Conservation Laws and Finite Volume Methods

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Notation and Derivation

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For more about these methods...

- AMath 574, Winter 2017
- Clawpack Software (Conservation Laws Package)  
  www.clawpack.org  
  www.clawpack.org/gallery
Note different notation...

Solution = $q(x, t)$,

Advection velocity = $u$,

Advection equation: $q_t + u q_x = 0$,

Linear hyperbolic system: $q_t + A q_x = 0$

Nonlinear hyperbolic system: $q_t + f(q)_x = 0$
Derivation of Conservation Laws

\( q(x, t) \) = density function for some conserved quantity.

**Integral form:**

\[
\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) \, dx = F_1(t) - F_2(t)
\]

where

\[ F_j = f(q(x_j, t)), \quad f(q) = \text{flux function.} \]
Derivation of Conservation Laws

If $q$ is smooth enough, we can rewrite

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) \, dx = f(q(x_1, t)) - f(q(x_2, t))$$

as

$$\int_{x_1}^{x_2} q_t \, dx = - \int_{x_1}^{x_2} f(q)_x \, dx$$

or

$$\int_{x_1}^{x_2} (q_t + f(q)_x) \, dx = 0$$

True for all $x_1, x_2$ $\implies$ differential form:

$$q_t + f(q)_x = 0.$$
Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values \( Q^n_i \approx q(x_i, t_n) \)
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: \( Q^n_i \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n) \, dx \)
- Integral form of conservation law,

\[
\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t) \, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))
\]

leads to conservation law \( q_t + f_x = 0 \) but also directly to numerical method.

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AMath 586
Shallow water equations

\[ h(x, t) = \text{depth} \]
\[ u(x, t) = \text{velocity (depth averaged, varies only with } x) \]

Conservation of mass and momentum \( hu \) gives system of two equations.

mass flux = \( hu \),
momentum flux = \( (hu)u + p \) where \( p = \text{hydrostatic pressure} \)

\[
\begin{align*}
ht + (hu)_x &= 0 \\
(hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x &= 0
\end{align*}
\]

Jacobian matrix:

\[
f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}.
\]
Linearized shallow water equations

\[ h(x, t) = h_0 + \tilde{h}(x, t) \quad \text{(with } |\tilde{h}| \ll h_0) \]
\[ u(x, t) = 0 + \tilde{u}(x, t) \quad \text{(linearized about ocean at rest)} \]

Insert into the nonlinear equations

\[ h_t + (hu)_x = 0 \]
\[ (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x = 0 \]
Linearized shallow water equations

\begin{align*}
  h(x, t) &= h_0 + \tilde{h}(x, t) \quad \text{(with } |\tilde{h}| \ll h_0) \\
  u(x, t) &= 0 + \tilde{u}(x, t) \quad \text{(linearized about ocean at rest)}
\end{align*}

Insert into the nonlinear equations

\begin{align*}
  h_t + (hu)_x &= 0 \\
  (hu)_t + (hu^2 + \frac{1}{2}gh^2)_x &= 0
\end{align*}

Then ignore quadratic terms like \( \tilde{u}\tilde{h}_x \) to obtain:

\begin{align*}
  \tilde{h}_t + h_0 \tilde{u}_x &= 0 \\
  h_0 \tilde{u}_t + g h_0 \tilde{h}_x &= 0
\end{align*}

\[ \Rightarrow \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix}_t + \begin{bmatrix} 0 & h_0 \\ g & 0 \end{bmatrix} \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix}_x = 0. \quad \text{Eigenvalues: } \pm \sqrt{gh_0} \]

Same structure as linear acoustics.
Compressible gas dynamics

Conservation laws:

\[ \rho_t + (\rho u)_x = 0 \]
\[ (\rho u)_t + (\rho u^2 + p)_x = 0 \]

Equation of state:

\[ p = P(\rho). \]

Same as shallow water if \( P(\rho) = \frac{1}{2} g \rho^2 \) (with \( \rho \equiv h \)).

Isothermal: \( P(\rho) = a^2 \rho \) (since \( T \) proportional to \( p/\rho \)).

Jacobian matrix:

\[ f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \quad \lambda = u \pm \sqrt{P'(\rho)}. \]
Linear acoustics

Example: Linear acoustics in a 1d gas tube

\[ q = \begin{bmatrix} p \\ u \end{bmatrix} \quad p(x, t) = \text{pressure perturbation} \]
\[ u(x, t) = \text{velocity} \]

Equations:

\[ p_t + \kappa u_x = 0 \quad \text{Change in pressure due to compression} \]
\[ \rho u_t + p_x = 0 \quad \text{Newton’s second law, } F = ma \]

where \( \kappa = \text{bulk modulus} \), and \( \rho = \text{unperturbed density of gas} \).

