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Abstract

A Wave Propagation Method with Constrained Transport for Ideal and Shallow Water Magnetohydrodynamics

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The ideal magnetohydrodynamic (MHD) equations are important in modeling phenomena in a wide range of applications including space weather, solar physics, laboratory plasmas, and astrophysical fluid flows. A reduced version of this system known as the shallow water magnetohydrodynamic (SMHD) equations contains many of the features of MHD and can be used as a mathematical model for nearly two-dimensional flows. Numerical methods for these systems must confront the challenge of producing approximate solutions that remain accurate near shock waves and that satisfy a divergence-free constraint on the magnetic field. Failure to accomplish this often leads to unphysical solutions. In this work, a high-resolution wave propagation method is developed that utilizes a novel constrained transport technique to keep the magnetic field divergence-free. This method is compared extensively to other constrained transport methods from the literature on several standard MHD and SMHD test problems. Although initially developed as a Cartesian grid method, this approach is later extended to arbitrary logically rectangular grids on general curved manifolds. As an application problem, the SMHD equations are solved on the surface of sphere using a cubed sphere grid. Such simulations could be used to model the dynamics of the solar tachocline.

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Chapter 1

INTRODUCTION

The goal of this work is to develop an accurate and robust numerical method for the approximate solution of the ideal magnetohydrodynamic (MHD) equations and the shallow water magnetohydrodynamic (SMHD) equations. This method should be applicable to flows in strictly Cartesian geometries as well as on general curved manifolds including the surface of a sphere. In this Chapter we give motivation for why it is important to study the equations of magnetohydrodynamics, discuss some of the issues that arise when numerically solving them, and give an overview of the contents of this thesis.

1.1 Motivation

In the context of physics, a plasma is an ionized gas that is electrically neutral and that consists of ions, neutrons, and electrons [67]. Astrophysical fluid dynamics is the branch of astronomy dealing with the study of the evolution of plasma in stars, galaxies, and interstellar and intergalactic media [9]. One of the simplest systems of equations for modeling the dynamics of plasmas are the ideal MHD equations. The equations model a non-relativistic, single species plasma that is assumed to be a perfect conductor, inviscid, and composed of a gas that obeys the ideal gas law. We should note here that when we refer to a *single species* plasma we are actually referring to an electrically neutral soup of electrons, neutrons, and ions. If we further assume that this fluid is of constant density and in hydrostatic balance in the vertical direction, we obtain the shallow water MHD (SMHD) equations [30].

Despite the simplicity of these equations from the point of view of physics, these equations not only pose a considerable mathematical challenge when one is attempting to solve them, but they can in many cases provide good models for various astrophysical phenomena. Below we briefly discuss two specific application problems where the ideal MHD and SMHD equations have been used as mathematical models.

1.1.1 Astrophysical jets

Highly collimated supersonic outflows are an important phenomena observed in many astrophysical settings. These outflows are known as *astrophysical jets* and "are observed propagating from sources as diverse as active galactic nuclei, young stellar objects, and Planetary Nebulae" [26]. An example of such a jet occurring in the formation of a new star is shown in Figure 1.1. It is believed that these jets contain important information about the sources from which they originated. Although astrophysical jets in general contain very complicated physics, the ideal magnetohydrodynamic equations can provide a simplified

