The Autocovariance of a Stationary or Non-Stationary Pure Power Law Process

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September 14, 2004

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Abstract

We generate numerical values of the autocovariance functions of time series whose spectral distribution functions are of pure power law form. We consider values of the power law exponent corresponding to stationary processes, and to non-stationary processes. Stationary cases include both long-memory processes, and processes that are deficient in low frequencies. Non-stationary cases are interpreted as those for which a first, or higher order, difference is stationary. In all these cases, the autocovariance functions are shown to be related to Lommel's Bessel functions. Accurate numerical values of the autocovariances are obtained by numerical implementations of series expansions and asymptotic expansions. Typical autocovariance sequences are generated for a number of cases, most of which have not been examined before.

1 Introduction

Long memory processes have been found to be useful as models for many time series encountered in metrology, geophysics, engineering and finance. In the literature a discrete parameter stationary process $\{X_t : t \in \mathbb{Z}\}$ is said to have long memory if its spectral density function (SDF) $S_X(\cdot)$ has a weak (integrable) singularity at zero frequency or if its autocovariance sequence (ACVS) $\{s_\tau : \tau \in \mathbb{Z}\}$ does not decay exponentially with time, but instead decreases more slowly, as a negative fractional power of the lag τ (here $ints$ is the set of all integers). A number of simple mathematical models have been proposed that generate long memory time series and that allow these processes to be studied in an idealized setting. Among these are the fractional Gaussian noise (FGN) process [8], the fractionally differenced (FD) process [4], [6], and the pure power law (PPL) process [2]. All these models generate time series that can exhibit long memory behavior, and all can be studied either in the time domain or the frequency domain. The relative ease with which numerical values of the ACVF or the SDF can be obtained depends on the particular model. For the FGN process, the ACVF is given by a simple closed form expression, but the SDF must be obtained from a slowly convergent infinite series. For the PPL and FD processes, it is the SDFs that are given by simple expressions while the ACVFs are more difficult to obtain. For the FD process, the ACVF can be generated recursively, one time step at a time; however, for the PPL process, which we focus on here, the ACVF has not been related to known special functions, and has not been generated previously in a systematic way.

Each of the simple models mentioned here (FGN, FD, and PPL) can be generalized to encompass non-stationary cases. In this extended realm, the original time series is non-stationary but, if that series is differenced a sufficient number of times, the result is a related time series that is stationary and does have a well-defined ACVF and SDF. In our present investigation, we include non-stationary PPL models in this way.

In the following, we show how the ACVF of a PPL process, in both the stationary and non-stationary cases, can be related to Lommel's functions. We use this connection, plus some results from the theory of asymptotic expansions of Fourier transforms, to obtain accurate numerical values of the ACVF. We present the numerical results for a range of cases, and investigate a novel way to display the values graphically. We begin by recalling the definition of a PPL process.

A time series $\{X_t : t = \ldots, -1,\ 0, 1, \ldots\}$ is a stationary PPL process if its SDF has the form

$$S_X(f) = C_0^{(\alpha)} |f|^\alpha, \quad |f| \leq \tfrac{1}{2}, \quad \alpha > -1. \tag{1}$$

Here, $C_0^{(\alpha)} > 0$ is a constant proportional to the process variance:

$$C_0^{(\alpha)} = 2^\alpha (1+\alpha)\, \mathrm{var}\{X_t\}. \tag{2}$$

For $-1 < \alpha < 0$ the SDF has a weak singularity at $f = 0$, and the process then is of long memory type. The case $\alpha = 0$ corresponds to a white noise process. The case $\alpha > 0$ also is well-defined, but the process then is deficient in low frequencies and is an anticorrelated process. In all these cases, the ACVF is given by the integral

$$s_\tau^{(\alpha)} = 2\, C_0^{(\alpha)} \int_0^{1/2} f^\alpha \cos(2\pi f \tau)\, df, \quad \tau \;=\; \ldots,\; -1,\; 0,\; 1,\; \ldots . \tag{3}$$

Equivalently, $s_\tau^{(\alpha)}$ is proportional to the inverse Fourier transform of $|f|^\alpha\ H(f+1/2)\ H(1/2-f)$, where H is the unit step function. In this latter interpretation, the integrand has jump discontinuities at $f = \pm 1/2$ and, unless $\alpha = 0$, has at $f = 0$ either a weak singularity when $\alpha < 0$, or a jump in the first derivative when $\alpha > 0$. The asymptotic form of the ACVF at large τ is related to the behavior of the integrand at these discontinuities. The details follow from the asymptotic theory of Fourier transforms of distributions (see, e.g., Lighthill [7], Ch. 4), and also have been investigated in the context of long memory time series by Beran [2].

By either of these routes it follows that, for large $|\tau|$,

$$s_\tau^{(\alpha)} \sim -\frac{C_0^{(\alpha)}\Gamma(\alpha+1)\sin(\pi\alpha/2)}{2^\alpha(\pi|\tau|)^{\alpha+1}}. \tag{4}$$

For small τ, series representations for the ACVS can be obtained by expanding the cosine in (3) and integrating. For example, for $\tau = 1$,

$$s_1^{(\alpha)} = \mathrm{var}\{X_t\}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\Big(\frac{\alpha+1}{\alpha+2m+1}\Big)\pi^{2m}. \tag{5}$$

The above expressions provide the limiting form at large τ and a series approach for small τ but, unless α is zero or a positive integer, the integral in (3) is not elementary, and the behavior of the ACVF at other values of τ is not readily obtained. In the following section, we show how the integral is related to Lommel's functions (Bessel function family), and this leads immediately to power series and asymptotic series representations that, with some effort, can be adapted for generation of numerical values for any value of $\alpha > -1$. In addition, we show these same expansions occur in the analysis of nonstationary PPL processes with $\alpha \in (-1-2d,\ 1-2d\ ]$ and $d = 1, 2, \ldots$, since the d^{th} difference of such a process is by definition a stationary process. Some of the derivations involve detailed manipulations. To improve readability, we present certain key results as theorems, with proofs given in the Appendix.

2 The Stationary Case

We begin the analysis of the stationary case ($\alpha > -1$) in a somewhat roundabout fashion by showing that, for the particular cases $\alpha = \pm 1/2$, the ACVS is related to the Fresnel integrals. The derivations are elementary. Without loss of generality, we assume in the following that $\mathrm{var}\{X_t\} = 1$. For $\alpha = -1/2$ it follows from Eqs. (3) and (2) that

$$s_\tau^{(-1/2)} = \frac{1}{\sqrt{2}}\int_0^{1/2} f^{-1/2}\cos(2\pi f\tau)\,df. \tag{6}$$

Note that $s_\tau^{(-1/2)}$, like all ACVFs, is an even function of τ. In the following, we derive expressions valid for $\tau \geq 0$, with the implicit understanding that they extend as even functions for negative τ. The change of variable $f = x^2/(4\tau)$ converts the integral in (6) to a canonical form that is recognized as the Fresnel integral C.

$$s_\tau^{(-1/2)} = \frac{1}{\sqrt{2\tau}} \int_0^{\sqrt{2\tau}} \cos(\frac{\pi}{2} x^2)\, dx = \frac{C(\sqrt{2\tau})}{\sqrt{2\tau}}. \tag{7}$$

Properties of C, and numerical tables, can be found, e.g., in [1], Ch. 7. (There are, however, alternative definitions of C, with different normalizations.)

The case $\alpha = +1/2$, which is PPL but not of long memory type, can be treated similarly and also leads to Fresnel integrals. We have

$$s_\tau^{(1/2)} = 3\sqrt{2} \int_0^{1/2} f^{1/2} \cos(2\pi f \tau)\, df. \tag{8}$$

An integration by parts, followed by the change of variable used in the previous case, shows that

$$s_\tau^{(1/2)} = \frac{3}{2} \Big[\, \delta_{\tau,0} - \frac{2}{\pi} \frac{S(\sqrt{2\tau})}{(2\tau)^{3/2}} \,\Big]. \tag{9}$$

Here δ is Kronecker's delta, and S is the Fresnel integral

$$S(z) = \int_0^z \sin(\frac{\pi}{2} x^2)\, dx. \tag{10}$$

For $\alpha = \pm 1/2$, the expressions above could be used to generate numerical values of the ACVS by reference to tables of C and S (e.g. [1], Ch. 5). The primary role of these representations, however, is to point toward a more general class of functions that encompasses other values of α. Special function affectionados may recall that Fresnel integrals occur as particular cases of Lommel's functions of two variables. The Lommel functions are peripheral members of the Bessel function family, and were introduced in the late 19th century by the physicist E. C. J. von Lommel in a study of the diffraction of light by a circular aperature.

Many properties of these functions are summarized in Watson's treatise on Bessel functions ([9], Section 16.5).

