Chapter 3

THE IDEAL MAGNETOHYDRODYNAMIC EQUATIONS

The ideal magnetohydrodynamic or MHD equations are given by

$$\partial_t \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \mathcal{E} \\ \mathbf{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\otimes\mathbf{u} + \tilde{p}\mathcal{I} - \mathbf{B}\otimes\mathbf{B} \\ \mathbf{u}(\mathcal{E}+\tilde{p}) - \mathbf{B}(\mathbf{u}\cdot\mathbf{B}) \\ \mathbf{u}\otimes\mathbf{B} - \mathbf{B}\otimes\mathbf{u} \end{bmatrix} = 0 \tag{3.1}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{3.2}$$

where

$$\tilde{p} = p + \frac{1}{2}|\mathbf{B}|^2 \tag{3.3}$$

$$p = (\gamma - 1)\left(\mathcal{E} - \frac{1}{2}\rho|\mathbf{u}|^2 - \frac{1}{2}|\mathbf{B}|^2\right). \tag{3.4}$$

In these equations, ρ is the density, $\mathbf{u} = (u_1, u_2, u_3)^t$ is the velocity field, $\mathcal{E}$ is the total energy, $\mathbf{B} = (B_1, B_2, B_3)^t$ is the magnetic field, p is the thermal pressure, $(1/2)|\mathbf{B}|^2$ is the magnetic pressure, $\tilde{p}$ is the total pressure, and γ is the ideal gas constant. Equation (3.4) is the *equation of state* for an ideal gas and closes the system by relating the pressure to the other unknowns. These equations model the dynamics of an electrically conducting fluid and are a combination of gas dynamics [53] and Maxwell's equations [41]. In this chapter we outline the derivation of the MHD system, explore its hyperbolic structure, and develop an approximate Riemann solver.

3.1 Derivation of the ideal MHD equations

The one-fluid magnetohydrodynamic equations model the dynamics of a collection of neutrons, electrons, and a species of ions (usually protons) that has an overall neutral charge. We refer to this collection of particles as a *plasma* and assume that it has a macroscopic mass density ρ, a velocity $\mathbf{u}$, a magnetic field $\mathbf{B}$, an electric field $\mathbf{\Omega}$, and a current density $\mathbf{J}$. The MHD fluid is assumed to be a good conductor; and thus, it will approximately obey Ohm's law:

$$\mathbf{J} = \sigma\left(\mathbf{\Omega} + \mathbf{u} \times \mathbf{B}\right), \tag{3.5}$$

where σ is the electrical conductivity. In the applications we would like to consider in this thesis, we will assume that the characteristic time variation of electromagnetic quantities is much longer than the time it takes light to propagate a characteristic distance; and therefore, we make the approximation that the speed of light is infinite in Maxwell's equations [67].

Under all of these assumptions, the collection of particles obey a reduced version of Maxwell's equations:

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = \frac{1}{\sigma} \nabla^2 \mathbf{B} \tag{3.6}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.7}$$

$$\nabla \times \mathbf{B} = \mathbf{J}\,. \tag{3.8}$$

We restrict ourselves further by considering the MHD fluid to be a perfect conductor ($\sigma = \infty$). This approximation differentiates *ideal* MHD from MHD. We are now ready to couple the reduced Maxwell's equations to the fluid dynamics of the plasma.

To obtain the correct fluid equations, we begin from the Euler equations for an ideal gas [53] and add the necessary modifications from electromagnetism. The conservation of mass equation is given by

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0\,. \tag{3.9}$$

The momentum equation is modified by the inclusion of the Lorentz force, which arises from the coupling between the magnetic field and the current density [67]:

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p\mathcal{I}) = \mathbf{J} \times \mathbf{B}\,. \tag{3.10}$$

Using (3.8), the Lorentz force can be rewritten as

$$\begin{aligned} \mathbf{J} \times \mathbf{B} &= (\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= -\nabla \left(\frac{1}{2}|\mathbf{B}|^2\right) + \mathbf{B} \cdot \nabla \mathbf{B} \\ &= -\nabla \left(\frac{1}{2}|\mathbf{B}|^2\right) + \nabla \cdot (\mathbf{B} \otimes \mathbf{B})\,. \end{aligned} \tag{3.11}$$

