ \Longrightarrow\; & \partial_t \left(h u_1\right) + \partial_x \left(h u_1^2 - h B_1^2 + \frac{1}{2} g\,h^2\right) + \partial_y \left(h u_1 u_2 - h B_1 B_2\right) \\ & \qquad - \left\{u_1 \partial_t h + \left(u_1^2 - B_1^2\right) \partial_x h + \left(u_1 u_2 - B_1 B_2\right) \partial_y h\right\} \\ & \qquad + u_1 \left(\partial_t h + u_1 \partial_x h + u_2 \partial_y h\right) - B_1 \left(B_1 \partial_x h + B_2 \partial_y h\right) = -gh\,\partial_x b \\ \Longrightarrow\; & \partial_t \left(h u_1\right) + \partial_x \left(h u_1^2 - h B_1^2 + \frac{1}{2} g\,h^2\right) + \partial_y \left(h u_1 u_2 - h B_1 B_2\right) = -gh\,\partial_x b\,. \end{aligned}$$

Integrating equation (4.17) yields

$$\begin{aligned} & h\partial_t u_2 + h\partial_x \left(u_1 u_2 - B_1 B_2\right) + h\partial_y \left(u_2^2 - B_2^2 + gh\right) \\ & \qquad + u_2 \left(u_3\big|_{z=h+b} - u_3\big|_{z=b}\right) + B_2 \left(B_3\big|_{z=h+b} - B_3\big|_{z=b}\right) = -gh\,\partial_y b \\ \Longrightarrow\; & \partial_t \left(h u_2\right) + \partial_x \left(h u_1 u_2 - h B_1 B_2\right) + \partial_y \left(h u_2^2 - h B_2^2 + \frac{1}{2} g\,h^2\right) \\ & \qquad - \left\{u_2 \partial_t h + \left(u_1 u_2 - B_1 B_2\right) \partial_x h + \left(u_2^2 - B_2^2\right) \partial_y h\right\} \\ & \qquad + u_2 \left(\partial_t h + u_1 \partial_x h + u_2 \partial_y h\right) - B_2 \left(B_1 \partial_x h + B_2 \partial_y h\right) = -gh\,\partial_y b \\ \Longrightarrow\; & \partial_t \left(h u_2\right) + \partial_x \left(h u_1 u_2 - h B_1 B_2\right) + \partial_y \left(h u_2^2 - h B_2^2 + \frac{1}{2} g\,h^2\right) = -gh\,\partial_y b\,. \end{aligned}$$

Integrating equation (4.18) yields

$$
\begin{aligned}
&h\partial_t B_1 + h\partial_y\left(u_2B_1 - u_1B_2\right) + B_1\left(u_3\big|_{z=h+b} - u_3\big|_{z=b}\right) - u_1\left(B_3\big|_{z=h+b} - B_3\big|_{z=b}\right) = 0\\
\Longrightarrow\quad &\partial_t\left(hB_1\right) + \partial_y\left(hu_2B_1 - hu_1B_2\right) - \left\{B_1\partial_t h + \left(u_2B_1 - u_1B_2\right)\partial_y h\right\}\\
&\qquad + B_1\left(\partial_t h + u_1\partial_x h + u_2\partial_y h\right) - u_1\left(B_1\partial_x h + B_2\partial_y h\right) = 0\\
&\Longrightarrow\quad \partial_t\left(hB_1\right) + \partial_y\left(hu_2B_1 - hu_1B_2\right) = 0\,.
\end{aligned}
$$

Integrating equation (4.19) yields

$$
\begin{aligned}
&h\partial_t B_2 + h\partial_x\left(u_1B_2 - u_2B_1\right) + B_2\left(u_3\big|_{z=h+b} - u_3\big|_{z=b}\right) - u_2\left(B_3\big|_{z=h+b} - B_3\big|_{z=b}\right) = 0\\
\Longrightarrow\quad &\partial_t\left(hB_2\right) + \partial_x\left(hu_1B_2 - hu_2B_1\right) - \left\{B_2\partial_t h + \left(u_1B_2 - u_2B_1\right)\partial_x h\right\}\\
&\qquad + B_2\left(\partial_t h + u_1\partial_x h + u_2\partial_y h\right) - u_2\left(B_1\partial_x h + B_2\partial_y h\right) = 0\\
&\Longrightarrow\quad \partial_t\left(hB_2\right) + \partial_x\left(hu_1B_2 - hu_2B_1\right) = 0\,.
\end{aligned}
$$