The Lommel function $U_\nu(w,z)$ is defined in general as the sum of the Neumann series of Bessel functions

$$U_\nu(w,z) = \sum_{m=0}^{\infty} (-1)^m (w/z)^{\nu+2m} J_{\nu+2m}(z). \tag{11}$$

In the present application $U_\nu(w,z)$ is needed only with the second argument, z, equal to zero. In the Appendix, we prove the following result:

Theorem I For $(\alpha > -1)$, the ACVS is given by

$$s_\tau^{(\alpha)} = \frac{C_0^{(\alpha)}\,\Gamma(\alpha+1)}{2^\alpha (\pi\tau)^{\alpha+1}}\, U_{\alpha+1}(2\pi\tau, 0)\,(-1)^\tau. \tag{12}$$

Power series expansions and asymptotic expansions of $U_\nu(w,0)$ are available, and lead to corresponding expansions of $s_\tau^{(\alpha)}$. In particular, $U_\nu(w,0)$ has the series expansion ([9], p. 540)

$$U_\nu(w,0) = \sum_{m=0}^{\infty} \frac{(-1)^m\,(w/2)^{\nu+2m}}{\Gamma(\nu+2m+1)} \tag{13}$$

which is convergent for all finite w. A corresponding expansion of $s_\tau^{(\alpha)}$ follows:

$$s_\tau^{(\alpha)} = \frac{C_0^{(\alpha)}\,\Gamma(\alpha+1)}{2^\alpha} \left(\sum_{m=0}^{\infty} \frac{(-1)^m\,(\pi\tau)^{2m}}{\Gamma(2m+\alpha+2)} \right) (-1)^\tau. \tag{14}$$

Because of the factor of $(-1)^\tau$ this actually is an expansion of $(-1)^\tau\, s_\tau^{(\alpha)}$ rather than of $s_\tau^{(\alpha)}$ itself. An alternative series can be obtained by expanding the cosine factor in Eq. (3) and integrating term by term:

$$s_\tau^{(\alpha)} = \frac{C_0^{(\alpha)}}{2^\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m\,(\pi\tau)^{2m}}{(2m)!\,(2m+\alpha+1)}. \tag{15}$$

Either series may be used to generate values of $s_\tau^{(\alpha)}$ for small τ and, while individual terms differ, both series are found to sum to the same results. A more general series approach, appropriate also for larger values of τ, is described later.

The function $U_\nu(w,0)$ has an asymptotic expansion for large arguments ([9], p. 550)

$$U_\nu(w,0) \sim \cos(w/2 - \nu\pi/2) + \sum_{m=0}^{N} \frac{(-1)^m}{\Gamma(\nu - 1 - 2m)} (2/w)^{2m-\nu+2}. \tag{16}$$

Because this series is asymptotic rather than convergent, the upper summation limit N is not specified precisely; rather, terms should be included only as long as they continue to decrease in size. From Eqs. (12) and (16), a corresponding expansion of $s_\tau^{(\alpha)}$ follows

$$s_\tau^{(\alpha)} \sim -\frac{C_0^{(\alpha)}\,\Gamma(\alpha+1)}{2^\alpha} \left\{ \frac{\sin(\pi\alpha/2)}{(\pi\tau)^{\alpha+1}} + (-1)^{\tau+1} \sum_{m=0}^{N} \frac{(-1)^m}{\Gamma(\alpha - 2m)\,(\pi\tau)^{2m+2}} \right\}. \tag{17}$$

The leading term of this expansion is the same as deduced from Fourier theory in Eq. (4).

The power series and asymptotic expansion results in Eqs. (15) and (17) hold for all values of $\alpha > -1$ and can be used to generate numerical values of $s_\tau^{(\alpha)}$. The case $\alpha = 0$ (white noise) is most easily treated directly by performing the integration in Eq. (3) to obtain the expected result $s_\tau^{(0)} = \delta_{\tau,0}$. Numerical values of $s_\tau^{(\alpha)}$ for other values of α will be presented and discussed in a later section. Before considering the numerical aspects, we examine the non-stationary case $\alpha < -1$.

3 The Non-Stationary Case

When $\alpha < -1$, the singularity of the integrand in Eq. (3) is no longer weak, and the integral no longer exists in the ordinary sense. For $d = 1, 2, \ldots$, however, any PPL process with $-1-2d < \alpha \leq 1-2d$ can be thought of as a process whose d^{th} difference is stationary. Thus, the autocovariance of the d^{th} difference is well-defined. A general theory of random processes that have stationary d^{th} increments has been developed by Yaglom ([10]). Differencing can be considered a form of linear filtering in which the first difference filter has an impulse response given by $\{1, -1\}$, while the d^{th} difference filter is a cascade of d first difference filters. The SDF of the filtered process is the product of the SDF of the original process with the squared gain function of the overall filter. Thus, the SDF of the d^{th} difference, say Y_t, of the PPL process X_t is

$$S_Y(f) = C_d^{(\alpha)} \sin^{2d}(\pi f)\, |f|^\alpha, \quad |f| \le \tfrac{1}{2}, \quad -1 - 2d < \alpha < 1 - 2d. \tag{18}$$

Here, the multiplying constant $C_d^{(\alpha)}$ is related to the variance of the d^{th} difference of the original process.

At this point, we extend the meaning of the symbol $s_\tau^{(\alpha)}$ to the non-stationary case, where it will denote the autocovariance of the d^{th} difference of a PPL process, where $-1 - 2d < \alpha \le 1 - 2d$:

$$s_\tau^{(\alpha)} = 2C_d^{(\alpha)} \int_0^{1/2} \cos(2\pi f \tau)\, \sin^{2d}(\pi f)\, f^\alpha \, df. \tag{19}$$

The notation does not display the value of d explicitly; rather, d is understood to be the greatest integer satisfying $d \le -\alpha/2 + 1/2$. When $d = 0$, Eq. (19) reduces to the stationary case. For $d > 0$ the integral seems more complicated, but the two cases are simply related, as we will show presently.

We begin by deducing the leading terms of the asymptotic expansion of $s_\tau^{(\alpha)}$ at large τ by use of Lighthill's method ([7], Ch. 4). Apart from the constant $C_d^{(\alpha)}$, the integral in (19) can be interpreted as a Fourier transform of $\sin^{2d}(\pi f)\, |f|^\alpha\ H(f + 1/2)\ H(1/2 - f)$. This function, say $y(f)$, has jump discontinuities at $f = \pm 1/2$. Also, unless $\alpha = -2d$, it has at $f = 0$ either a weak singularity, or a jump in its first derivative. The asymptotic form of the ACVF at large τ is related to the behavior of the integrand near these discontinuities, and the following result is proved in the Appendix:

Theorem II At large $|\tau|$,

$$s_\tau^{(\alpha)} \sim C_d^{(\alpha)} \left\{ \frac{(-1)^{d+1}\Gamma(\alpha + 2d + 1)\sin(\pi\alpha/2)}{2^{\alpha+2d}\pi^{\alpha+1}|\tau|^{\alpha+2d+1}} \left[1 + \frac{d(\alpha + 2d + 1)(\alpha + 2d + 2)}{12|\tau|^2} \right] + \frac{(-1)^\tau \alpha}{2^\alpha \pi^2 |\tau|^2} \right\}. \tag{20}$$

We note that, for $d = 0$, the leading term is the same as for the stationary case, Eq. (4).

We now return to Eq. (19), and show how the expressions for $s_\tau^{(\alpha)}$ in the stationary and non-stationary cases are related. While our approach was developed independently, it has some points of contact with the work of Greenhall [5], who developed a theory of generalized autocovariances of continuous-time, non-stationary processes that have a stationary d^{th} difference. Here, we focus on the d^{th} difference itself, and work in the realm of discrete-time processes.

It is an elementary fact that sines and cosines are eigenfunctions of the second derivative operator, and an analogous result holds in the discrete case. Specifically, we let δ_τ^2 denote the second centered difference operator defined, for an arbitrary sequence $h(\tau)$, as:

$$\delta_\tau^2 \, h(\tau) \equiv h(\tau+1) - 2h(\tau) + h(\tau-1). \tag{21}$$

Then, it follows by use of addition formulas for the trigonometric functions or, more simply, by using complex exponentials, that

$$\delta_\tau^2 \cos(2\pi f \tau) = -4 \, \sin^2(\pi f) \, \cos(2\pi f \tau). \tag{22}$$

Thus, taking the second difference of a cosine (or sine) sequence returns the same sequence multiplied by $-4 \, \sin^2(\pi f)$. Consequently, Eq. (19) for the ACVF of the d^{th} difference of a PPL process may be written as

$$s_\tau^{(\alpha)} = \frac{(-1)^d \, C_d^{(\alpha)}}{2^{2d-1}} \int_0^{1/2} \delta_\tau^{2d} \, \cos(2\pi f \tau) \, f^\alpha \, df. \tag{23}$$

While this expression has been derived for the non-stationary PPL process, an analogous expression would hold for any other model of a non-stationary processes whose d^{th} difference is stationary. In all such cases, the effects of the d-stage difference filter may be replaced by the $2d$ order difference operator acting on the cosine function inside the integral representation of the autocovariance.