The momentum equation can then be written as

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} + \big(p + \frac{1}{2}|\mathbf{B}|^2\big)\mathcal{I} - \mathbf{B} \otimes \mathbf{B}\right) = 0\,. \tag{3.12}$$

The total energy in this system is comprised of the kinetic energy, the thermal energy, and the energy density of magnetic field:

$$\mathcal{E} = \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{p}{\gamma - 1} + \frac{1}{2}|\mathbf{B}|^2\,, \tag{3.13}$$

where γ is the ideal gas constant. Using the mass equation (3.10), the momentum equation (3.12), and the fact that entropy must remain constant along streamlines for smooth flows,

$$\partial_t \left(\frac{p}{\rho^\gamma}\right) + \mathbf{u} \cdot \nabla \left(\frac{p}{\rho^\gamma}\right) = 0\,, \tag{3.14}$$

it can be shown that the equation for the conservation of total energy is given by (see [67])

$$\partial_t \mathcal{E} + \nabla \cdot \left(\mathbf{u}\big(\mathcal{E} + p + \frac{1}{2}|\mathbf{B}|^2\big) - \mathbf{B}(\mathbf{u} \cdot \mathbf{B})\right) = 0\,. \tag{3.15}$$

Maxwell's equations (3.6) and (3.7) can be rewritten as

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) = 0 \tag{3.16}$$
$$\nabla \cdot \mathbf{B} = 0\,. \tag{3.17}$$

Equations (3.10), (3.12), (3.15), and (3.16) produce a system of 8 evolution equations for the 8 primitive variables $(\rho, \mathbf{u}, p, \mathbf{B})$. The total energy is related to these variables through (3.13). The divergence-free constraint on the magnetic field (3.17) must be satisfied for all time.

Combining all of these equations, the ideal MHD equations can be written compactly as

$$\partial_t \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \mathcal{E} \\ \mathbf{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\otimes\mathbf{u} + \tilde{p}\mathcal{I} - \mathbf{B}\otimes\mathbf{B} \\ \mathbf{u}\big(\mathcal{E} + \tilde{p}\big) - \mathbf{B}(\mathbf{u}\cdot\mathbf{B}) \\ \mathbf{u}\otimes\mathbf{B} - \mathbf{B}\otimes\mathbf{u} \end{bmatrix} = 0 \tag{3.18}$$
$$\nabla \cdot \mathbf{B} = 0\,, \tag{3.19}$$

where

$$\tilde{p} = p + \frac{1}{2}|\mathbf{B}|^2 \tag{3.20}$$
$$p = (\gamma - 1)\left(\mathcal{E} - \frac{1}{2}\rho|\mathbf{u}|^2 - \frac{1}{2}|\mathbf{B}|^2\right)\,. \tag{3.21}$$

In component form, the 2D ideal MHD equations are

$$\partial_t \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \mathcal{E} \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} + \partial_x \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + \tilde{p} - B_1^2 \\ \rho u_1 u_2 - B_1 B_2 \\ \rho u_1 u_3 - B_1 B_3 \\ u_1\big(\mathcal{E} + \tilde{p}\big) - B_1\,(\mathbf{u}\cdot\mathbf{B}) \\ 0 \\ u_1 B_2 - u_2 B_1 \\ u_1 B_3 - u_3 B_1 \end{bmatrix} + \partial_y \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 - B_1 B_2 \\ \rho u_2^2 + \tilde{p} - B_2^2 \\ \rho u_2 u_3 - B_2 B_3 \\ u_2\big(\mathcal{E} + \tilde{p}\big) - B_2\,(\mathbf{u}\cdot\mathbf{B}) \\ u_2 B_1 - u_1 B_2 \\ 0 \\ u_2 B_3 - u_3 B_2 \end{bmatrix} = 0 \tag{3.22}$$
$$\partial_x B_1 + \partial_y B_2 = 0\,. \tag{3.23}$$