Integrating equation (4.20) yields

$$
\begin{aligned}
&h\partial_x B_1 + h\partial_y B_2 + B_3\big|_{z=h+b} - B_3\big|_{z=b} = 0\\
\Longrightarrow\ &h\partial_x B_1 + h\partial_y B_2 + B_1\,\partial_x h + B_2\,\partial_y h = 0\\
\Longrightarrow\ &\partial_x\left(hB_1\right) + \partial_y\left(hB_2\right) = 0\,.
\end{aligned}
$$

After combining all these results, the SMHD equations are given in vector form by

$$
\partial_t\begin{bmatrix} h\\ h\mathbf{u}\\ h\mathbf{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix} h\mathbf{u}\\ h\mathbf{u}\otimes\mathbf{u} - h\mathbf{B}\otimes\mathbf{B} + \tilde{p}\mathcal{I}\\ h\mathbf{u}\otimes\mathbf{B} - h\mathbf{B}\otimes\mathbf{u}\end{bmatrix} = \begin{bmatrix} 0\\ -gh\,\nabla b\\ 0\end{bmatrix} \tag{4.22}
$$

$$
\nabla\cdot\left(h\mathbf{B}\right) = 0\,, \tag{4.23}
$$

where $\nabla \equiv (\partial_x, \partial_y)$ and

$$
\tilde{p} = \frac{1}{2}gh^2 \tag{4.24}
$$

is the *hydrodynamic pressure*. In component form, these equations can be written as

$$
\partial_t\begin{bmatrix} h\\ hu_1\\ hu_2\\ hB_1\\ hB_2\end{bmatrix} + \partial_x\begin{bmatrix} hu_1\\ hu_1^2 - hB_1^2 + \tilde{p}\\ hu_1u_2 - hB_1B_2\\ 0\\ hu_1B_2 - hu_2B_1\end{bmatrix} + \partial_y\begin{bmatrix} hu_2\\ hu_1u_2 - hB_1B_2\\ hu_2^2 - hB_2^2 + \tilde{p}\\ hu_2B_1 - hu_1B_2\\ 0\end{bmatrix} = \begin{bmatrix} 0\\ -gh\,\partial_x b\\ -gh\,\partial_y b\\ 0\\ 0\end{bmatrix} \tag{4.25}
$$

$$
\partial_x\left(hB_1\right) + \partial_y\left(hB_2\right) = 0. \tag{4.26}
$$

We refer to the above derived equations as the shallow magnetohydrodynamic equations or SMHD equations.

4.2 Hyperbolic structure of the SMHD equations

In this section we derive the flux Jacobian for the SMHD equations and compute its eigendecomposition. Notice that in each direction there are only four non-trivial equations and one for which the time derivative is identically zero. Because of this, the flux Jacobian we compute in each direction will be a 5×5 matrix, but with zeros placed in the row and column that corresponds to the trivial equation. Furthermore, we define the following quantities:

$$u_N \equiv \begin{cases} u_1 & \text{in 1-direction} \\ u_2 & \text{in 2-direction} \end{cases} \qquad u_T \equiv \begin{cases} u_2 & \text{in 1-direction} \\ u_1 & \text{in 2-direction} \end{cases} \tag{4.27}$$

$$B_N \equiv \begin{cases} B_1 & \text{in 1-direction} \\ B_2 & \text{in 2-direction} \end{cases} \qquad B_T \equiv \begin{cases} B_2 & \text{in 1-direction} \\ B_1 & \text{in 2-direction} \end{cases}, \tag{4.28}$$

where the subscripts denote the components normal (N) and tangent (T) to the coordinate lines. With this in mind, the flux Jacobian in the x-direction is given by

$$A_1 \equiv \partial_q f = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ c_g^2 - u_N^2 & 2u_N & 0 & 0 & 0 \\ B_N B_T - u_N u_T & u_T & u_N & 0 & -B_N \\ 0 & 0 & 0 & 0 & 0 \\ u_T B_N - u_N B_T & B_T & -B_N & 0 & u_N \end{bmatrix} \tag{4.29}$$

and the flux Jacobian in the y-direction is given by

$$A_2 \equiv \partial_q g = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ B_N B_T - u_N u_T & u_N & u_T & -B_N & 0 \\ c_g^2 - u_N^2 & 0 & 2u_N & 0 & 0 \\ u_T B_N - u_N B_T & -B_N & B_T & u_N & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.30}$$