Hyperbolic system:

\[ \begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0. \]
Linear system of \( m \) equations: \( q(x, t) \in \mathbb{R}^m \) for each \((x, t)\) and

\[
q_t + Aq_x = 0, \quad -\infty < x, \infty, \ t \geq 0.
\]

\( A \) is \( m \times m \) with eigenvalues \( \lambda^p \) and eigenvectors \( r^p \), for \( p = 1, 2, \ldots, m \):

\[
Ar^p = \lambda^p r^p.
\]

Combining these for \( p = 1, 2, \ldots, m \) gives

\[
AR = R\Lambda
\]

where

\[
R = [r^1 \ r^2 \ \ldots \ r^m], \quad \Lambda = \text{diag}(\lambda^1, \lambda^2, \ldots, \lambda^m).
\]

The system is hyperbolic if the eigenvalues are real and \( R \) is invertible. Then \( A \) can be diagonalized:

\[
R^{-1}AR = \Lambda
\]
Let $R$ be matrix of right eigenvectors and $v(x, t) = R^{-1}q(x, t)$. Multiply system $q_t + Aq_x = 0$ by $R^{-1}$ on left to obtain

$$R^{-1}q_t + R^{-1}AR^{-1}q_x = 0$$

Since $R^{-1}AR = \Lambda$, this diagonalizes the system:

$$w_t + \Lambda w_x = 0.$$ 

This is a system of $m$ decoupled advection equations

$$w^p_t + \lambda^p w^p_x = 0.$$ 

So

$$w^p(x, t) = w^p(x - \lambda^p t, 0)$$

where $w(x, 0) = R^{-1}q(x, 0) = R^{-1}\eta(x)$. 
\[
\begin{bmatrix}
 p \\
 u
\end{bmatrix}_t + \begin{bmatrix}
 0 & \kappa \\
 1/\rho & 0
\end{bmatrix} \begin{bmatrix}
 p \\
 u
\end{bmatrix}_x = 0.
\]

This has the form \( q_t + A q_x = 0 \) with

\[
\text{eigenvalues: } \lambda^1 = -c, \quad \lambda^2 = +c,
\]

where \( c = \sqrt{\kappa/\rho} = \text{speed of sound} \).

\[
\text{eigenvectors: } \quad r^1 = \begin{bmatrix}
 -Z \\
 1
\end{bmatrix}, \quad r^2 = \begin{bmatrix}
 Z \\
 1
\end{bmatrix}
\]

where \( Z = \rho c = \sqrt{\rho \kappa} = \text{impedance} \).

\[
R = \begin{bmatrix}
 -Z & Z \\
 1 & 1
\end{bmatrix}, \quad R^{-1} = \frac{1}{2Z} \begin{bmatrix}
 -1 & Z \\
 1 & Z
\end{bmatrix}.
\]
Linear acoustics

\[ p_t + \kappa u_x = 0 \]  \hspace{1cm} \text{Change in pressure due to compression}
\[ \rho u_t + p_x = 0 \]  \hspace{1cm} \text{Newton’s second law, } F = ma

This is a first-order hyperbolic system \( q_t + Aq_x = 0 \).

Second-order form:
Can combine equations to obtain wave equation:

\[ p_{tt} = c^2 p_{xx} \]

since

\[ p_{tt} = -\kappa u_{xt}, \]
\[ u_{tx} = -1/\rho \ p_{xx} \]

and so

\[ p_{tt} = -\kappa (-1/\rho) p_{xx} = c^2 p_{xx}. \]
Acoustic waves

\[ q(x, 0) = \begin{bmatrix} p_0(x) \\ 0 \end{bmatrix} = -\frac{p_0(x)}{2Z} \begin{bmatrix} -Z \\ 1 \end{bmatrix} + \frac{p_0(x)}{2Z} \begin{bmatrix} Z \\ 1 \end{bmatrix} = w^1(x, 0)r^1 + w^2(x, 0)r^2 = \begin{bmatrix} p_0(x)/2 \\ -p_0(x)/(2Z) \end{bmatrix} + \begin{bmatrix} p_0(x)/2 \\ p_0(x)/(2Z) \end{bmatrix}. \]
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\[ = w^1(x, 0)r^1 + w^2(x, 0)r^2 \]

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Shock formation

For nonlinear problems wave speed generally depends on $q$.

Waves can steepen up and form shocks
$\implies$ even smooth data can lead to discontinuous solutions.
Shock formation

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Waves can steepen up and form shocks$$
\Longrightarrow \text{ even smooth data can lead to discontinuous solutions.}
$$
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Computational challenges!

Need to capture sharp discontinuities.

PDE breaks down, standard finite difference approximation to
$q_t + f(q)_x = 0$ can fail badly: nonphysical oscillations,
convergence to wrong weak solution.