In the equation above, the difference operator must remain part of the integrand since, if the operations of differencing and integration were interchanged, the resulting integral would not exist in an ordinary sense. There is, however, a simple device that will allow the difference operator to be taken outside the integral without recourse to generalized function theory: Recall that each differencing of a polynomial lowers its degree by one, so δ_τ^{2d} annihilates any polynomial of degree less than $2d$. Thus, without changing the value of the integral, we may subtract from the integrand a polynomial $P(2\pi f\tau)$ of degree $< 2d$, and write

$$s_\tau^{(\alpha)} = \frac{(-1)^d\, C_d^{(\alpha)}}{2^{2d-1}} \int_0^{1/2} \delta_\tau^{2d} \left[\, \cos(2\pi f\tau) - P(2\pi f\tau) \,\right] f^\alpha \, df. \tag{24}$$

In particular, the polynomial may be chosen to cancel the first d terms of the Maclaurin expansion of the cosine factor, i.e.,

$$P(2\pi f\tau) = \sum_{m=0}^{d-1} \frac{(-1)^m\, (2\pi f\tau)^{2m}}{(2m)!}. \tag{25}$$

At this point the integrand, apart from the second difference operation, has only a weak singularity at $f = 0$, so the difference operation in Eq. (24) may be taken outside the integral, yielding

$$s_\tau^{(\alpha)} = \frac{(-1)^d\, C_d^{(\alpha)}}{2^{2d+\alpha}}\, \delta_\tau^{2d} \left\{ 2^{\alpha+1} \int_0^{1/2} \left[\, \cos(2\pi f\tau) - P(2\pi f\tau) \,\right] f^\alpha \, df \right\}. \tag{26}$$

Here, for later convenience, the factor involving a power of 2 has been split into separate factors inside and outside the difference operator. Also, it is advantageous to add another polynomial of order $2d - 2$, say $Q(\pi\tau)$, inside the brackets of Eq. (26) to obtain

$$s_\tau^{(\alpha)} = \frac{(-1)^d\, C_d^{(\alpha)}}{2^{2d+\alpha}}\, \delta_\tau^{2d}\, g_\tau^{(\alpha)}. \tag{27}$$

where

$$g_\tau^{(\alpha)} = 2^{\alpha+1} \int_0^{1/2} \left[\, \cos(2\pi f\tau) - P(2\pi f\tau) \,\right] f^\alpha \, df \; + \; Q(\pi\tau). \tag{28}$$

or, with the change of variable from f to $x = 2\pi f \tau$,

$$g_\tau^{(\alpha)} = \frac{1}{(\pi\tau)^{\alpha+1}} \int_0^{\pi\tau} \left[\, \cos(x) - P(x) \,\right] x^\alpha \, dx \; + \; Q(\pi\tau). \tag{29}$$

The polynomial Q, like P, is annihilated by δ_τ^{2d} so it contributes nothing to the ACVF, but Q can be chosen judiciously later to simplify the final expressions. We consider first $-1-2d < \alpha < 1-2d$, i.e., we exclude the limiting case $\alpha = 1-2d$. That particular case, and a few other special cases, will be treated in a later section. For the special case $\alpha = 1-2d$, we will take $Q = 0$ and obtain $g_\tau^{(\alpha)}$ directly from Eq. (28); for the other values of α considered here, a different choice will be shown to be more useful:

$$Q(\pi\tau) = \sum_{m=0}^{d-1} \frac{(-1)^m (\pi\tau)^{2m}}{(2m)!\,(2m+\alpha+1)}. \tag{30}$$

If the $s_\tau^{(\alpha)}$ are thought of as a family of functions of order $d = 1, 2, 3, \ldots$ then, from the viewpoint of special function theory, Eq. (27) may be thought of as a type of discrete Rodrigues' formula that generates the entire family by successive differencing of $g_\tau^{(\alpha)}$.

The asymptotic forms of $g_\tau^{(\alpha)}$ at large and small τ are derived in the Appendix, where we show Theorem III The function $g_\tau^{(\alpha)}$ has the series expansion

$$g_\tau^{(\alpha)} = \sum_{m=0}^{\infty} \frac{(-1)^m \, (\pi\tau)^{2m}}{(2m)!\,(2m+\alpha+1)}. \tag{31}$$

which is convergent for all finite τ. Also, for large τ, this function has the asymptotic expansion

$$g_\tau^{(\alpha)} \approx -\frac{\Gamma(\alpha+1)\sin(\pi\alpha/2)}{(\pi\tau)^{\alpha+1}} + \frac{\alpha\,(-1)^\tau}{(\pi\tau)^2}\left\{ 1 - \frac{(\alpha-1)(\alpha-2)}{(\pi\tau)^2} + \frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(\pi\tau)^4} - \cdots \right\}. \tag{32}$$

A comparison of Eqs. (31) and (15) reveals that the series expansion of $g_\tau^{(\alpha)}$ is formally identical to the expansion of $s_\tau^{(\alpha)}$ for the case $d = 0$, with the value of the constant $C_0^{(\alpha)}$ set to 2^α. Also, by the properties of the Gamma function, the products occurring in the large τ asymptotic expansion can be written as

$$\alpha\,(\alpha-1)\cdots(\alpha-2k)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-2k)}. \tag{33}$$

When this substitution is made in Eq. (32), and the result is compared to Eq. (17), it is seen that the asymptotic expansion of $g_\tau^{(\alpha)}$ is formally identical to the expansion of $s_\tau^{(\alpha)}$ for the case $d=0$, with the multiplying constant $C_0^{(\alpha)}$ set equal to 2^α. Thus, essentially the same function has to be evaluated no matter what value of d is of interest. Then, if $d>0$, a $(2d)^{th}$ order difference of the resulting function must be taken to obtain the final ACVS. As we show later, there may be a loss of significant figures in the evaluation of these differences and, for sufficiently large τ, it will be advantageous to evaluate $s_\tau^{(\alpha)}$ directly from the asymptotic expansion (20) rather than by differencing $g_\tau^{(\alpha)}$.

To summarize, we have shown that the ACVS of a stationary ($d=0$) or non-stationary ($d>0$) PPL process can be written as the $(2d)^{th}$ difference of a sequence $g_\tau^{(\alpha)}$. The sequence may be evaluated from convergent expansions at small τ or from asymptotic expansions at large τ. These expansions are formally equivalent to the expansions obtained via Lommel's U-function in Eq. (12), even though those initial expansions were derived rigorously only for the stationary case $d=0$.

While the small τ series expansion is "convergent" for all finite τ, it is not likely to be useful as written except for quite small values, say $\tau=1,2,3,4$, and perhaps 5. In the following section on numerical implementation we offer an alternative series-based method of evaluating $g_\tau^{(\alpha)}$ that will work also for larger values of τ. Of course, for sufficiently large τ the asymptotic series, Eq. (32), will be preferable.

4 Numerical Implementation

For $d = 0, 1, 2, \ldots$ and, correspondingly, $-1 - 2d < \alpha < 1 - 2d$, the ACVS $s_\tau^{(\alpha)}$ is expressed in terms of the sequence $g_\tau^{(\alpha)}$ as in Eqs. (27) and (28). The $g_\tau^{(\alpha)}$ are defined in Eq. (28) or, equivalently, by the power series and asymptotic series in Eqs. (31) and (32). In this section, we indicate some refinements to these representations that improve numerical accuracy, and we investigate possible loss of accuracy in the evaluation of the differences in Eq. (27).

We begin with the series representations at small τ. From Eqs. (28) and (30),

$$g_0^{(\alpha)} = 1/(\alpha + 1) \tag{34}$$

This value is correct except for the case $\alpha = -1$, which requires an alternative choice of the polynomial Q, as indicated in the previous section. From Eq. (31) with $\tau = 1$,

$$g_1^{(\alpha)} = \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m}}{(2m)!\,(2m + \alpha + 1)} \tag{35}$$

This series may be summed as written. For $-10 \le \alpha \le 1$, and α not an odd integer, the ratio of the n^{th} term to the first term is less than 10^{-15} for $n > 14$. Some minor improvement in convergence rate can be obtained by averaging successive (opposite sign) terms and summing the averages.

For $\tau > 1$, the series of Eq. (31) could still be used, but the convergence rate decreases with increasing τ so, e.g., when $\tau = 10$, the ratio of the n^{th} term to the first term is less than 10^{-15} only after 43 terms are included.

A modification to the series provides better convergence, and allows it to be used for larger values of τ. To obtain the modification, we write the integral in Eq. (29) as a sum of integrals over $(0, \pi), (\pi, 2\pi)$, etc, expand the first of these integrals and combine it with the Q-polynomial by using Eq. (30). The result, for $\tau > 1$, is

$$g_\tau^{(\alpha)} = \frac{1}{(\pi\tau)^{\alpha+1}} \left\{ I_1 + \sum_{k=2}^{\tau} I_k \right\} \tag{36}$$

Here,

$$I_1 = \pi^{\alpha+1}\, g_1^{(\alpha)} \tag{37}$$

and

$$I_k = \int_{(k-1)\pi}^{k\pi} \cos(x)\, x^\alpha\, dx \ , \quad k \geq 2 \tag{38}$$

In the Appendix, we show how the I_k can be evaluated:

Theorem IV

$$I_k = (-1)^{k+1} \sum_{m=1}^{\infty} f^{(2m-1)}(u_k)\, J_m \tag{39}$$

Here,

$$u_k = (k - 1/2)\, \pi \tag{40}$$

Also,

$$f(u) = (u_k + u)^\alpha \tag{41}$$

and $f^{(n)}(u_k)$ is the n^{th} derivative of $f(u)$ evaluated at $u = u_k$, i.e.,

$$f^{(n)}(u_k) = \alpha(\alpha - 1) \cdots (\alpha - n + 1)\, u_k^{\alpha - k} \tag{42}$$

Also, the J_m are the rescaled integrals

$$J_m = \frac{2}{(2m-1)!} \int_0^{\pi/2} u^{2m-1}\, \sin(u)\, du \tag{43}$$

An explicit expression is

$$J_m = 2(-1)^{m+1} \Bigg\{ 1 \ - \ \frac{1}{1 \cdot 2} \Big(\frac{\pi}{2}\Big)^2 \Bigg[1 - \frac{1}{3 \cdot 4} \Big(\frac{\pi}{2}\Big)^2 \Big(1 - \cdots \{ \ 1 \ - \frac{1}{(2m-3)(2m-2)} \Big(\frac{\pi}{2}\Big)^2 \} \cdots \Big) \Bigg] \Bigg\} \tag{44}$$

Table 1: Values of the First 10 Jm Integrals

m	Jm
1	2.00000000000e+000
2	4.67401100272e-001
3	3.99379155298e-002
4	1.78904599695e-003
5	4.94745527293e-005
6	9.29532016691e-007
7	1.26429389180e-008
8	1.30267352461e-010
9	1.05226938274e-012
10	6.66133814775e-015

Note that the J_m do not depend on d or on α so they may be evaluated once and for all. In practice, no more than ten terms are needed to obtain good accuracy from Eq. (39). For convenience, the first ten J_m, evaluated from Eq. (44), are provided in Table 1.