In the above expressions,

$$c_g \equiv \sqrt{B_N^2 + gh}\,. \tag{4.31}$$

4.2.1 The eigendecomposition

We now seek a decomposition of these matrices such that

$$A = R\,\Lambda\,R^{-1}, \tag{4.32}$$

where Λ is the diagonal matrix of eigenvalues, R is the matrix of right eigenvectors, and R^{-1} is the matrix of left eigenvectors. These matrices can be written as follows:

$$R = \left[r^1 \middle| r^2 \middle| r^3 \middle| r^4 \right], \quad \Lambda = \text{diag}\left(\lambda^1, \lambda^2, \lambda^3, \lambda^4\right), \quad R^{-1} = \left[l^1 \middle| l^2 \middle| l^3 \middle| l^4 \right]^t, \tag{4.33}$$

where r^ℓ and l^ℓ are of dimension 5×1 and denote the ℓ^{th} right and left eigenvectors, respectively. The right and left eigenvectors are orthonormal to each other; and therefore, they satisfy

$$r^i \cdot l^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases} \tag{4.34}$$

The eigenvalues are given by

$$\lambda^1 = u_N - c_g \tag{4.35}$$
$$\lambda^2 = u_N - B_N \tag{4.36}$$
$$\lambda^3 = u_N + B_N \tag{4.37}$$
$$\lambda^4 = u_N + c_g \tag{4.38}$$

and always satisfy the property

$$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4\,. \tag{4.39}$$

In order to avoid writing separate expressions for the eigenvectors of A_1 and A_2, we will make use of the $\circlearrowright$ symbol to indicate that the two rows immediately above and below the symbol must be switched when considering the decomposition in the y-direction. The right eigenvectors are

$$r^{1,4} = \begin{bmatrix} 1 \\ u_N \mp c_g \\ u_T \\ 0 \\ B_T \end{bmatrix} \begin{matrix} \\ \circlearrowright \\ \\ \circlearrowright \\ \\ \end{matrix} \qquad r^{2,3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \pm 1 \end{bmatrix} \begin{matrix} \\ \circlearrowright \\ \\ \circlearrowright \\ \\ \end{matrix}. \tag{4.40}$$

The left eigenvectors are

$$l^{1,4} = \begin{bmatrix} \frac{1}{2c_g}\left(c_g \pm u_N\right) \\ \mp \frac{1}{2c_g} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \circlearrowright \\ \\ \circlearrowright \\ \\ \end{matrix} \qquad l^{2,3} = \begin{bmatrix} -\frac{1}{2}\left(u_T \pm B_T\right) \\ 0 \\ \frac{1}{2} \\ 0 \\ \pm \frac{1}{2} \end{bmatrix} \begin{matrix} \\ \circlearrowright \\ \\ \circlearrowright \\ \\ \end{matrix}. \tag{4.41}$$

4.2.2 The SMHD waves

As shown in Figure 4.2, the Riemann solution to the SMHD system is composed of four waves each propagating at a finite speed:

$$\mathcal{W}^{1,4}: \ \text{Magnetogravity waves,} \tag{4.42}$$
$$\mathcal{W}^{2,3}: \ \text{Alfvén waves.} \tag{4.43}$$

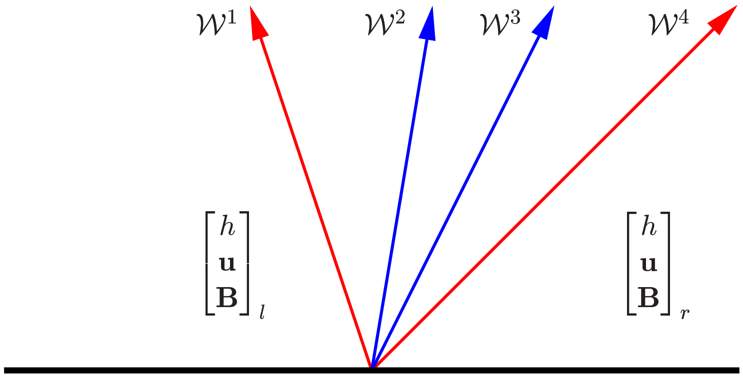

Figure 4.2: Wave structure for the shallow water MHD equations. $\mathcal{W}^1$ and $\mathcal{W}^4$ are the genuinely nonlinear magnetogravity waves and $\mathcal{W}^2$ and $\mathcal{W}^3$ are the linearly degenerate Alvén waves.