3.2 Hyperbolic structure of the ideal MHD equations

In this section we derive the flux Jacobian for the MHD equations and compute its eigen-decomposition. Notice that in each direction there are only seven non-trivial equations and one for which the time derivative is identically zero. Because of this, the flux Jacobian we compute in each direction will be an 8×8 matrix, but with zeros placed in the row and column that corresponds to the trivial equation. Furthermore, we define the following quantities:

$$u_N \equiv \begin{cases} u_1 & \text{in 1-direction} \\ u_2 & \text{in 2-direction} \end{cases} \qquad u_T \equiv \begin{cases} u_2 & \text{in 1-direction} \\ u_1 & \text{in 2-direction} \end{cases} \tag{3.24}$$

$$B_N \equiv \begin{cases} B_1 & \text{in 1-direction} \\ B_2 & \text{in 2-direction} \end{cases} \qquad B_T \equiv \begin{cases} B_2 & \text{in 1-direction} \\ B_1 & \text{in 2-direction} \end{cases} \tag{3.25}$$

and

$$\gamma_n \equiv n - \gamma\,, \qquad n \in \text{integer} \tag{3.26}$$

$$a_1 \equiv \frac{1}{2}(\gamma - 3)u_N^2 + \frac{1}{2}(\gamma - 1)(u_T^2 + u_3^2) \tag{3.27}$$

$$a_2 \equiv -u_N \left(\frac{\gamma - 1}{2\rho}(\mathcal{E} + \tilde{p})|\mathbf{u}|^2 + \frac{B_N}{\rho}(\mathbf{u} \cdot \mathbf{B}) \right) \tag{3.28}$$

$$a_3 \equiv \frac{1}{\rho}(\mathcal{E} + \tilde{p}) - \frac{B_N^2}{\rho} - (\gamma - 1)u_N^2 \tag{3.29}$$

$$a_4 \equiv (1 - \gamma)u_N u_T - \frac{1}{\rho} B_N B_T \tag{3.30}$$

$$a_5 \equiv (1 - \gamma)u_N u_3 - \frac{1}{\rho} B_N B_3 \tag{3.31}$$

$$a_6 \equiv (2 - \gamma)u_N B_T - u_T B_N \tag{3.32}$$

$$a_7 \equiv (2 - \gamma)u_N B_3 - u_3 B_N\,, \tag{3.33}$$

where the subscripts denote the components normal (N) and tangent (T) to the coordinate lines. With this in mind, the flux Jacobian in the x-direction is given by (see [13])

$$A_1 \equiv \partial_q f = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & \gamma_3 u_N & \gamma_1 u_T & \gamma_1 u_3 & -\gamma_1 & 0 & \gamma_2 B_2 & \gamma_2 B_3 \\ -u_N u_T & u_T & u_N & 0 & 0 & 0 & -B_N & 0 \\ -u_N u_3 & u_3 & 0 & u_N & 0 & 0 & 0 & -B_N \\ a_2 & a_3 & a_4 & a_5 & \gamma u_N & 0 & a_6 & a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{u_T B_N - u_N B_T}{\rho} & \frac{B_T}{\rho} & -\frac{B_N}{\rho} & 0 & 0 & 0 & u_N & 0 \\ \frac{u_3 B_N - u_N B_3}{\rho} & \frac{B_3}{\rho} & 0 & -\frac{B_N}{\rho} & 0 & 0 & 0 & u_N \end{bmatrix} \tag{3.34}$$

and the flux Jacobian in the y-direction is given by

$$A_2 \equiv \partial_q g = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -u_N u_T & u_N & u_T & 0 & 0 & -B_N & 0 & 0 \\ a_1 & \gamma_1 u_T & \gamma_3 u_N & \gamma_1 u_3 & -\gamma_1 & \gamma_2 B_2 & 0 & \gamma_2 B_3 \\ -u_N u_3 & 0 & u_3 & u_N & 0 & 0 & 0 & -B_N \\ a_2 & a_4 & a_3 & a_5 & \gamma u_N & a_6 & 0 & a_7 \\ \frac{u_T B_N - u_N B_T}{\rho} & -\frac{B_N}{\rho} & \frac{B_T}{\rho} & 0 & 0 & u_N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{u_3 B_N - u_N B_3}{\rho} & 0 & \frac{B_3}{\rho} & -\frac{B_N}{\rho} & 0 & 0 & 0 & u_N \end{bmatrix}. \tag{3.35}$$