Once the J_m are available, the overall procedure is to evaluate the $g_\tau^{(\alpha)}$ from Eq. (36), with the I_m given by Eq. (39). This is done for each value of $\tau = 2, 3, \ldots$ up to some moderate value beyond which the asymptotic formula of Eq. (32) is used instead.

We now examine the large τ asymptotic formula. A slightly more compact version of Eq. (32) results by nesting the multiplications:

$$g_\tau^{(\alpha)} = \frac{\Gamma(\alpha+1)\cos(\pi(\alpha+1)/2)}{(\pi\tau)^{\alpha+1}} + \frac{\alpha\cos(\pi\tau)}{(\pi\tau)^2}\left\{1 - \frac{(\alpha-1)(\alpha-2)}{(\pi\tau)^2}\left[1 - \frac{(\alpha-3)(\alpha-4)}{(\pi\tau)^2}\left(1 - \cdots\right)\right]\right\} \tag{45}$$

Terms should be included only as long as they continue to decrease in size. A rule of thumb follows from Eq. (45). Terms through the factor $(\alpha - k)$ are included as long as

$$\left| \frac{(\alpha - k + 1)(\alpha - k)}{(\pi\tau)^2} \right| < 1 \tag{46}$$

After a bit of algebra, this is seen to be equivalent to

$$k \; < \; \alpha \; + \; 1/2 \; + \; \sqrt{(\pi\tau)^2 + 1/4} \tag{47}$$

This bound becomes more restrictive as α becomes more negative and, of course, it is most constraining when τ is small. At large τ, the bound is approximately

$$k \; < \; \pi\tau \; + \; \alpha \; + \; 1/2 \tag{48}$$

At this point, we have indicated how the $g_\tau^{(\alpha)}$ may be evaluated for both small and large τ. One point remains to be examined: the possible loss of significant figures in the evaluation of the differences in Eq. (27).

Empirically, we find that, while the asymptotic expansion of Eq. (32) gives accurate values of $g_\tau^{(\alpha)}$ when τ is large, some cancelation does occur in the evaluation of the differences in Eq. (27). These effects become more pronounced as α becomes more negative. The reasons for this, and the remedy, are not hard to find. We consider first the leading term in Eq. (32), and the impact of differencing on the factor that involves τ.

We have

$$\begin{aligned} \delta_\tau^2 \left\{ \frac{1}{(\pi\tau)^{\alpha+1}} \right\} \; &= \; \frac{1}{(\pi\tau)^{\alpha+1}} \left\{ \frac{1}{(1+1/\tau)^{\alpha+1}} - 2 + \frac{1}{(1-1/\tau)^{\alpha+1}} \right\} \\ &\approx \; \frac{1}{(\pi\tau)^{\alpha+1}} \left\{ \left(1 - \frac{(\alpha+1)}{\tau} + \frac{(\alpha+1)(\alpha+2)}{2\tau^2} - \cdots \right) \right. \\ &- \; \left. 2 + \left(1 + \frac{(\alpha+1)}{\tau} + \frac{(\alpha+1)(\alpha+2)}{2\tau^2} + \cdots \right) \right\} \end{aligned} \tag{49}$$

Note that, in the expansions, terms of the highest two orders cancel identically. The dominant surviving term gives

$$\delta_\tau^2 \left\{ \frac{1}{(\pi\tau)^{\alpha+1}} \right\} \approx (\alpha+1)(\alpha+2)\, \pi^2 \frac{1}{(\pi\tau)^{\alpha+3}} \tag{50}$$

The process can be repeated and, with each application of δ_τ^2, leading terms continue to cancel. This is the reason the observed numerical cancelations become more pronounced when higher order differences are required. After application of δ_τ^{2d}, the dominant surviving term is

$$\delta_\tau^{2d}\left\{\frac{1}{(\pi\tau)^{\alpha+1}}\right\} \approx \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+2d)\,\pi^{2d}}{(\pi\tau)^{\alpha+2d+1}} \tag{51}$$

Thus, the $(2d)^{th}$ difference of the first term in Eq. (32) is, approximately,

$$\delta_\tau^{2d}\left\{\frac{\Gamma(\alpha+1)\sin(\pi\alpha/2)}{(\pi\tau)^{\alpha+1}}\right\} \approx \frac{\Gamma(\alpha+2d+1)\sin(\pi\alpha/2)}{\pi^{\alpha+1}\tau^{\alpha+2d+1}} \tag{52}$$

When this result is used in Eqs. (32) and (27), it reproduces exactly the leading term of the asymptotic expansion of $s_\tau^{(\alpha)}$ given in Eq. (20).

We now consider the effects of differencing on the remaining terms in Eq. (32). We have

$$\begin{aligned}\delta_\tau^2\left\{\frac{\cos(\pi\tau)}{(\pi\tau)^2}\right\} &= \delta_\tau^2\left\{\frac{(-1)^\tau}{(\pi\tau)^2}\right\} \\ &\approx \frac{(-1)^{\tau+1}}{(\pi\tau)^2}\left\{1-2/\tau+\cdots+2+1+2/\tau+\cdots\right\}\end{aligned} \tag{53}$$

Note that the effects of cancelation are much less severe than for the first term of Eq. (32): Here, the leading terms do not cancel and the first cancelation is in the following term. Thus, the dominant surviving term is the leading term itself, and

$$\delta_\tau^2\left\{\frac{\cos(\pi\tau)}{(\pi\tau)^2}\right\} \approx -4\left\{\frac{\cos(\pi\tau)}{(\pi\tau)^2}\right\} \tag{54}$$

The same pattern is observed in differencing the higher order terms in Eq. (32). For example,

$$\delta_\tau^2\left\{\frac{\cos(\pi\tau)}{(\pi\tau)^4}\right\} \approx -4\left\{\frac{\cos(\pi\tau)}{(\pi\tau)^4}\right\} \tag{55}$$

Thus, by retaining the highest order terms at each level, we approximate the effects of differencing on the second part of the right hand side of Eq. (32) by

$$\begin{aligned}&\delta_\tau^{2d}\left\{\frac{\alpha\cos(\pi\tau)}{(\pi\tau)^2}\left[1-\frac{(\alpha-1)(\alpha-2)}{(\pi\tau)^2}(1-\cdots)\right]\right\} \approx \\ &(-4)^d\left\{\frac{\alpha\cos(\pi\tau)}{(\pi\tau)^2}\left[1-\frac{(\alpha-1)(\alpha-2)}{(\pi\tau)^2}(1-\cdots)\right]\right\}\end{aligned} \tag{56}$$

If the leading term of this expansion is used in Eqs. (32) and (27), it reproduces exactly the second term of the asymptotic expansion of $s_\tau^{(\alpha)}$ given in Eq. (20). Thus, differencing the asymptotic expansion of $g_\tau^{(\alpha)}$ reproduces the asymptotic expansion of $s_\tau^{(\alpha)}$, as it should. For numerical purposes, it will be more convenient to work directly with the expansion of $s_\tau^{(\alpha)}$ for large values of τ.

In summary, the overall algorithm for generating $s_\tau^{(\alpha)}$ is as follows: We obtain $g_0^{(\alpha)}$ and $g_1^{(\alpha)}$ from Eqs. (34) and (35). The next few $g_\tau^{(\alpha)}$ are obtained from Eq. (36). For larger τ, say $\tau > 15$, we obtain $g_\tau^{(\alpha)}$ from the asymptotic formula of Eq. (32). Then, the $s_\tau^{(\alpha)}$ follow from Eq. (27). However, effects of cancelations may be evident for sufficiently large τ, somewhere in the range 50 - 500, depending on the value of d. For those larger values of τ, values of $s_\tau^{(\alpha)}$ will be obtained directly from (20).

5 Particular Cases

For a few special values of α the function $s_\tau^{(\alpha)}$, or the related function $g_\tau^{(\alpha)}$, can be written in terms of functions that may be more familiar than those of Lommel. For the stationary cases $d = 0$, $\alpha = \pm 1/2$, as noted earlier, the ACVS can be written in terms of the Fresnel integrals C and S. We show here that relationship extends to the non-stationary cases $d \geq 1$, $\alpha = -2d \pm 1/2$, where $g_\tau^{(\alpha)}$ can be expressed in terms of C and S. Analogously, for the non-stationary cases $\alpha = -2d$ and $\alpha = -2d + 1$, the $g_\tau^{(\alpha)}$ can be expressed in terms of Sine integral and Cosine integral functions, Si and Ci. These expressions do provide a connection to familiar special functions. Also, with careful numerical implementation, they can provide test cases for the more general algorithm.

In the following, we make use of a recursion relation for the $g_\tau^{(\alpha)}$ that is derived in the

Appendix: Theorem V

$$g_\tau^{(\alpha)} = \frac{-(\pi\tau)^2}{(\alpha+1)(\alpha+2)}\, g_\tau^{(\alpha+2)} + \frac{1}{\alpha+1}\, \cos(\pi\tau) \tag{57}$$

From Eqs. (7) and (9), with $C_0^{(\alpha)}$ set to 2^α, we have

$$g_\tau^{(-1/2)} = 2\, C(\sqrt{2\tau})/\sqrt{2\tau} \tag{58}$$

$$g_\tau^{(1/2)} = \delta_{0,\tau} \; - \; \left(\frac{2}{\pi}\right) \frac{S(\sqrt{2\tau})}{(2\tau)^{3/2}} \tag{59}$$

Thus, from Eq. (57), we obtain

$$g_\tau^{(-3/2)} = 4(\pi\tau)^2 \left[\delta_{0,\tau} - \left(\frac{2}{\pi}\right) \frac{S(\sqrt{2\tau})}{(2\tau)^{3/2}} \right] - 2\, \cos(\pi\tau) \tag{60}$$

and

$$g_\tau^{(-5/2)} = -(2/3) \left[\pi^2\, (2\tau)^{3/2}\, C(\sqrt{2\tau}) \; + \; \cos(\pi\tau) \right] \tag{61}$$

This approach could be continued to generate expressions for $g_\tau^{(-7/2)}$, etc.