To test if these waves are genuinely nonlinear or linearly degenerate, we compute the quantity $\partial_q \lambda^p \cdot r^p$ [53]. After some algebra, we obtain that

$$\partial_q \lambda^{1,4} \cdot r^{1,4} = \pm \frac{3}{2} \frac{g}{\sqrt{B_N^2 + gh}} \neq 0 \quad \forall q \tag{4.44}$$

$$\partial_q \lambda^{2,3} \cdot r^{2,3} = 0 \quad \forall q \,. \tag{4.45}$$

This result implies that the magnetogravity waves are genuinely nonlinear and the Alfvén waves are linearly degenerate. Note that $\partial_q \lambda^1 \cdot r^1 > 0$ and $\partial_q \lambda^4 \cdot r^4 < 0$ for all q. This implies that the magnetogravity waves are convex; and thus, compound waves cannot occur [83].

The decomposition shown above is valid for all $h > 0$. If $h > 0$, the SMHD system is strictly hyperbolic as long as $B_N \neq 0$. If $B_N = 0$, then the two Alfvén waves merge into one to become the Rossby wave. This is a merger of two linearly degenerate waves; and therefore, it does not cause any numerical difficulties. Notice also that the SMHD system and the corresponding eigendecomposition trivially reduce to the standard shallow water (SW) equations if $\mathbf{B} \equiv 0$ [53].

4.2.3 A Riemann solver based on Roe averages

In order to complete our development of an approximate Riemann solver for the SMHD equations, we will need to compute the Roe averages. We present here a derivation similar to the one found in De Sterck [83]. It turns out that in order to obtain the correct Roe averages, we will only need to consider the flux in the x-direction. The Roe averages generate an approximate flux Jacobian at the cell interface $x = x_{i-1/2}$ that satisfies the following condition [53]:

$$[\hat{A}_1]_{i-1/2} \left(Q_i - Q_{i-1}\right) = [f_1]_i - [f_1]_{i-1} \,. \tag{4.46}$$

The approximate flux Jacobian that satisfies this condition can be computed using the decomposition

$$[\hat{A}_1]_{i-1/2} = \hat{T}_{i-1/2}\,\hat{S}^{-1}_{i-1/2}\,, \tag{4.47}$$

where

$$\hat{S}_{i-1/2} \equiv \int_0^1 \frac{\partial q}{\partial z}(z(\xi))\,d\xi \tag{4.48}$$

$$\hat{T}_{i-1/2} \equiv \int_0^1 \frac{\partial f}{\partial z}(z(\xi))\,d\xi\,. \tag{4.49}$$

Using an analogous approach to that used for the SW equations, we will take for the SMHD equations the following choice for z [53]:

$$z = h^{-1/2}\,q\,, \text{ so that } \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} \sqrt{h} \\ \sqrt{h}u_N \\ \sqrt{h}u_T \\ \sqrt{h}B_N \\ \sqrt{h}B_T \end{bmatrix}. \tag{4.50}$$

From this definition, q and f can be written as functions of z:

$$q(z) = \begin{bmatrix} z_1 z_1 \\ z_1 z_2 \\ z_1 z_3 \\ z_1 z_4 \\ z_1 z_5 \end{bmatrix} \quad \Longrightarrow \quad \frac{\partial q}{\partial z} = \begin{bmatrix} 2z_1 & 0 & 0 & 0 & 0 \\ z_2 & z_1 & 0 & 0 & 0 \\ z_3 & 0 & z_1 & 0 & 0 \\ z_5 & 0 & 0 & z_1 & 0 \\ z_5 & 0 & 0 & 0 & z_1 \end{bmatrix} \tag{4.51}$$

$$f(z) = \begin{bmatrix} z_1 z_2 \\ z_2^2 - z_4^2 + \frac{1}{2} g z_1^4 \\ z_2 z_3 - z_4 z_5 \\ 0 \\ z_2 z_5 - z_3 z_4 \end{bmatrix} \quad \Longrightarrow \quad \frac{\partial f}{\partial z} = \begin{bmatrix} z_2 & z_1 & 0 & 0 & 0 \\ 2g z_1^3 & 2z_2 & 0 & -2z_4 & 0 \\ 0 & z_3 & z_2 & -z_5 & -z_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & z_5 & -z_4 & -z_3 & z_2 \end{bmatrix}. \tag{4.52}$$

