Chapter 7 Exercises

From: *Finite Difference Methods for Ordinary and Partial Differential Equations*
by R. J. LeVeque, SIAM, 2007. `http://www.amath.washington.edu/~rjl/fdmbook`

Exercise 7.1 *(Convergence of midpoint method)*

Consider the midpoint method $U^{n+1} = U^{n-1} + 2kf(U^n)$ applied to the test problem $u' = \lambda u$. The method is zero-stable and second order accurate, and hence convergent. If $\lambda < 0$ then the true solution is exponentially decaying.

On the other hand, for $\lambda < 0$ and $k > 0$ the point $z = k\lambda$ is never in the region of absolute stability of this method (see Example 7.7), and hence the numerical solution should be growing exponentially for any nonzero time step. (And yet it converges to a function that is exponentially decaying.)

Suppose we take $U^0 = \eta$, use Forward Euler to generate U^1, and then use the midpoint method for $n = 2, 3, \ldots$. Work out the exact solution U^n by solving the linear difference equation and explain how the apparent paradox described above is resolved.

Exercise 7.2 *(Example 7.10)*

Perform numerical experiments to confirm the claim made in Example 7.10.

Exercise 7.3 *(stability on a kinetics problem)*

Consider the kinetics problem (7.8) with $K_1 = 3$ and $K_2 = 1$ and initial data $u_1(0) = 3$, $u_2(0) = 4$, and $u_3(0) = 2$ as shown in Figure 7.4. Write a program to solve this problem using the forward Euler method.

(a) Choose a time step based on the stability analysis indicated in Example 7.12 and determine whether the numerical solution remains bounded in this case.

(b) How large can you choose k before you observe instability in your program?

(c) Repeat parts (a) and (b) for $K_1 = 300$ and $K_2 = 1$.

Exercise 7.4 *(damped linear pendulum)*

The m-file `ex7p11.m` implements several methods on the damped linear pendulum system (7.11) of Example 7.11.

(a) Modify the m-file to also implement the 2-step explicit Adams-Bashforth method AB2.

(b) Test the midpoint, trapezoid, and AB2 methods (all of which are second order accurate) for each of the following case (and perhaps others of your choice) and comment on the behavior of each method.

(i) $a = 100,\ b = 0$ (undamped),
(ii) $a = 100,\ b = 3$ (damped),

(iii) $a = 100,\ b = 10$ (more damped).

Exercise 7.5 *(fixed point iteration of implicit methods)*

Let $g(x) = 0$ represent a system of s nonlinear equations in s unknowns, so $x \in \mathbb{R}^s$ and $g : \mathbb{R}^s \to \mathbb{R}^s$. A vector $\bar{x} \in \mathbb{R}^s$ is a *fixed point* of $g(x)$ if

$$\bar{x} = g(\bar{x}). \tag{E7.5a}$$

One way to attempt to compute $\bar{x}$ is with *fixed point iteration*: from some starting guess x^0, compute

$$x^{j+1} = g(x^j) \tag{E7.5b}$$

for $j = 0,\ 1,\ \ldots$.

(a) Show that if there exists a norm $\|\cdot\|$ such that $g(x)$ is Lipschitz continuous with constant $L < 1$ in a neighborhood of $\bar{x}$, then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (E7.5a) from (E7.5b).

(b) Suppose $g(x)$ is differentiable and let $g'(x)$ be the $s \times s$ Jacobian matrix. Show that if the condition of part (a) holds then $\rho(g'(\bar{x})) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix.

(c) Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make N correction iterations, i.e., we set

$$\begin{aligned}
&\hat{U}^0 = U^n + kf(U^n)\\
&\text{for } j = 0,\ 1,\ \ldots,\ N-1\\
&\qquad \hat{U}^{j+1} = U^n + kf(\hat{U}^j)\\
&\qquad \text{end}\\
&U^{n+1} = \hat{U}^N.
\end{aligned}$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region S_N of this method for $N = 2,\ 5,\ 10,\ 20$ (perhaps using `plotS.m` from the webpage) and observe that in fact the stability region does not grow much in size.

(d) Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if $|k\lambda| < 1$ for all eigenvalues λ of the Jacobian matrix $f'(u)$.

(e) Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region S_N of part (c) to converge to as $N \to \infty$?