Another set of particular cases that can be related to tabulated functions are those for $\alpha = -2d$. For those cases, we return to Eq. (28). When $d = 1$ and $\alpha = -2$, we have

$$g_\tau^{(-2)} = \pi\tau \int_0^{\pi\tau} \left[\frac{\cos(x) - 1}{x^2} \right] dx - 1 \tag{62}$$

A single integration by parts leads to

$$g_\tau^{(-2)} = -\left[\, (\pi\tau)\, \mathrm{Si}(\pi\tau) \; + \; \cos(\pi\tau) \,\right] \tag{63}$$

Here, Si denotes the Sine integral (See [1], Ch. 5)

$$\mathrm{Si}(z) = \int_0^z \frac{\sin(x)}{x}\, dx \tag{64}$$

Similarly, for $d = 2$ and $\alpha = -4$, it follows from Eqs. (57) and (63) that

$$g_\tau^{(-4)} = \frac{(\pi\tau)^3}{6}\, \mathrm{Si}(\pi\tau) \; + \; \left[(\pi\tau)^2/6 - 1/3 \right] \cos(\pi\tau) \tag{65}$$

The corresponding expressions for $d = 3$ and $\alpha = -6$, etc, then could be obtained in the same way.

It also is possible to generate the ACVS for the cases $d > 0$ and $\alpha = -2d + 1$, but only by departing from the choice of Q used above. The problem for these cases is the presence of factors of $\alpha + 1$ in denominators of expressions such as Eq. (57), or in the $m = 0$ term of Eq. (31). However, Q can be chosen for convenience to be any polynomial of order less than $2d$; thus, for the cases $\alpha = -2d+1$, we choose $Q = 0$ and obtain $g_\tau^{(\alpha)}$ directly from Eq. (28). For $d = 1$ and $\alpha = -1$, this gives

$$g_\tau^{(-1)} = \int_0^{\pi\tau} \Big[\frac{\cos(x) - 1}{x} \Big] \, dx \tag{66}$$

The integral may be expressed in terms of the Cosine integral Ci (See, e.g. [1], Ch. 5)

$$\mathrm{Ci}(z) = \int_0^{z} \Big[\frac{\cos(x) - 1}{x} \Big] \, dx \; + \; \gamma \; + \; \ln(z) \tag{67}$$

(Here, γ is Euler's constant). This leads to

$$g_\tau^{(-1)} = \mathrm{Ci}(\pi\tau) - \gamma - \ln(\pi\tau) \tag{68}$$

Similarly, for the case $d = 2$ and $\alpha = -3$, Eq. (28) gives

$$g_\tau^{(-3)} = (\pi\tau)^2 \int_0^{\pi\tau} \Big[\frac{\cos(x) - 1 + x^2/2}{x^3} \Big] \, dx \tag{69}$$

After two integrations by parts, this leads to

$$g_\tau^{(-3)} = \frac{1}{2} \Big\{ 1 - \cos(\pi\tau) - (\pi\tau)^2 \Big[\mathrm{Ci}(\pi\tau) + 3/2 - \gamma - \ln(\pi\tau) \Big] \Big\} \tag{70}$$

The procedure could be continued to obtain $g_\tau^{(\alpha)}$ for $d = 3$ and $\alpha = -5$, etc. The corresponding $s_\tau^{(\alpha)}$ can be obtained by differencing, as in Eq. (27).

In each of the special cases presented in this section, closed forms are obtained for $s_\tau^{(\alpha)}$ or $g_\tau^{(\alpha)}$ in terms of known special functions. Thus, in principle, these expressions could be used to generate numerical values for these particular cases, which then could serve as test cases for the general algorithm. But we must emphasize that, even for these special cases, the expressions in the present section offer no advantages over the general algorithm. Except for the cases $\alpha = \pm 1/2$, if accurate values are to be obtained from these expressions more effort is required than just coding them as written, and inserting a call to an existing function routine to evaluate, e.g., $S(x)$ or Ci(x). As an example of the sort of pitfall that can occur, we note that, in Eq. (63), if the argument $\pi\tau$ is large, the leading term from the asymptotic expansion of $\mathrm{Si}(\pi\tau)$, when multiplied by $\pi\tau$, exactly cancels the term $\cos(\pi\tau)$ in the equation. Thus, if loss of accuracy is to be avoided for large argrments, a separate expansion needs to be worked out for the function $x\,\mathrm{Si}(x) + \cos(x)$ with the cancelation of leading terms done analytically. Similar concerns arise for other expressions in this section. Also, for all the cases with $d > 0$, the potential loss of accuracy from differencing via application of δ_τ^{2d} is as real here as when the general algorithm is used.

The general algorithm described in the previous section was implemented in the S-Plus programming language. As a test of the general algorithm, several of the particular cases noted here were run.

For $d = 0$ and $\alpha = \pm 1/2$ numerical values were obtained from Eqs. (7) and (9) by use of series and asymptotic expansions for the Fresnel integrals given in Ch. 7 of [1]. The two sets of results were compared for $0 \leq \tau \leq 100$, and were found to agree to at least six figures.

For $d = 1$ and $\alpha = -3/2$ numerical values were obtained from Eqs. (27) and (61). For $d = 1$ and $\alpha = -2$, Eq. (63) was used with values of the Sine integral generated as described in Ch. 5 of [1]. These results agreed with those obtained from the general algorithm.

A few cases with larger values of τ also were run without numerical difficulty, although the most interesting features of the ACVS are found at more modest values of τ.

6 Selected Results

We examine here the behavior of the ACVS for $d = 0, 1, 2, 3$, and for corresponding ranges of α that include both long memory cases $-1 - 2d < \alpha < -2d$, and cases deficient in low frequencies $-2d < \alpha < 1 - 2d$. Also, we consider different graphical formats that can best display their interesting features.

We begin with the stationary cases, for which $d = 0$, and look first at a case rich in long memory components, $\alpha = -0.99$.

Figure 1 shows a plot of the ACVS, $s_\tau^{(-0.99)}$, for $\tau = 0, 1, 2, \ldots, 100$. It is clear that the ACVS decays to zero rather slowly, but the simple form of the long memory component is not particularly obvious on the linear plot. When the same points are displayed on a log-log plot, as in Fig. 2, the long memory component appears as the linear trend seen at large τ, consistent with Eq. (4).

When analogous plots are generated for other values of α in the range $-1 < \alpha < 0$ they are found to be qualitatively similar to the two shown here. In contrast, when we consider positive values $0 < \alpha < 1$, the situation is different. These cases are deficient in low frequencies, and the ACVS exhibits anticorrelated, oscillating behavior. We consider here how this behavior can best be displayed.

Figure 3 shows the ACVS for $\alpha = +0.99$ displayed on a linear plot. Because the ordinates change rapidly, we have added a broken line to connect successive points. This line is intended only to guide the eye, and the ACVS is defined solely at the integer values of τ. Clearly, $s_\tau^{(0.99)}$ becomes negative, oscillates, and decays rapidly to zero. But, as in Fig. 1, the exact form of the large τ behavior is not evident. If α were negative, this behavior could be displayed on a log-log plot, as in Fig 2. But, for positive α, with some values of the ACVS negative, a log-log plot is no longer an option. A semilog plot provides a modicum of improvement, as seen in Figure 4. This view expands the negative ordinate part of the plot somewhat, but it still fails to display interesting features of the large τ behavior.

To cope with this situation, we need to devise a different type of plot, one that shares the log-log plots' ability to compress magnitudes, but one that also can tolerate ordinates that may be negative or zero. A non-traditional plot that accomplishes this is one we term the *asinh-log* plot, or simply the *AL* plot.

The AL plot makes use of the inverse hyperbolic sin function, $\operatorname{asinh}(x)$. Recall that $\operatorname{asinh}(x)$ is an odd function that is asymptotic to $\log(2x)$ for large positive x, and is asymptotic to x for small x. Thus, for some function $y(\tau)$, $\operatorname{asinh}(y(\tau))$ scales logarithmically when $y(\tau)$ is large and linearly when $y(\tau)$ is small. To obtain logarithmic scaling for a function such as $s_\tau^{(\alpha)}$, which never exceeds 1, we multiply by a scale factor that we denote by A and consider $\operatorname{asinh}(As_\tau^{(\alpha)})$. To obtain a convenient normalization, we divide this quantity by $\operatorname{asinh}(A)$. Thus, the AL plot of $s_\tau^{(\alpha)}$ is a plot of $\operatorname{asinh}(A\ s_\tau^{(\alpha)})\ /\ \operatorname{asinh}(A)$ versus $\log(\tau)$. The division by $\operatorname{asinh}(A)$ ensures that, when $s_\tau^{(\alpha)}$ is 1, the plotted ordinate also is equal to 1. The choice of scale factor A determines the degree to which the smaller ordinates in the tail of the AVCS will be expanded visually on the plot. Since the scaling is logarithmic, generally A will need to be fairly large, say $10^4 - 10^6$, to produce a significant visual impact.