3.2.1 The eigendecomposition

We now seek a decomposition of these matrices such that

$$A = R\,\Lambda\,R^{-1}\,, \tag{3.36}$$

where Λ is the diagonal matrix of eigenvalues, R is the matrix of right eigenvectors, and R^{-1} is the matrix of left eigenvectors. These matrices can be written as follows:

$$\begin{gathered} R = \left[r^1 \,\middle|\, r^2 \,\middle|\, r^3 \,\middle|\, r^4 \,\middle|\, r^5 \,\middle|\, r^6 \,\middle|\, r^7 \right], \;\; \Lambda = \text{diag}\left(\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \lambda^6, \lambda^7\right), \\ R^{-1} = \left[l^1 \,\middle|\, l^2 \,\middle|\, l^3 \,\middle|\, l^4 \,\middle|\, l^5 \,\middle|\, l^6 \,\middle|\, l^7 \right]^t, \end{gathered} \tag{3.37}$$

where r^ℓ and l^ℓ are of dimension 8×1 and denote the ℓ^{th} right and left eigenvectors, respectively. The right and left eigenvectors are orthonormal to each other; and therefore, they satisfy

$$r^i \cdot l^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases} \tag{3.38}$$

The eigenvalues (see for example Ryu and Jones [76] or De Sterck [82]) are given by

$$\lambda^1 = u_N - c_f \tag{3.39}$$
$$\lambda^2 = u_N - c_a \tag{3.40}$$
$$\lambda^3 = u_N - c_s \tag{3.41}$$
$$\lambda^4 = u_N \tag{3.42}$$
$$\lambda^5 = u_N + c_s \tag{3.43}$$
$$\lambda^6 = u_N + c_a \tag{3.44}$$
$$\lambda^7 = u_N + c_f\,, \tag{3.45}$$

where

$$a \equiv \sqrt{\frac{\gamma p}{\rho}} \tag{3.46}$$

$$c_a \equiv \sqrt{\frac{B_N^2}{\rho}} \tag{3.47}$$

$$c_f \equiv \left\{ \frac{1}{2} \left[a^2 + \frac{|\mathbf{B}|^2}{\rho} + \sqrt{\left(a^2 + \frac{|\mathbf{B}|^2}{\rho} \right)^2 - 4a^2 \frac{B_N^2}{\rho}} \right] \right\}^{1/2} \tag{3.48}$$

$$c_s \equiv \left\{ \frac{1}{2} \left[a^2 + \frac{|\mathbf{B}|^2}{\rho} - \sqrt{\left(a^2 + \frac{|\mathbf{B}|^2}{\rho} \right)^2 - 4a^2 \frac{B_N^2}{\rho}} \right] \right\}^{1/2} . \tag{3.49}$$

These eigenvalues always satisfy the property

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda_5 \le \lambda_6 \le \lambda_7 \,. \tag{3.50}$$

We make the following definitions:

$$\mu_f^{\mp} \equiv \frac{\alpha_f c_f^2}{\gamma - 1} \mp \alpha_f c_f u_N \pm \alpha_s c_a \beta_1 (\beta_2 u_T + \beta_3 u_3) + \frac{\gamma - 2}{\gamma - 1} \alpha_f (c_f^2 - a^2) \tag{3.51}$$

$$\mu_s^{\mp} \equiv \frac{\alpha_s c_s^2}{\gamma - 1} \mp \alpha_s c_s u_N \mp \alpha_f a \beta_1 (\beta_2 u_T + \beta_3 u_3) + \frac{\gamma - 2}{\gamma - 1} \alpha_s (c_s^2 - a^2) \tag{3.52}$$

$$\alpha_f \equiv \begin{cases} 1 & \text{if} \quad B_T = B_3 = 0 \quad \text{and} \quad c_a^2 = a^2 \\ \sqrt{(c_f^2 - c_a^2)/(c_f^2 - c_s^2)} & \text{otherwise} \end{cases} \tag{3.53}$$

$$\alpha_s \equiv \begin{cases} 1 & \text{if} \quad B_T = B_3 = 0 \quad \text{and} \quad c_a^2 = a^2 \\ \sqrt{(c_f^2 - a^2)/(c_f^2 - c_s^2)} & \text{otherwise} \end{cases} \tag{3.54}$$

and

$$\beta_1 \equiv \operatorname{sign}(B_N) \tag{3.55}$$

$$\beta_2 \equiv \begin{cases} 1/\sqrt{2} & \text{if} \quad B_T = B_3 = 0 \\ B_T/\sqrt{B_T^2 + B_3^2} & \text{otherwise} \end{cases} \tag{3.56}$$

