

Euler

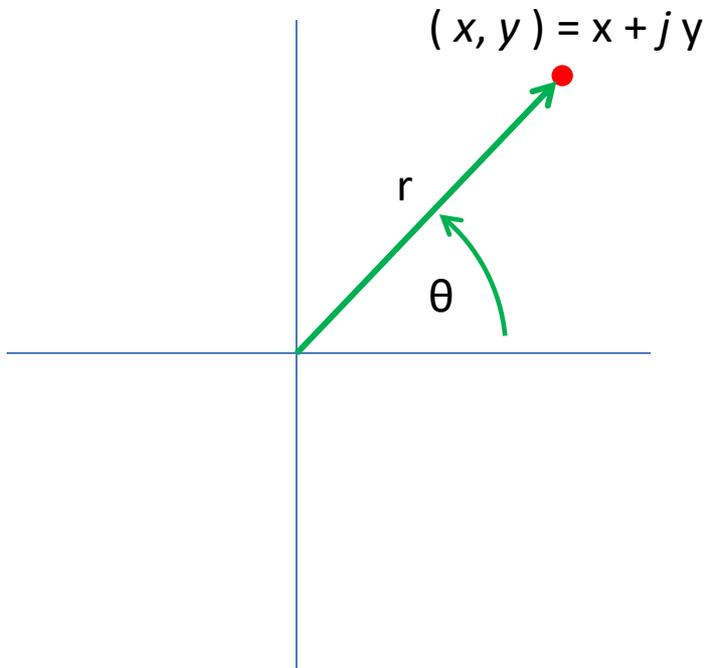
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Let's start with Euler

$$e^{jx} = \cos x + j \sin x$$

First, let's prove Euler



Let's assume that e^{jx} has a value somewhere on the real and imaginary plane, we just don't know what that value is.

If it has a value, we must be able to represent it in either polar or rectangular coordinates.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

So let's assume:

$$e^{jx} = r (\cos \theta + j \sin \theta)$$

for some r and θ that are functions of x , and test whether that's true.

Assume there exist some functions r and θ of x such that:

$$e^{jx} = r(\cos \theta + j \sin \theta)$$

Taking the derivative of each side:

$$\frac{d}{dx} e^{jx} = \frac{d}{dx} (r(\cos \theta + j \sin \theta))$$

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx} e^{jx} = j e^{jx}$$

$$\frac{d}{dx} \cos \theta = -\sin \theta \frac{d\theta}{dx}$$

$$\frac{d}{dx} \sin \theta = \cos \theta \frac{d\theta}{dx}$$

$$j e^{jx} = r(-\sin \theta + j \cos \theta) \frac{d\theta}{dx} + (\cos \theta + j \sin \theta) \frac{dr}{dx}$$

$$je^{jx} = r(-\sin \theta + j \cos \theta) \frac{d\theta}{dx} + (\cos \theta + j \sin \theta) \frac{dr}{dx}$$

Substituting our original assumption:

$$e^{jx} = r(\cos \theta + j \sin \theta)$$

$$jr(\cos \theta + j \sin \theta) = r(-\sin \theta + j \cos \theta) \frac{d\theta}{dx} + (\cos \theta + j \sin \theta) \frac{dr}{dx}$$

Separating the real and imaginary parts:

$$-r \sin \theta = -r \sin \theta \frac{d\theta}{dx} + \cos \theta \frac{dr}{dx}$$

$$jr \cos \theta = jr \cos \theta \frac{d\theta}{dx} + j \sin \theta \frac{dr}{dx}$$

means

$$\frac{d\theta}{dx} = 1$$

$$\frac{dr}{dx} = 0$$

Our original assumption plus what we now know:

$$e^{jx} = r(\cos \theta + j \sin \theta)$$

$$\frac{d\theta}{dx} = 1 \quad \text{means } \theta = x + b \quad \text{where } b \text{ is a constant.}$$

$$\frac{dr}{dx} = 0 \quad \text{means } r \text{ is a constant.}$$

If this is true for any x , it must also be true for $x = 0$.

$$e^{j0} = e^0 = 1 = r(\cos \theta + j \sin \theta)$$

$$r = 1 \quad \theta \Big|_{x=0} = 0$$

Therefore $b = 0$

$$\theta = x$$

$$e^{jx} = \cos x + j \sin x$$

Now that we trust Euler, let's use it to do some derivations.