Figure 5 shows an AL plot of the AVCS for $\alpha = +0.99$ with the scale factor A set to 10^5. This plot reveals a feature the former plots could not: that, on an AL plot, the ACVS exhibits a damped oscillatory behavior about a linear trend for intermediate τ. (The oscillation also is present in Eq. (20) as the term involving the factor $(-1)^\tau$.) Because of the large scale factor A, this trend occurs in the range where the asinh is logarithmic. At larger values of τ, the ordinates are small enough that the asinh departs from its asymptotic form, and the plotted values then depart from that trend. It follows from the large argument asymptotic form of the asinh function that, if the ACVS oscillates about a power law trend function τ^β then, on an AL plot, this trend appears as a straight line of slope $\beta/\log(2A)$, as in Fig. 5.

We next examine the case of white noise, $\alpha = 0$, and the two nearby cases, $\alpha = \pm 0.001$. These three cases are shown in the AL plot of Figure 6. For the white noise case, the ACVF is zero except at $\tau = 0$, as expected. Interestingly, the two nearby cases have, for $\tau > 0$, nearly mirror symmetry about the white noise case.

We now turn to the first nonstationary cases with $d = 1$. To provide a contrast to the cases shown in Fig. 6, we consider the nonstationary counterpart of white noise, namely $\alpha = -2$, plus the two nearby cases $\alpha = -1.9$ and $\alpha = -2.1$. These cases are shown in the AL plot of Figure 7.

The most striking aspect of this plot is the amount of structure seen in the ACVS for the nonstationary white noise case, $\alpha = -2$. This result is quite different from stationary white noise, $\alpha = 0$. We note, though, that for $\tau > 0$, the ACVS does oscillate about the stationary white noise result. The most likely reason for this is that $s_\tau^{(-2)}$ is obtained as a second difference of the sequence $g_\tau^{(-2)}$, so values of the ACVS for, say, τ and $\tau+1$ involve two of the same values of $g_\tau^{(-2)}$, but with opposite algebraic signs. Thus, the second difference operation introduces negative correlations between successive values of $s_\tau^{(-2)}$. While the ACVS for $\alpha = -2$ exhibits these large oscillations, it is interesting that the two "nearby" cases, $\alpha = -1.9, \; -2.1$, have oscillations that are relatively surpressed. In fact, it is not clear from the figure just what will happen when we consider cases that lie between, say, $\alpha = -2.0$ and $\alpha = -2.1$. This question can be answered by examining the AL plot for the case $\alpha = -2.01$ shown in Figure 8. Interestingly, for small τ, the ACVS looks much the same as for $\alpha = -2$. But, for $\tau > 20$, the ACVS makes a transition and begins to look much as it does for $\alpha = -2.1$. If plots are examined for values of α approaching more closely to -2, the transition is found to occur at larger and larger τ until, in the limit, no transition is seen.

We have not shown the ACVS for $\alpha = -1.99$ because it is completely analogous to the result for $\alpha = -2.01$, but with the transition occuring below the x-axis, so for large τ the ACVS begins to looks much like the $\alpha = -1.9$ case seen in Figure 7.

The results for nonstationary cases that involve larger values of d show some of the same types of features seen for the $d = 1$ case. As examples, we show non-stationary white noise cases for $d = 2$ ($\alpha = -4$) and $d = 3$ ($\alpha = -6$), plus nearby cases, in Figures 9 and 10, respectively.

The results for $d = 2$ are close to those shown for $d = 1$ in Fig. 7, although the numbers are not identical. The results for $d = 3$ are somewhat similar to those for both $d = 2$ and $d = 1$. The most notable difference is the delayed onset of oscillations for $s_\tau^{(-6)}$. We note that $s_2^{(-6)}$ is positive, while $s_2^{(-4)}$ and $s_2^{(-2)}$ both are negative.

7 Discussion

We have obtained explicit expressions for the ACVS of stationary and nonstationary PPL process, including both long memory cases, and cases deficient in low frequencies. Among our principal results are

- The ACVS of a stationary PPL process can be expressed in terms of Lommel's Bessel functions of two variables.
- The stationary AVCS can be evaluated from power series or asymptotic series obtained from Lommel's functions.
- The ACVS of the stationary d^{th} difference of a nonstationary PPL processes can be expressed as the $(2d)^{th}$ difference of a related sequence $g_\tau^{(\alpha)}$.
- The sequence $g_\tau^{(\alpha)}$ can be evaluated from essentially the same power series and asymptotic series that apply in the stationary case.

- A comprehensive numerical algorithm exists for evaluating the ACVS for both stationary and non-stationary cases. This scheme is based on refined versions of the power series and asymptotic series, and includes a method to cope with the potential loss of accuracy in the evaluation of the $(2d)^{th}$ differences.

- The detailed structure present in the tail of the ACVS for many cases can be revealed by use of a novel display, the scaled asinh-log plot.

- Intriguing features of the ACVS are presented for a number of cases.

Acknowledgement

We gratefully acknowledge partial support of this work from ONR Grant No. N00014–03–1–0614 (University of Washington) and NSF Grant No. DMI–9801614 (Insightful).

8 Appendix

To prove Theorem I, we need to show that $s_\tau^{(\alpha)}$ in Eq. (3) can be written in terms of Lommel's functions. To begin, we make the change of variable $f = x^2/2$ to obtain

$$s_\tau^{(\alpha)} = C_0^{(\alpha)}\, 2^{1-\alpha} \int_0^1 x^{2\alpha+1} \cos(\pi\tau x^2)\, dx. \tag{71}$$

From Watson ([9], p. 544) we have the general result, valid for $\nu > 0$,

$$\begin{aligned} \frac{w^\nu}{2^{\nu-1}\Gamma(\nu)} \int_0^1 x^{2\nu-1} \cos(wx^2/2)\, dx &= U_\nu(w,0)\, \cos(w/2) \\ &\quad + U_{\nu+1}(w,0)\, \sin(w/2). \end{aligned} \tag{72}$$

Here, $U_\nu(w,z)$ denotes Lommel's U-function. In the present application $U_\nu(w,z)$ is needed only with the second argument, z, equal to zero. Because $U_\nu(w,0)$ is bounded at $w=0$, the second term in Eq. (72) does not contribute when $w = 2\pi\tau$. With $\nu = \alpha+1$, and we obtain Eq. (12). Q.E.D.

To prove Theorem II, we show the asymptotic expansion of $s_\tau^{(\alpha)}$ is given by Eq. (20). As noted, the integral in (19) can be interpreted as a Fourier transform of $\sin^{2d}(\pi f)\,|f|^\alpha\, H(f+1/2)\, H(1/2-f)$. This function, say $y(f)$, has jump discontinuities at $f = \pm 1/2$. Also, unless $\alpha = -2d$, it has at $f=0$ either a weak singularity, or a jump in its first derivative. The asymptotic form of the ACVF at large τ is related to the behavior of the integrand near these discontinuities, and the leading term can be obtained easily. Near $f=0$,

$$y(f) \sim \pi^{2d}|f|^{\alpha+2d}\,\{1 - \mathrm{O}(f^2)\}. \tag{73}$$

Near $f = 1/2$,

$$y(f) \sim (1/2)^\alpha\, H(1/2-f)\,\{1 - 2\alpha\,(1/2-f) + \mathrm{O}((1/2-f)^2)\}. \tag{74}$$

And, near $f=-1/2$,

$$y(f) \sim (1/2)^\alpha\, H(f+1/2)\,\{1 - 2\alpha\,(f+1/2) + \mathrm{O}((f+1/2)^2)\}. \tag{75}$$

Thus, from Lighthill [7] (Theorem 19 and Table 1) Eq. (20) follows. Q.E.D.

To prove Theorem III, we need to derive the asymptotic expansion and series expansion of $s_\tau^{(\alpha)}$, as given in Eqs. (32) and (31). We first investigate $g_\tau^{(\alpha)}$ for large values of τ and, with this in mind, rewrite the integral in Eq. (29) as

$$\begin{aligned} g_\tau^{(\alpha)} \;=\; & \frac{1}{(\pi\tau)^{\alpha+1}}\left\{\int_0^\infty \big[\,\cos(x) - P(x)\,\big]\, x^\alpha\, dx \right. \\ & \left. - \int_{\pi\tau}^\infty \cos(x)\, x^\alpha\, dx + \int_{\pi\tau}^\infty P(x)\, x^\alpha\, dx \right\} \;+\; Q(\pi\tau). \end{aligned} \tag{76}$$

As noted, we consider here $-1-2d < \alpha < 1-2d$, i.e., we exclude the limiting case $\alpha = 1-2d$ that is treated separately. It then follows that all three integrals in (76) exist. At this point, and for increased simplicity, we choose Q to cancel the third integral, i.e.,

$$\begin{aligned} Q(\pi\tau) &= -\frac{1}{(\pi\tau)^{\alpha+1}} \int_{\pi\tau}^{\infty} P(x)\, x^{\alpha}\, dx \\ &= -\sum_{m=0}^{d-1} \frac{(-1)^m}{(2m)!\,(\pi\tau)^{\alpha+1}} \int_{\pi\tau}^{\infty} x^{2m+\alpha}\, dx \\ &= \sum_{m=0}^{d-1} \frac{(-1)^m (\pi\tau)^{2m}}{(2m)!\,(2m+\alpha+1)}. \end{aligned} \tag{77}$$

We next evaluate the first integral in Eq. (76) for various values of d. For the lowest order case, $d = 1$ and $-3 < \alpha < -1$, the integral is

$$I_0^{(\alpha)} = \int_0^{\infty} \left[\, \cos(x) - 1 \,\right] x^{\alpha}\, dx. \tag{78}$$

A single integration by parts results in

$$I_0^{(\alpha)} = \frac{1}{\alpha+1} \int_0^{\infty} \sin(x)\, x^{\alpha+1}\, dx. \tag{79}$$

This integral can be found in integral transform tables (e.g. [3]), either as a Fourier sine transform or as a Mellin transform. With either of these interpretations we find

$$I_0^{(\alpha)} = -\Gamma(\alpha+1)\, \sin(\pi\alpha/2). \tag{80}$$

For $\alpha = -2$, the integral in (79) is known and yields $I_0^{(-2)} = -\pi/2$. The same result follows from (80) by a limiting procedure.