$$\beta_3 \equiv \begin{cases} 1/\sqrt{2} & \text{if} \quad B_T = B_3 = 0 \\ B_3/\sqrt{B_T^2 + B_3^2} & \text{otherwise} \end{cases} \tag{3.57}$$

$$\theta_1 \equiv \frac{1}{2} \left[\alpha_f^2 a^2 \left(c_f^2 - \frac{\gamma - 2}{\gamma - 1} a^2 \right) + \alpha_s^2 c_f^2 \left(c_s^2 - \frac{\gamma - 2}{\gamma - 1} a^2 \right) \right]^{-1} \tag{3.58}$$

$$\theta_2 \equiv \frac{1}{2} \left[\alpha_f^2 c_f a \beta_1 + \alpha_s^2 c_s c_a \beta_1 \right]^{-1} . \tag{3.59}$$

In order to avoid writing separate expressions for the eigenvectors of A_1 and A_2, we will make use of the $\circlearrowright$ symbol to indicate that the two rows immediately above and below the symbol must be switched when considering the decomposition in the y-direction. Using these definitions, the right eigenvectors are

$$r^{1,7} = \begin{bmatrix} \alpha_f \\ \alpha_f(u_N \mp c_f) \\ \alpha_f u_T \pm \alpha_s \beta_1 \beta_2 c_a \\ \alpha_f u_3 \pm \alpha_s \beta_1 \beta_3 c_a \\ \frac{1}{2}\alpha_f |\mathbf{u}|^2 + \mu_f^{\mp} \\ 0 \\ \frac{1}{\sqrt{\rho}}\alpha_s \beta_2 c_f \\ \frac{1}{\sqrt{\rho}}\alpha_s \beta_3 c_f \end{bmatrix} \tag{3.60}$$

$$r^{2,6} = \begin{bmatrix} 0 \\ 0 \\ \pm\beta_1\beta_3 \\ \mp\beta_1\beta_2 \\ \pm(\beta_3 u_T - \beta_2 u_3)\beta_1 \\ 0 \\ \frac{1}{\sqrt{\rho}}\beta_3 \\ -\frac{1}{\sqrt{\rho}}\beta_2 \end{bmatrix} \qquad r^4 = \begin{bmatrix} 1 \\ u_N \\ u_T \\ u_3 \\ \frac{1}{2}|\mathbf{u}|^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.61}$$

$$r^{3,5} = \begin{bmatrix} \alpha_s \\ \alpha_s(u_N \mp c_s) \\ \alpha_s u_T \mp \alpha_f \beta_1 \beta_2 a \\ \alpha_s u_3 \mp \alpha_f \beta_1 \beta_3 a \\ \frac{1}{2}\alpha_s |\mathbf{u}|^2 + \mu_s^{\mp} \\ 0 \\ -\frac{1}{c_f\sqrt{\rho}}\alpha_f \beta_2 a^2 \\ -\frac{1}{c_f\sqrt{\rho}}\alpha_f \beta_3 a^2 \end{bmatrix}. \tag{3.62}$$

The left eigenvectors are

$$
l^{1,7} = \begin{bmatrix} \frac{1}{2}\theta_1 \alpha_f a^2 |\mathbf{u}|^2 \pm \theta_2 \left[\alpha_f a u_N \beta_1 - \alpha_s c_s \left(\beta_2 u_T + \beta_3 u_3\right)\right] \\ -\theta_1 \alpha_f a^2 u_N \mp \theta_2 \alpha_f a \beta_1 \\ -\theta_1 \alpha_f a^2 u_T \pm \theta_2 \alpha_s c_s \beta_2 \\ -\theta_1 \alpha_f a^2 u_3 \pm \theta_2 \alpha_s c_s \beta_3 \\ \theta_1 \alpha_f a^2 \\ 0 \\ \theta_1 \sqrt{\rho}\, \alpha_s \beta_2 c_f \left(c_s^2 - \frac{\gamma-2}{\gamma-1} a^2\right) \\ \theta_1 \sqrt{\rho}\, \alpha_s \beta_3 c_f \left(c_s^2 - \frac{\gamma-2}{\gamma-1} a^2\right) \end{bmatrix} \tag{3.63}
$$

