and Chan (1995), is an 'exact' technique for simulating fGn and certain other stationary processes. Although DHM is well-known in the statistical literature, it was not included in a recent major review of simulation techniques for power-law and related processes in the engineering literature (Kasadin, 1995), nor was it mentioned in a recent paper in this journal (Yin, 1996). Since DHM is an exact method and can produce a realization of fGn in the same amount of computer time as GSSM can, it would appear that DHM eliminates the need for GSSM; however, as we show in this paper, there is no real practical difference in the statistical properties of realizations produced by GSSM and DHM, so either method is a reasonable choice. In addition, for processes other than fGn, GSSM might be preferable for two reasons. First, unlike GSSM, DHM is not a completely general method in that it requires a nonnegativity constraint to hold, which can fail for certain stationary processes. Second, DHM requires knowledge of the ACVS (this is readily available for fGn – see Section 2), whereas GSSM requires knowledge of the spectral properties for a process, so the latter method is easier to work with for processes that are specified directly in terms of their spectra.

The remainder of this paper is organized as follows. Following a review of the basic properties of fGn and fBm in Section 2, we introduce GSSM in Section 3 and provide a useful measure of how close it comes to being exact. In Section 4 we survey other SSMs that have been discussed in the literature (including Yin, 1996), pointing out why GSSM has better theoretical properties than these other SSMs. In Section 4 we review DHM, indicate why it is considered an exact method, and compare it to GSSM.

2. Fractional Gaussian Noise and Fractional Brownian Motion

By definition the sequence $\{X_t : t = 0, 1, 2, \ldots\}$ of RVs is said to be a zero mean fractional Gaussian noise (fGn) if (i) any finite subset of $\{X_t\}$ obeys a multivariate normal (Gaussian) distribution; (ii) the expected value of any member of the sequence is zero, i.e., $E\{X_t\} = 0$ for all t; and (iii) the covariance between any two RVs in $\{X_t\}$, say X_t and $X_{t+\tau}$, depends just on their separation τ in the following manner:

$$s_{X,\tau} \equiv \operatorname{cov}\{X_t, X_{t+\tau}\} = \frac{\sigma_X^2}{2}\left(|\tau+1|^{2H} - 2|\tau|^{2H} + |\tau-1|^{2H}\right), \quad \tau = 0, \pm 1, \pm 2, \ldots,$$

where $0 < H < 1$ is the Hurst (or self-similarity) parameter, and $\sigma_X^2 \equiv \text{var}\{X_t\} = s_{X,0}$ is the process variance (we assume $0 < \sigma_X^2 < \infty$). The sequence $\{s_{X,\tau}\}$ is known as the autocovariance sequence (ACVS). Because $E\{X_t\}$ and $\text{cov}\,\{X_t, X_{t+\tau}\}$ for any t and τ are finite and do not depend on t and because the RVs in $\{X_t\}$ are multivariate Gaussian, an fGn satisfies the definition of a Gaussian stationary process.

An fGn can be regarded as increments of fractional Brownian motion (fBm) $\{B_H(t) : 0 \leq t < \infty\}$ with parameter H; i.e.,

$$X_t = B_H(t+1) - B_H(t).$$

A precise definition of fBm is given in, e.g., Beran (1994), from which we note that, for $t > 0$, $B_H(t)$ is a zero mean Gaussian process with variance $\sigma_X^2 t^{2H}$ and that $B_H(0) \equiv 0$. We can thus create samples of fBm by cumulatively summing $\{X_t\}$:

$$B_t \equiv B_H(t) = \sum_{u=0}^{t-1} X_t, \quad t = 1, 2, \ldots. \qquad (i)$$

With $B_0 \equiv 0$, we refer to $\{B_t : t = 0, 1, \ldots\}$ as discrete fractional Brownian motion (dfBm) to distinguish from fBm, which is defined over the entire real axis. The spectral density function (SDF) for $\{B_H(t)\}$ can be taken to be

$$S_B(f) = \frac{\sigma_X^2 C_H}{|f|^{2H+1}}, \quad -\infty < f < \infty, \text{ where } C_H \equiv \frac{\Gamma(2H+1)\sin(\pi H)}{(2\pi)^{2H+1}}$$

(Mandelbrot and van Ness and/or Flandrin). It follows from filtering and aliasing considerations that the SDF for $\{B_t\}$ is given by

$$S_B(f) = \sigma_X^2 C_H \sum_{j=-\infty}^{\infty} \frac{1}{|f+j|^{2H+1}}, \quad |f| \leq 1/2,$$

while the SDF for $\{X_t\}$ is given by

$$S_X(f) = 4\,\sigma_X^2 C_H \sin^2(\pi f) \sum_{j=-\infty}^{\infty} \frac{1}{|f+j|^{2H+1}}, \quad |f| \leq 1/2,$$

(Sinai, 197?; Beran, 1994, p. 53, Equation (2.17)). For $f \neq 0$, these SDFs both involve a converging infinite summation that can be effectively approximated via a Euler–Maclaurin summation:

$$\sum_{j=-\infty}^{\infty} \frac{1}{|f+j|^{2H+1}} \approx \sum_{j=-J}^{J} \frac{1}{|j+f|^{2H+1}} + \sum_{l=-1,1} \Big[\frac{1}{2H(J+1+lf)^{2H}} + \frac{1}{2(J+1+lf)^{2H+1}} + \frac{2H+1}{12(J+1+lf)^{2H+2}} - \frac{(2H+1)(2H+2)(2H+3)}{720(J+1+lf)^{2H+4}} \Big]$$