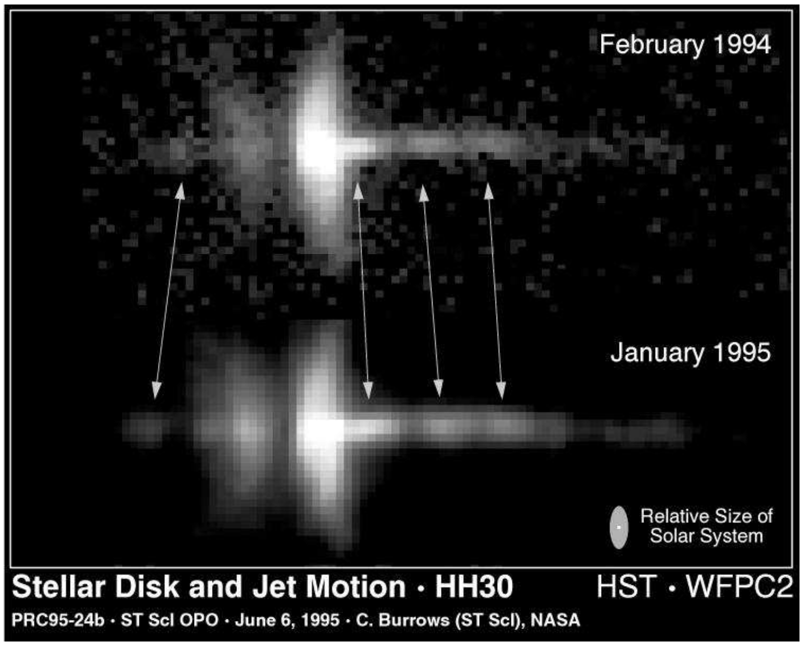

Figure 1.1: Hubble Telescope images of a newly forming star called HH-30. The star lies 450 light-years away in the constellation Taurus. The formation of the star is characterized by an accretion disk and a stellar jet. (This image is taken from `http://www.ast.cam.ac.uk`.)

framework for understanding their qualitative behavior. Some examples from the literature in which ideal MHD equations have been used as a mathematical model for astrophysical jets include [26], [27], [44], and [60].

1.1.2 The solar tachocline

The solar interior is composed of several zones including the core, the radiative zone, and the convective zone (see Figure 1.2). The radiative and convective zones are separated by a thin layer known as the solar tachocline. It is believed that "the solar tachocline [contains] a rather strong toroidal [magnetic] field that is the source for sunspot fields" [30]. For this reason there is strong interest in the solar physics community to consider fluid dynamic models of the solar tachocline that include the effects of the magnetic field [29]. In particular, Gilman [30] derived from the ideal MHD equations a simplified set of equations known as the shallow water MHD (SMHD) equations. The additional approximations required in order to obtain the SMHD equations are that the fluid has constant density and is in hydrostatic balance in the vertical (radial) direction. Because the solar tachocline is believed to have a thickness of approximately 2%-5% of the solar radius, these assumptions provide a physically plausible model [30]. Furthermore, since the SMHD equations govern an electrically conducting fluid, the fluid dynamics remains coupled to the magnetic field.

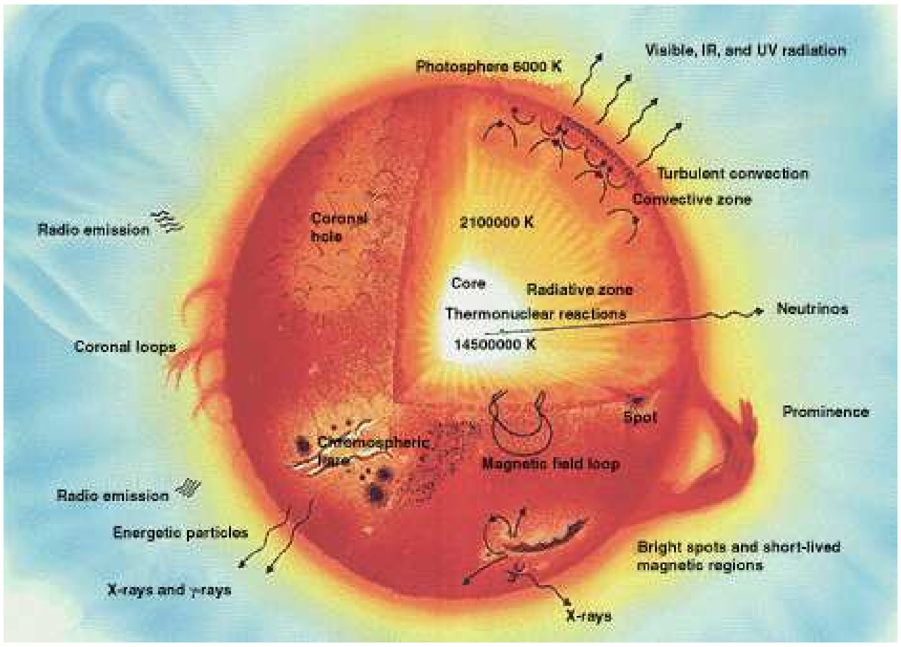

Figure 1.2: A diagram of the solar interior. The solar tachocline is the thin region separating the radiative zone from the convective zone. (This image is taken from `http://imagine.gsfc.nasa.gov`.)