The case $d = 2$, $-5 < \alpha < -3$, is done similarly. The integral then is

$$I_0^{(\alpha)} = \int_0^{\infty} \left[\, \cos(x) - 1 + x^2/2 \,\right] x^{\alpha}\, dx. \tag{81}$$

Two integrations by parts convert this to

$$I_0^{(\alpha)} = -\frac{1}{(\alpha+1)(\alpha+2)} \int_0^{\infty} \left[\, \cos(x) - 1 \,\right] x^{\alpha+2}\, dx. \tag{82}$$

The remaining integral is the same form as in the case $d = 1$, from which it follows that

$$I_0^{(\alpha)} = \frac{\Gamma(\alpha+3)}{(\alpha+1)(\alpha+2)} \sin(\pi(\alpha+2)/2) = -\Gamma(\alpha+1)\sin(\pi\alpha/2). \tag{83}$$

This expression is formally identical to the one obtained for the case $d = 1$. The cases for $d = 3, 4, \ldots$ can be obtained recursively. The integral for general d is

$$I_0^{(\alpha)} = \int_0^\infty \left[\, \cos(x) - P(x) \,\right] x^\alpha \, dx. \tag{84}$$

Two integrations by parts yield

$$I_0^{(\alpha)} = -\frac{1}{(\alpha+1)(\alpha+2)} \int_0^\infty \left[\, \cos(x) - P''(x) \,\right] x^{\alpha+2} \, dx. \tag{85}$$

The remaining integral is the same form as in Eq. (84), so

$$I_0^{(\alpha)} = -\frac{1}{(\alpha+1)(\alpha+2)} \, I_0^{(\alpha+2)}. \tag{86}$$

It follows by recursion, and the properties of the Gamma function, that Eq. (83) is valid for all $d = 1, 2, 3, \ldots$. Thus, interestingly, the same expression holds for all these cases even though polynomials of different degree occur in the integrand for different values of d and corresponding ranges of α. This phenomenon can be understood most easily from the viewpoint of the theory of distributions, in which all these integrals may be regarded as the Hadamard finite part of one singular integral (see, e.g. [7], Table I).

The case $\alpha = -2d$ requires some care, since the product $\Gamma(\alpha+1)\,\sin(\pi\alpha/2)$ is indeterminate. The Gamma function has a simple pole at each negative integer $z = -n$, with residue $(-1)^n/n!$ (See, e.g., [1], p. 255.) Thus, by use of the Taylor series expansion of the sine, it follows that the limit is finite:

$$I_0^{(-2d)} = \lim_{\alpha \to -2d} \{\, \Gamma(\alpha+1)\, \sin(\pi\alpha/2) \,\} = \frac{(-1)^d \pi}{2\,(2d-1)!}. \tag{87}$$

The case $d = 1$ was noted earlier, and is consistent with this more general result.

Finally, we derive an asymptotic expansion for the second integral in Eq. (76), namely

$$\int_{\pi\tau}^\infty \cos(x)\, x^\alpha \, dx \equiv I_\tau^{(\alpha)}. \tag{88}$$

Two integrations by parts show that

$$I_\tau^{(\alpha)} = -\alpha\,(\pi\tau)^{\alpha-1}\,(-1)^\tau - \alpha\,(\alpha-1)\,I_\tau^{(\alpha-2)}, \tag{89}$$

whence, by iteration,

$$I_\tau^{(\alpha)} = -\alpha\,(\pi\tau)^{\alpha-1}\,(-1)^\tau \Bigg\{ 1 \;-\; \frac{(\alpha-1)(\alpha-2)}{(\pi\tau)^2} \;+\; \frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(\pi\tau)^4} - \cdots \Bigg\}. \tag{90}$$

By using the expressions in Eqs. (83) and (90) in Eq. (76), we obtain the asymptotic expansion of $g_\tau^{(\alpha)}$ as given in Eq. (eq:gbig). This proves the asymptotic expansion part of the Theorem.

We now turn to the case of small τ. We have, from Eqs. (29) and (30), and for $d = 1, 2, \ldots$

$$g_\tau^{(\alpha)} = \frac{1}{(\pi\tau)^{\alpha+1}} \int_0^{\pi\tau} \big[\,\cos(x) - P(x)\,\big]\, x^\alpha\, dx \;+\; \sum_{m=0}^{d-1} \frac{(-1)^m\,(\pi\tau)^{2m}}{(2m)!\,(2m+\alpha+1)}. \tag{91}$$

(In order that the small τ series represent the same function as the large τ asymptotic expansion, we require Q to be the same polynomial in both cases, even though Q does not contribute to the $(2d)^{th}$ difference.) In the limit as $\tau \to 0$, the term involving the integral is zero, and it follows that

$$g_0^{(\alpha)} = 1/(\alpha+1). \tag{92}$$

For $\tau = 1, 2, \ldots$ the integrand in Eq. (91) can be expanded and integrated term by term to yield

$$g_\tau^{(\alpha)} = \sum_{m=d}^{\infty} \frac{(-1)^m\,(\pi\tau)^{2m}}{(2m)!\,(2m+\alpha+1)} \;+\; \sum_{m=0}^{d-1} \frac{(-1)^m\,(\pi\tau)^{2m}}{(2m)!\,(2m+\alpha+1)}. \tag{93}$$

Note that the second sum, which comes from the polynomial Q, supplies the terms that are missing at the beginning of the first sum so the two sums can be combined to obtain Eq. (31). Q.E.D.

To prove Theorem IV we begin, as noted, by we write the integral in Eq. (91) as a sum of integrals over $(0, \pi), (\pi, 2\pi)$, etc, expand the first of these integrals and combine it with the existing Q-sum in Eq. (91). The result, for $\tau > 1$, is

$$g_\tau^{(\alpha)} = \frac{1}{(\pi\tau)^{\alpha+1}} \left\{ I_1 + \sum_{k=2}^{\tau} I_k \right\} \tag{94}$$

Here,

$$I_1 = \pi^{\alpha+1} \, g_1^{(\alpha)} \tag{95}$$

and

$$I_k = \int_{(k-1)\pi}^{k\pi} \cos(x) \, x^\alpha \, dx \ , \quad k \geq 2 \tag{96}$$

In the integrals I_k, the range of integration may be shifted back to $(-\pi/2, \pi/2)$ to yield

$$I_k = (-1)^{k+1} \int_{-\pi/2}^{\pi/2} \sin(u) \, (u_k + u)^\alpha \, du \tag{97}$$

Here,

$$u_k = (k - 1/2) \, \pi \tag{98}$$

We expand the second factor in the integrand as

$$f(u) \equiv (u_k + u)^\alpha \approx u_k^\alpha \ + \ f'(u_k) \, u \ + \ \frac{f''(u_k)}{2!} \, u^2 \ + \cdots \tag{99}$$

where

$$f^{(n)}(u_k) = \alpha(\alpha - 1) \cdots (\alpha - n + 1) \, u_k^{\alpha - k} \tag{100}$$

Since the sine factor is odd, only the odd order terms contribute to the integral. Thus, we have

$$I_k = (-1)^{k+1} \sum_{m=1}^{\infty} f^{(2m-1)}(u_k) \, J_m \tag{101}$$

Here, the J_m are the rescaled integrals

$$J_m = \frac{2}{(2m-1)!} \int_0^{\pi/2} u^{2m-1} \sin(u)\, du \tag{102}$$

In particular,

$$J_1 \;=\; 2 \int_0^{\pi/2} u \sin(u)\, du \;=\; 2 \tag{103}$$

and, by reference to tables, or after two integrations by parts,

$$J_m \;=\; \frac{2\,(\pi/2)^{2m-2}}{(2m-2)!} \;-\; J_{m-1} \tag{104}$$

Thus, e.g.,

$$J_2 = (1/4)\pi^2 - 2 \;\approx 0.46740011 \tag{105}$$

$$J_3 = (1/220)[(5/8)\pi^4 - 30\pi^2 + 240] \;\approx\; 0.03993792 \tag{106}$$

and additional J_m may be evaluated recursively. Alternatively, the expression Eq. (104) may be iterated, yielding

$$J_m = \frac{2}{(2m-1)!}\Big\{ (2m-1)\,(\pi/2)^{2m-2} \;-\; (2m-1)(2m-2)(2m-3)\,(\pi/2)^{2m-4} + \cdots \pm (2m-1)!\,(\pi/2)^0 \Big\} \tag{107}$$

This result may be reversed to obtain the more stable nested version shown in Eq. (44). Q.E.D.

To prove Theorem V, we first derive a general expression for $I_\tau^{(\alpha)}$ in terms of $I_\tau^{(\alpha+2)}$. By solving Eq. (89) for $I_\tau^{(\alpha-2)}$, and replacing α by $\alpha+2$, we obtain

$$I_\tau^{(\alpha)} = \frac{(-1)}{(\alpha+1)(\alpha+2)}\left\{ I_\tau^{(\alpha+2)} \;+\; (\alpha+2)\,(\pi\tau)^{(\alpha+1)}\cos(\pi\tau) \right\} \tag{108}$$

From Eqs. (76) and (88), we have

$$g_\tau^{(\alpha)} = \frac{1}{(\pi\tau)^{\alpha+1}}\Bigg\{\Gamma(\alpha+1)\,\cos(\pi(\alpha+1)/2) + \frac{1}{(\alpha+1)(\alpha+2)}\left[\,I_\tau^{(\alpha+2)} + (\alpha+2)(\pi\tau)^{\alpha+1}\cos(\pi\tau)\,\right]\Bigg\} \tag{109}$$

After minor algebra it follows that

$$g_\tau^{(\alpha)} = \frac{-(\pi\tau)^2}{(\alpha+1)(\alpha+2)}\,g_\tau^{(\alpha+2)} + \frac{1}{\alpha+1}\,\cos(\pi\tau) \tag{110}$$

Q.E.D.