$$
l^{2,6} = \begin{bmatrix} \mp\frac{1}{2}\beta_1 \left(\beta_3 u_T - \beta_2 u_3\right) \\ 0 \\ \pm\frac{1}{2}\beta_1\beta_3 \\ \mp\frac{1}{2}\beta_1\beta_2 \\ 0 \\ 0 \\ \frac{1}{2}\sqrt{\rho}\,\beta_3 \\ -\frac{1}{2}\sqrt{\rho}\,\beta_2 \end{bmatrix} \qquad l^4 = \begin{bmatrix} 1 - \theta_1 \left(\alpha_f^2 a^2 + \alpha_s^2 c_f^2\right) |\mathbf{u}|^2 \\ 2\theta_1 \left(\alpha_f^2 a^2 + \alpha_s^2 c_f^2\right) u_N \\ 2\theta_1 \left(\alpha_f^2 a^2 + \alpha_s^2 c_f^2\right) u_T \\ 2\theta_1 \left(\alpha_f^2 a^2 + \alpha_s^2 c_f^2\right) u_3 \\ -2\theta_1 \left(\alpha_f^2 a^2 + \alpha_s^2 c_f^2\right) \\ 0 \\ 2\theta_1 \sqrt{\rho}\, \alpha_f \alpha_s \beta_2 c_f \left(c_f^2 - c_s^2\right) \\ 2\theta_1 \sqrt{\rho}\, \alpha_f \alpha_s \beta_3 c_f \left(c_f^2 - c_s^2\right) \end{bmatrix} \tag{3.64}
$$

$$
l^{3,5} = \begin{bmatrix} \frac{1}{2}\theta_1 \alpha_s c_f^2 |\mathbf{u}|^2 \pm \theta_2 \left[\alpha_s c_a u_N \beta_1 + \alpha_f c_f \left(\beta_2 u_T + \beta_3 u_3\right)\right] \\ -\theta_1 \alpha_s c_f^2 u_N \mp \theta_2 \alpha_s c_a \beta_1 \\ -\theta_1 \alpha_s c_f^2 u_T \mp \theta_2 \alpha_f c_f \beta_2 \\ -\theta_1 \alpha_s c_f^2 u_3 \mp \theta_2 \alpha_f c_f \beta_3 \\ \theta_1 \alpha_s c_f^2 \\ 0 \\ -\theta_1 \sqrt{\rho}\, \alpha_f \beta_2 c_f \left(c_f^2 - \frac{\gamma-2}{\gamma-1} a^2\right) \\ -\theta_1 \sqrt{\rho}\, \alpha_f \beta_3 c_f \left(c_f^2 - \frac{\gamma-2}{\gamma-1} a^2\right) \end{bmatrix} . \tag{3.65}
$$

3.2.2 The MHD waves

As shown in Figure 3.1, the Riemann solution to the MHD system is composed of seven waves each propagating at a finite speed:

$$\mathcal{W}^{1,7}: \text{ Fast magnetosonic waves,} \tag{3.66}$$

$$\mathcal{W}^{2,6}: \text{ Alfvén waves,} \tag{3.67}$$

$$\mathcal{W}^{3,5}: \text{ Slow magnetosonic waves,} \tag{3.68}$$

$$\mathcal{W}^{4}: \text{ Entropy wave.} \tag{3.69}$$

To test if these waves are genuinely nonlinear or linearly degenerate, we compute the quantity $\partial_q \lambda^p \cdot r^p$ [53]. After some algebra it can be shown that (see Jeffrey and Taniuti [42])

$$\partial_q \lambda^{1,7} \cdot r^{1,7} \neq 0 \quad \forall q \tag{3.70}$$
$$\partial_q \lambda^{2,6} \cdot r^{2,6} = 0 \quad \forall q \tag{3.71}$$
$$\partial_q \lambda^{3,5} \cdot r^{3,5} \neq 0 \quad \forall q \tag{3.72}$$
$$\partial_q \lambda^{4} \cdot r^{4} = 0 \quad \forall q \,. \tag{3.73}$$

This result implies that the magnetosonic waves are genuinely nonlinear, while the Alfvén waves and the entropy wave are linearly degenerate. For certain values of the primitive variables, the ideal MHD system can become non-strictly hyperbolic. In particular, in the following cases two or more of the waves travel at the same wave speed:

- if $\mathbf{B}_N = 0$ then
  - $c_s = c_a = 0$ and $c_f = \sqrt{a^2 + \frac{|\mathbf{B}|^2}{\rho}}$
  - 2,3,4,5,6-waves propagate at the same speed
- if $B_T = B_3 = 0$, $|\mathbf{B}| \neq 0$, and $c_a \neq a$ then
  - $c_s = c_a = \sqrt{B_N^2/\rho}$ and $c_f = a$
  - 2,3-waves propagate at the same speed
  - 5,6-waves propagate at the same speed
- if $B_T = B_3 = 0$, $|\mathbf{B}| \neq 0$, and $c_a = a$ then
  - $c_s = c_a = c_f = a$
  - 1,2,3-waves propagate at the same speed
  - 5,6,7-waves propagate at the same speed

The right and left eigenvectors have been normalized in such a way that they remain linearly independent for all physically meaningful values of the primitive variables, including the above outlined non-strictly hyperbolic cases [13].

3.2.3 A Riemann solver based on arithmetic averages

In order to complete our development of an approximate Riemann solver for the ideal MHD equations, we will need to construct an approximate flux Jacobian at each cell edge. Because the ideal MHD equations have a fairly complicated flux Jacobian and eigenstructure, the Roe averages will not be used to compute an approximate flux Jacobian. Instead, we consider a

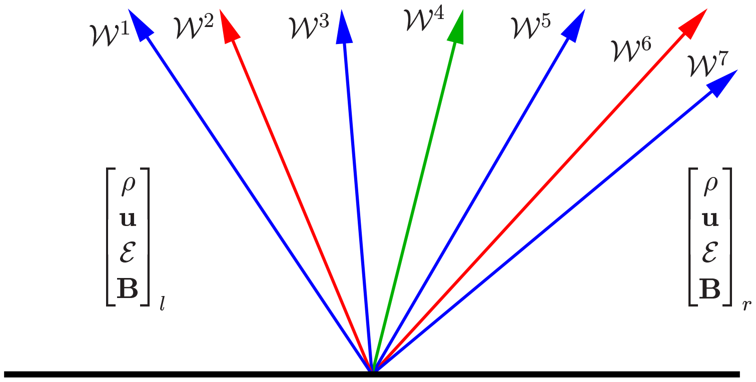

Figure 3.1: Wave structure for the ideal MHD equations. $\mathcal{W}^1$ and $\mathcal{W}^7$ are the fast magnetosonic waves, $\mathcal{W}^2$ and $\mathcal{W}^6$ are the Alvén waves, $\mathcal{W}^3$ and $\mathcal{W}^5$ are the slow magnetosonic waves, and $\mathcal{W}^4$ is the entropy wave.

consistent flux Jacobian based on the arithmetic averaged quantities:

$$\hat{\rho} = \frac{1}{2}\left(\rho_i + \rho_{i-1}\right) \tag{3.74}$$

$$\hat{u}_\star = \frac{1}{2}\left([u_\star]_i + [u_\star]_{i-1}\right) \tag{3.75}$$

$$\hat{p} = \frac{1}{2}\left(p_i + p_{i-1}\right) \tag{3.76}$$

$$\hat{B}_\star = \frac{1}{2}\left([B_\star]_i + [B_\star]_{i-1}\right) . \tag{3.77}$$

The main disadvantage in using the arithmetic averages over the Roe averages is that the Roe averaged flux Jacobian satisfies

$$[\hat{A}_1]_{i-1/2}\left(Q_i - Q_{i-1}\right) = [f_1]_i - [f_1]_{i-1} , \tag{3.78}$$

while the flux Jacobian based on the arithmetic average does not. This may cause strong shocks to be more smeared out in the case of arithmetic averages [53]. In general, however, the two approaches will produce comparable results [76, 77]. Further discussions on the Roe averages for ideal MHD can be found in [14].

3.3 Example: a 1D Riemann problem

As an example, we consider the following initial value Riemann problem taken from Ryu and Jones [76]:

$$\left[\rho, u_1, u_2, u_3, p, B_1, B_2, B_3\right]\Big|_{t=0} = \begin{cases} \left[1, 0, 0, 0, 1, 0.7, 0, 0\right] & \text{if} \quad x < 0 \\ \left[0.3, 0, 0, 1, 0.2, 0.7, 1, 0\right] & \text{if} \quad x > 0 . \end{cases} \tag{3.79}$$

The solution on a 100 point and a 5000 point grid are compared in Figure 3.2. These results compare well to those shown in Ryu and Jones [76]. The following quantities are plotted: 3.2(a) the density, 3.2(b) the thermal pressure, 3.2(c) the total energy, 3.2(d) the x-velocity, 3.2(e) the y-velocity, 3.2(f) the z-velocity, 3.2(g) the y-component of the magnetic field, and 3.2(h) the z-component of the magnetic field.