1.2 Numerical challenges

In the process of developing accurate and robust numerical methods for solving the equations of magnetohydrodynamics one is faced with several numerical challenges. The main challenges which we are concerned with in this work are the following:

- How to accurately solve the equations of magnetohydrodynamics in such a way as to maintain the divergence-free constraint on the magnetic field.

- How to extend such methods to solve the equations of magnetohydrodynamics in generalized coordinates.

Although many types of numerical methods have been developed in order to deal with the divergence-free constraint in magnetohydrodynamics, questions about accuracy and robustness still remain in all of these approaches. Furthermore, in many physical applications we are not only interested in simple Cartesian geometries, but also in flows on curvilinear grids and on curved manifolds.

When solving hyperbolic equations in generalized coordinates, two more fundamental challenges arise:

- How to accurately solve hyperbolic conservation laws with a spatially varying flux function.
- How to solve balance laws in such a way that steady states are accurately maintained.

Both a spatially varying flux function as well as a geometric source term are induced by the metric of the underlying coordinate system [5].

1.3 Overview of this thesis

In this work we consider each of the above numerical challenges in the context of the LeVeque wave propagation method [51]. This method has in the past been successfully applied to several application problems including acoustics [24, 25], linear elasticity [23], nonlinear elasticity [54], shallow water waves [52], incompressible fluid dynamics [48], combustion and detonation waves [39, 40], relativistic hydrodynamics [3], and numerical relativity [8]. We concentrate in this work on the application to magnetohydrodynamics. An overview of the contents of each chapter is given below:

Chapter 2. In this chapter the wave propagation method as it applies to general hyperbolic systems is derived. We introduce the notion of a Riemann problem and how this can be used to obtain accurate numerical solutions to problems in which shock waves and other types of discontinuities may form. We then discuss some recently published modifications [5, 58] to the original wave propagation method [51]. We describe how these modifications allow us to compute accurate solutions to conservation laws with spatially varying flux functions and provide us with a technique for solving balance laws without the need for operator splitting.

Chapter 3. In this chapter the ideal magnetohydrodynamic equations are introduced. We derive the ideal MHD system and discuss the physical assumptions necessary to accomplish this. Once derived, the hyperbolic structure of the MHD equations is explored and an approximate Riemann solver based on arithmetic averages is reviewed.

Chapter 4. This chapter is concerned with the derivation and hyperbolic structure of the shallow water magnetohydrodynamic equations. This chapter on SMHD is analogous to the previous chapter on ideal MHD.

Chapter 5. In this chapter we consider a technique that can be coupled with the wave propagation method for keeping the magnetic field divergence-free in the ideal MHD and SMHD equations. We argue why it is necessary to numerically impose this divergence-free constraint and later show that failure to do so can lead to unphysical solutions. We review several types of methods found in the literature and discuss their strengths and weaknesses. In particular, it is argued that the constrained transport method is a natural choice for solving the MHD and SMHD equations. Despite this claim, we show that the constrained transport methods proposed by Balsara and Spicer [7] and Ryu et al. [78] are both inconsistent and that the Dai and Woodward [20] approach introduces large unphysical oscillations near shocks. To overcome these difficulties, we propose a novel approach based on TVD ideas. We demonstrate the accuracy and robustness of this method by applying it to several ideal MHD problems from the literature. The SMHD system has only recently been introduced by Gilman [30] and numerical solutions to these equations have previously only been considered by De Sterck [83, 84]. We demonstrate that the wave propagation method with constrained transport can also be applied to the SMHD system by considering the same test cases found in [83, 84].

Chapter 6. We focus in this chapter on extending the previously developed wave propagation method with constrained transport to generalized coordinates. In particular, we modify the curved manifold wave propagation method developed by Bale [3]. The ideal MHD and the SMHD equations and the previously developed constrained transport method are all extended to generalized coordinates.

Chapter 7. As an application problem of the method developed in the previous chapter, we consider the problem of solving the SMHD equations on the surface of a sphere. Such simulations could be used to model the dynamics of the solar tachocline. In order to avoid the grid singularities that are introduced by a latitude-longitude grid at the North and South Poles, we make use of a “cubed sphere” grid [75]. The cubed sphere grid is a collection of six smaller grid patches that cover the entire surface of the sphere without overlap. Special boundary conditions must be applied across the interfaces separating the six patches in order to guarantee an accurate mass and momentum transfer. On this grid we solve the linear advection equation, the shallow water equations, and the shallow water magnetohydrodynamic equations. For each system we consider test problems that validate the numerical method.