(in practise, $J = 100$ yields sufficient accuracy). The ACVS and the SDF for $\{X_t\}$ contain the same 'information' so that, e.g., we can recover the ACVS from the SDF via the inverse Fourier relationship

$$s_{X,\tau} = \int_{-1/2}^{1/2} S_X(f) e^{i2\pi f\tau}\, df. \tag{c}$$

It should be clear from the above that, given a simulation of fGn, it is an easy matter to create a simulaton of dfBm by using Equation (i). In what following, we thus concentrate on simulation of fGn. Our task is to define a process $\{U_t\}$ that can be easily simulated using pseudo-random numbers on a digital computer such that $U_0, U_1, \ldots, U_{N-1}$ have statistical properties that closely match those of $X_0, X_1, \ldots, X_{N-1}$ from an fGn.

3. Gaussian Spectral Synthesis Method

We consider here the construction of a zero mean Gaussian process $\{U_t^{(M)}\}$ whose ACVS $\{s_{U^{(M)},\tau}\}$ agrees with $\{s_{X,\tau}\}$ out to lag $N-1$ to a certain quantifiable degree of approximation. Let M be any even integer such that $M \geq N$. Let W_j, $j = 0, \ldots, M-1$, be a set of M independent and identically distributed Gaussian RVs with zero mean and unit variance. For $0 < f \leq \frac{1}{2}$, let $S'_X(f) \equiv S_X(f)$, and let $S'_X(0)$ be a finite constant to be determined shortly. Let $f_j \equiv \frac{j}{M}$. We define the process $\{U_t^{(M)}\}$ via

$$U_t^{(M)} \equiv \frac{1}{M} \sum_{j=0}^{M-1} \mathcal{U}_j e^{-i2\pi f_j t}, \qquad t = 0, \ldots, M-1, \tag{f}$$

where

$$\mathcal{U}_j \equiv \begin{cases} W_0\sqrt{MS'_X(0)}, & j=0; \\ (W_{2j-1}+iW_{2j})\sqrt{\frac{M}{2}S'_X(f_j)}, & 1 \le j < \frac{M}{2}; \\ W_{M-1}\sqrt{MS'_X(\frac{1}{2})}, & j=\frac{M}{2}; \\ \mathcal{U}^*_{M-j}, & \frac{M}{2} < j \le M-1 \end{cases}$$

(note that, if we let M be a power of 2, Equation (f) can be quickly computed using a conventional FFT algorithm so realizations of $\{U_t^{(M)}\}$ of length M can thus be readily generated). By construction, $\{U_t^{(M)}\}$ is a real-valued Gaussian process. A straight-forward exercise shows that $\{U_t^{(M)}\}$ is a real-valued stationary process with zero mean and ACVS $\{s_{U^{(M)},\tau}\}$ given by

$$s_{U^{(M)},\tau} = \frac{1}{M}\sum_{j=0}^{M-1} S'_X(f_j)e^{i2\pi f_j\tau}. \qquad (b)$$

We set the constant $S'_X(0)$ to the value that minimizes

$$\begin{aligned} g(S'_X(0)) &\equiv \sum_{\tau=-(N-1)}^{N-1}\left(s_{U^{(M)},\tau} - s_{X,\tau}\right)^2 \\ &= \sum_{\tau=-(N-1)}^{N-1}\left(\frac{S'_X(0)}{M} + \frac{1}{M}\sum_{j=1}^{M-1} S_X(f_j)e^{i2\pi f_j\tau} - s_{X,\tau}\right)^2. \end{aligned}$$

Differentiating g with respect to $S'_X(0)$ and setting the result to 0 yields the solution

$$S'_X(0) = \frac{M}{2N-1}\left(\sum_{\tau=-(N-1)}^{N-1} s_{X,\tau} - \frac{1}{M}\sum_{j=1}^{M-1} S_X(f_j)\frac{\sin([2N-1]\pi f_j)}{\sin(\pi f_j)}\right).$$

Using the ACVS for fGn, we can reduce the above somewhat by noting that

$$\sum_{\tau=-(N-1)}^{N-1} s_{X,\tau} = \sigma_X^2\left[N^{2H} - (N-1)^{2H}\right].$$

Because the right-hand side of Equation (b) can be regarded as a Riemann sum approximation to the integral of Equation (c), we have $s_{U^{(M)},\tau} \approx s_{X\tau}$ for possibly some values of τ, but certainly not all: whereas $\{U_t^{(M)}\}$ is a harmonic process and hence its ACVS is periodic, the ACVS for $\{X_t\}$ damps down to 0 as $\tau \to \infty$. By leaving N fixed and increasing M, however, we can make $s_{U^{(M)},\tau}$ arbitrarily close to $s_{X,\tau}$ for $\tau = 0, \ldots, N-1$ (this follows from the convergence of the N Riemann sums to the corresponding integrals as $M \to \infty$). Hence, if M is picked sufficiently large,