The path of integration for (4.48) and (4.49) is along the line

$$z_p = [Z_p]_{i-1} + \xi\left([Z_p]_i - [Z_p]_{i-1}\right)\,, \tag{4.53}$$

where $p = 1, 2, 3, 4, 5$. The integration results in

$$\hat{S}_{i-1/2} = \begin{bmatrix} 2\bar{Z}_1 & 0 & 0 & 0 & 0 \\ \bar{Z}_2 & \bar{Z}_1 & 0 & 0 & 0 \\ \bar{Z}_3 & 0 & \bar{Z}_1 & 0 & 0 \\ \bar{Z}_5 & 0 & 0 & \bar{Z}_1 & 0 \\ \bar{Z}_5 & 0 & 0 & 0 & \bar{Z}_1 \end{bmatrix} \tag{4.54}$$

$$\hat{T}_{i-1/2} = \begin{bmatrix} \bar{Z}_2 & \bar{Z}_1 & 0 & 0 & 0 \\ 2g\bar{Z}_1\hat{h} & 2\bar{Z}_2 & 0 & -2\bar{Z}_4 & 0 \\ 0 & \bar{Z}_3 & \bar{Z}_2 & -\bar{Z}_5 & -\bar{Z}_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{Z}_5 & -\bar{Z}_4 & -\bar{Z}_3 & \bar{Z}_2 \end{bmatrix}, \tag{4.55}$$

where $\bar{Z}_p \equiv \frac{1}{2}\left([Z_p]_i + [Z_p]_{i-1}\right)$ and $\hat{h} \equiv \frac{1}{2}(h_i + h_{i-1})$. From formula (4.47) the approximate flux Jacobian is therefore given by

$$\begin{aligned}
[\hat{A}_1]_{i-1/2} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{\bar{Z}_4^2}{\bar{Z}_1^2} + g\hat{h} - \frac{\bar{Z}_2^2}{\bar{Z}_1^2} & 2\frac{\bar{Z}_2}{\bar{Z}_1} & 0 & 0 & 0 \\ \frac{\bar{Z}_5\bar{Z}_4}{\bar{Z}_1^2} - \frac{\bar{Z}_3\bar{Z}_2}{\bar{Z}_1^2} & \frac{\bar{Z}_3}{\bar{Z}_1} & \frac{\bar{Z}_2}{\bar{Z}_1} & 0 & -\frac{\bar{Z}_4}{\bar{Z}_1} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\bar{Z}_4\bar{Z}_3}{\bar{Z}_1^2} - \frac{\bar{Z}_5\bar{Z}_2}{\bar{Z}_1^2} & \frac{\bar{Z}_5}{\bar{Z}_1} & -\frac{\bar{Z}_4}{\bar{Z}_1} & 0 & \frac{\bar{Z}_2}{\bar{Z}_1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \hat{B}_N^2 + g\hat{h} - \hat{u}_N^2 & 2\hat{u}_N & 0 & 0 & 0 \\ \hat{B}_N\hat{B}_T - \hat{u}_N\hat{u}_T & \hat{u}_T & \hat{u}_N & 0 & -\hat{B}_N \\ 0 & 0 & 0 & 0 & 0 \\ \hat{u}_T\hat{B}_N - \hat{u}_N\hat{B}_T & \hat{B}_T & -\hat{B}_N & 0 & \hat{u}_N \end{bmatrix},
\end{aligned} \tag{4.56}$$

where the Roe average quantities are thus given by

$$\hat{h} = \frac{1}{2}(h_i + h_{i-1}) \tag{4.57}$$

$$\hat{u}_\star = \frac{\sqrt{h_i}\,[u_\star]_i + \sqrt{h_{i-1}}\,[u_\star]_{i-1}}{\sqrt{h_i} + \sqrt{h_{i-1}}} \tag{4.58}$$

$$\hat{B}_\star = \frac{\sqrt{h_i}\,[B_\star]_i + \sqrt{h_{i-1}}\,[B_\star]_{i-1}}{\sqrt{h_i} + \sqrt{h_{i-1}}}. \tag{4.59}$$

With this last result there is enough information to construct an approximate Riemann solver for use in a wave propagation method. Note that the Roe average analysis could also be carried out in the y-direction. Since this will yield no new information, however, it will be omitted.