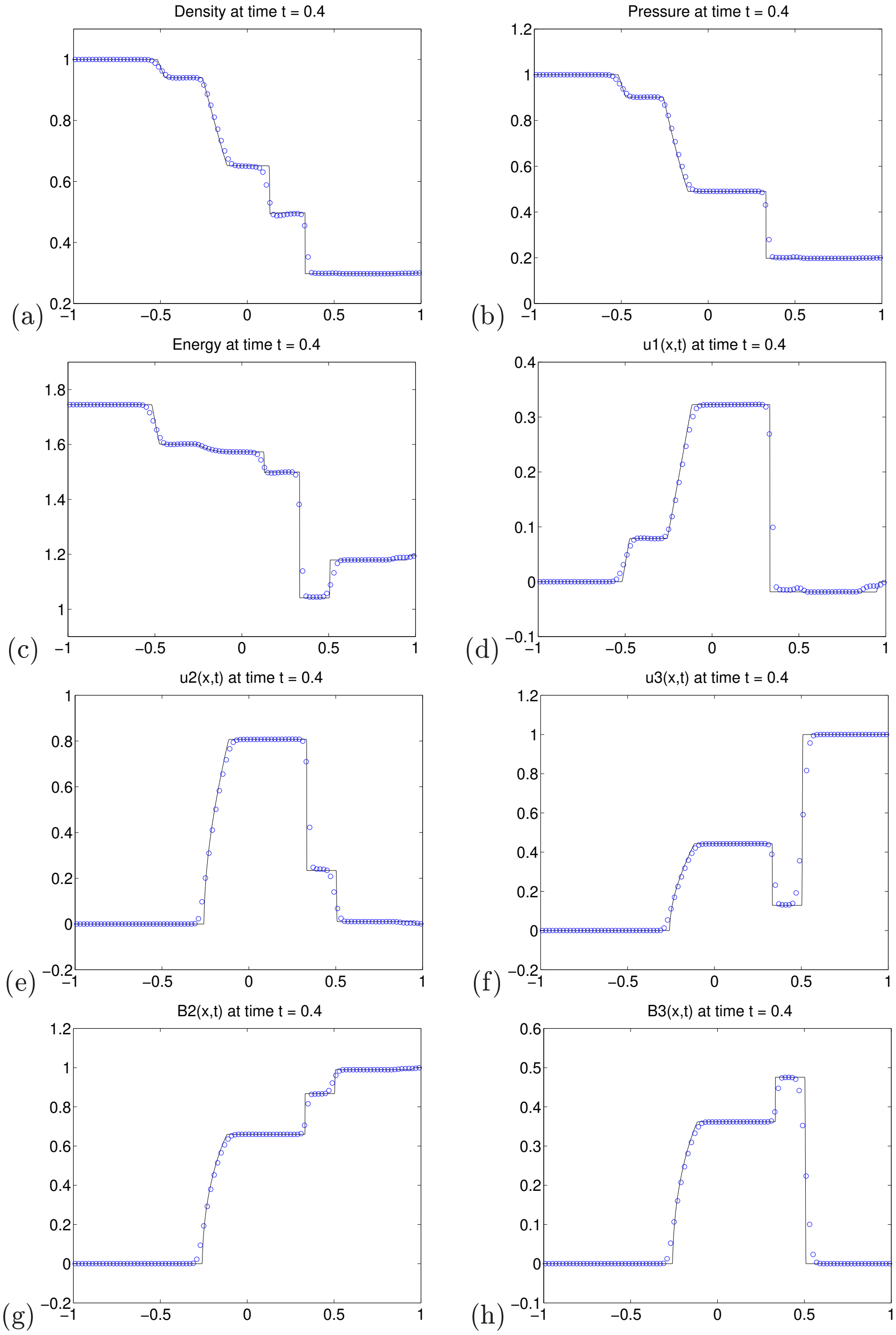

Figure 3.2: An example of a Riemann solution for the ideal MHD equations.