$$e^{-jx}$$

$$\cos(x)$$

$$\sin(x)$$

$$\cos(x + \theta)$$

$$\sin(x + \theta)$$

$$\cos^2(x)$$

$$\sin^2(x)$$

$$e^{-jx}$$

$$e^{jx} = \cos x + j \sin x$$

$$e^{-jx} = \cos(-x) + j \sin(-x)$$

$$\cos(-x) = \cos x$$

$$\sin(-x) = -\sin x$$

$$e^{-jx} = \cos x + j(-\sin x)$$

$$e^{-jx} = \cos x - j \sin x$$

$\cos(x)$ and $\sin(x)$

$$e^{jx} = \cos x + j \sin x$$

$$e^{-jx} = \cos x - j \sin x$$

$$e^{jx} + e^{-jx} = 2 \cos x$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$e^{jx} - e^{-jx} = 2j \sin x$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\cos(x + \theta)$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\cos(x + \theta) = \frac{e^{j(x + \theta)} + e^{-j(x + \theta)}}{2}$$

$$x^{(b+c)} = x^b x^c$$

$$\cos(x + \theta) = \frac{e^{jx} e^{j\theta} + e^{-jx} e^{-j\theta}}{2}$$

$$2 \cos(x + \theta) = e^{jx} e^{j\theta} + e^{-jx} e^{-j\theta}$$

$$2 \cos(x + \theta) = e^{jx} e^{j\theta} + e^{-jx} e^{-j\theta}$$

$$e^{jx} = \cos x + j \sin x$$

$$e^{-jx} = \cos x - j \sin x$$

$$2 \cos(x + \theta) = (\cos x + j \sin x)(\cos \theta + j \sin \theta) \\ + (\cos x - j \sin x)(\cos \theta - j \sin \theta)$$

$$2 \cos(x + \theta) = \cos x \cos \theta + j \cos x \sin \theta \\ + j \sin x \cos \theta - \sin x \sin \theta \\ + \cos x \cos \theta - j \cos x \sin \theta \\ - j \sin x \cos \theta - \sin x \sin \theta$$

$$2 \cos(x + \theta) = 2 \cos x \cos \theta - 2 \sin x \sin \theta$$

$$\cos(x + \theta) = \cos x \cos \theta - \sin x \sin \theta$$

similarly for $\sin(x + \theta)$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x + \theta) = \frac{e^{j(x+\theta)} - e^{-j(x+\theta)}}{2j}$$

$$x^{(b+c)} = x^b x^c$$

$$\sin(x + \theta) = \frac{e^{jx} e^{j\theta} - e^{-jx} e^{-j\theta}}{2j}$$

$$2j \sin(x + \theta) = e^{jx} e^{j\theta} - e^{-jx} e^{-j\theta}$$

$$2j \sin(x + \theta) = e^{jx} e^{j\theta} - e^{-jx} e^{-j\theta}$$

$$e^{jx} = \cos x + j \sin x$$

$$e^{-jx} = \cos x - j \sin x$$

$$2j \sin(x + \theta) = (\cos x + j \sin x)(\cos \theta + j \sin \theta) \\ - (\cos x - j \sin x)(\cos \theta - j \sin \theta)$$

$$2j \sin(x + \theta) = \cos x \cos \theta + j \cos x \sin \theta \\ + j \sin x \cos \theta - \sin x \sin \theta \\ - \cos x \cos \theta + j \cos x \sin \theta \\ + j \sin x \cos \theta + \sin x \sin \theta$$

$$2j \sin(x + \theta) = 2j \sin x \cos \theta - 2j \cos x \sin \theta$$

$$\sin(x + \theta) = \sin x \cos \theta - \cos x \sin \theta$$

$$\cos^2(x)$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\cos^2 x = \left(\frac{e^{jx} + e^{-jx}}{2} \right) \left(\frac{e^{jx} + e^{-jx}}{2} \right)$$

$$\cos^2 x = \frac{e^{j2x} + 2e^{jx}e^{-jx} + e^{-j2x}}{4}$$

$$\cos^2 x = \frac{e^{j2x} + e^{-j2x} + 2}{4}$$

$$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

similarly for $\sin^2(x)$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin^2 x = \left(\frac{e^{jx} - e^{-jx}}{2j} \right) \left(\frac{e^{jx} - e^{-jx}}{2j} \right)$$

$$\sin^2 x = \frac{e^{j2x} - 2e^{jx}e^{-jx} + e^{-j2x}}{-4}$$

$$\sin^2 x = \frac{2 - e^{j2x} + e^{-j2x}}{4}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$