4.3 Example: a 1D Riemann problem

As an example, we consider the following initial value Riemann problem taken from De Sterck [83]:

$$\left[h, u_1, u_2, B_1, B_2\right]\Big|_{t=0} = \begin{cases} \left[1, 0, 0, 1, 0\right] & \text{if} \quad x < 0 \\ \left[2, 0, 0, 0.5, 1\right] & \text{if} \quad x > 0\,. \end{cases} \tag{4.60}$$

The solution on a 100 point and a 5000 point grid are compared in Figure 4.3. These results compare well to those shown in De Sterck [83]. In Figures 4.3(a), 4.3(b), 4.3(c), and 4.3(e) the 1-magnetogravity wave appears as a left traveling shock and the 4-magnetogravity wave as a right traveling rarefaction. In Figures 4.3(b) and 4.3(d) the slower moving Alfvén waves are evident.

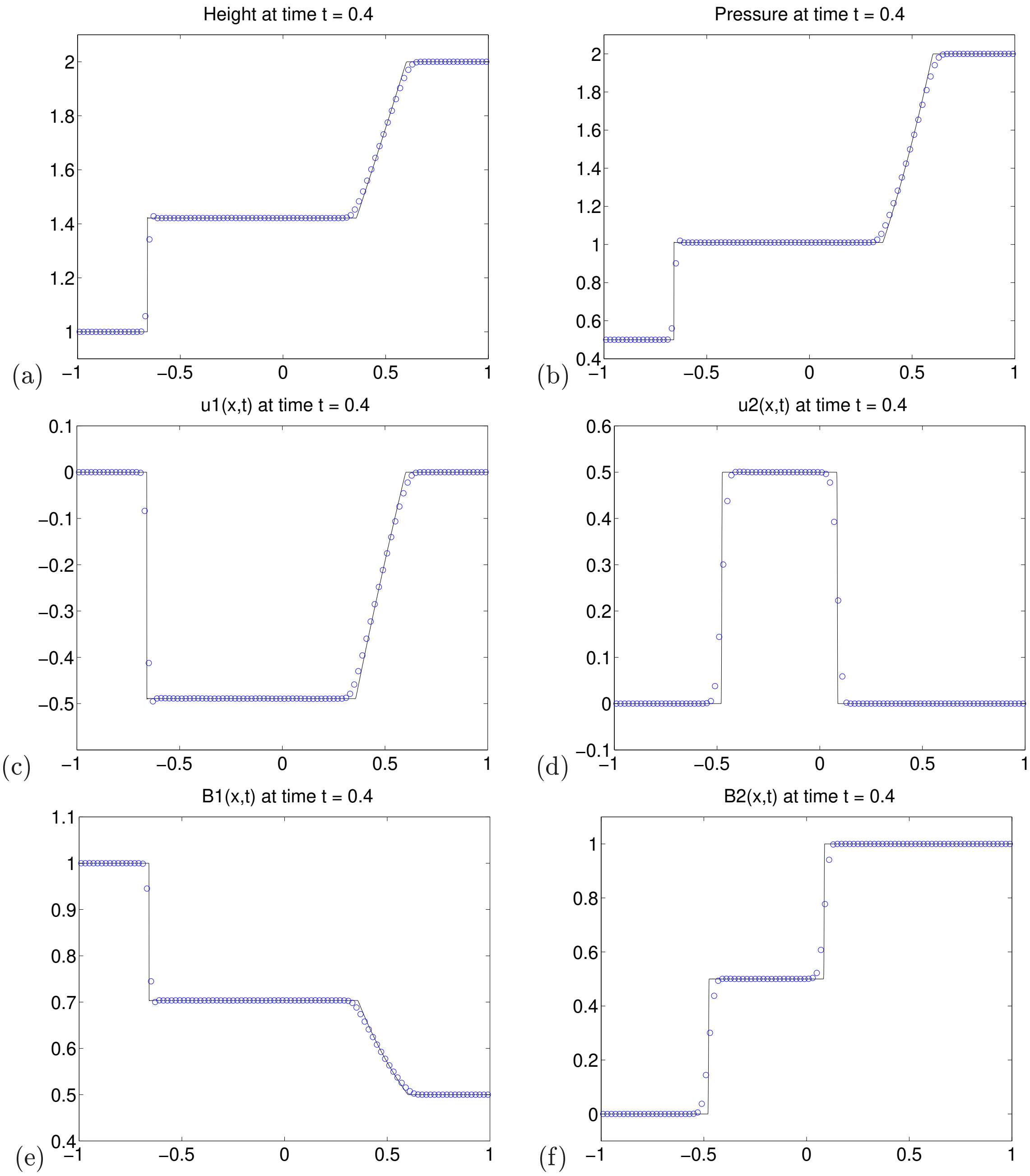

Figure 4.3: An example of a Riemann solution for the shallow water MHD equations.