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Figure Captions

Figure 1. ACVS for Stationary Case $\alpha = -0.99$ (linear plot)

Figure 2. ACVS for Stationary Case $\alpha = -0.99$ (log-log plot)

Figure 3. ACVS for Stationary Case $\alpha = +0.99$ (linear plot)

Figure 4. ACVS for Stationary Case $\alpha = +0.99$ (semilog plot)

Figure 5. ACVS for Stationary Case $\alpha = +0.99$ (AL plot, $A = 10^5$)

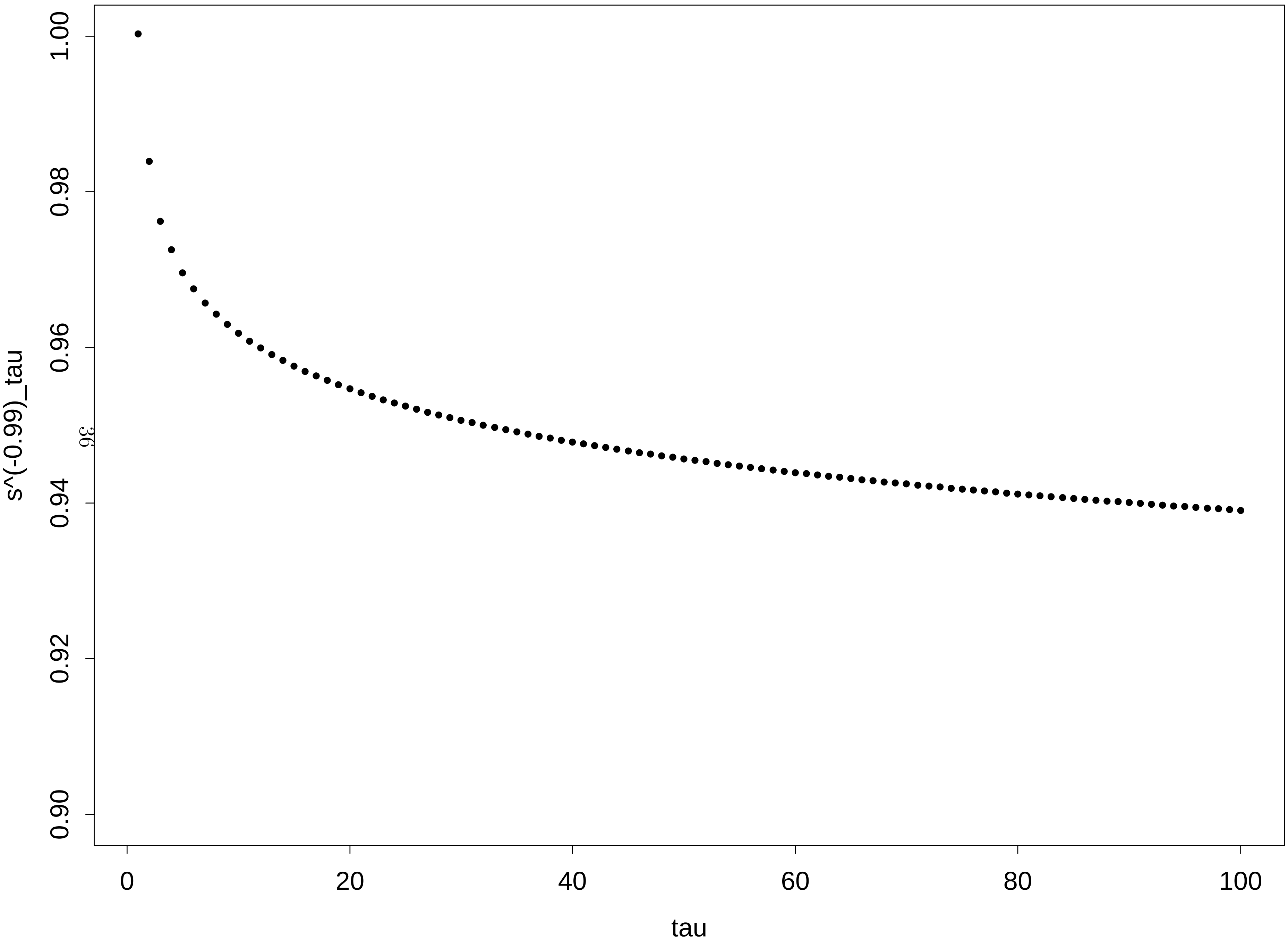
Figure 6. ACVS for Stationary Cases $\alpha = 0$ and $\alpha = \pm 0.001$ (AL plot, $A = 10^5$)

Figure 7. ACVS for Nonstationary Cases $\alpha = -1.9, -2.0, -2.1$ (AL plot, $A = 10^5$)

Figure 8. ACVS for Nonstationary Case $\alpha = -2.01$ (AL plot, $A = 10^5$)

Figure 9. ACVS for Nonstationary Cases $\alpha = -3.9, -4.0, -4.1$ (AL plot, $A = 10^5$)

Figure 10. ACVS for Nonstationary Cases $\alpha = -5.9, -6.0, -6.1$ (AL plot, $A = 10^5$)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0.94
0.92
0.90
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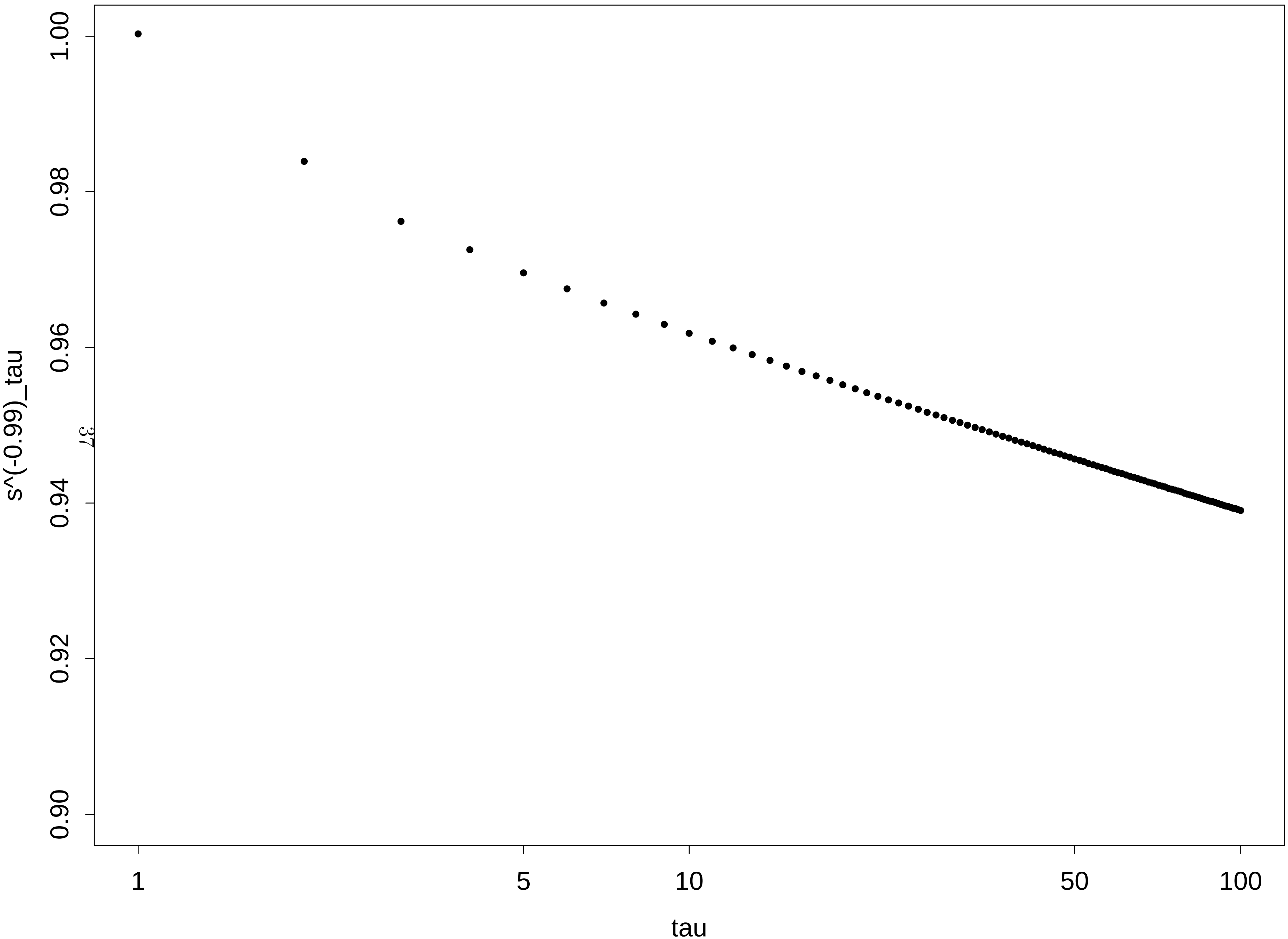
40
60
80
100
tau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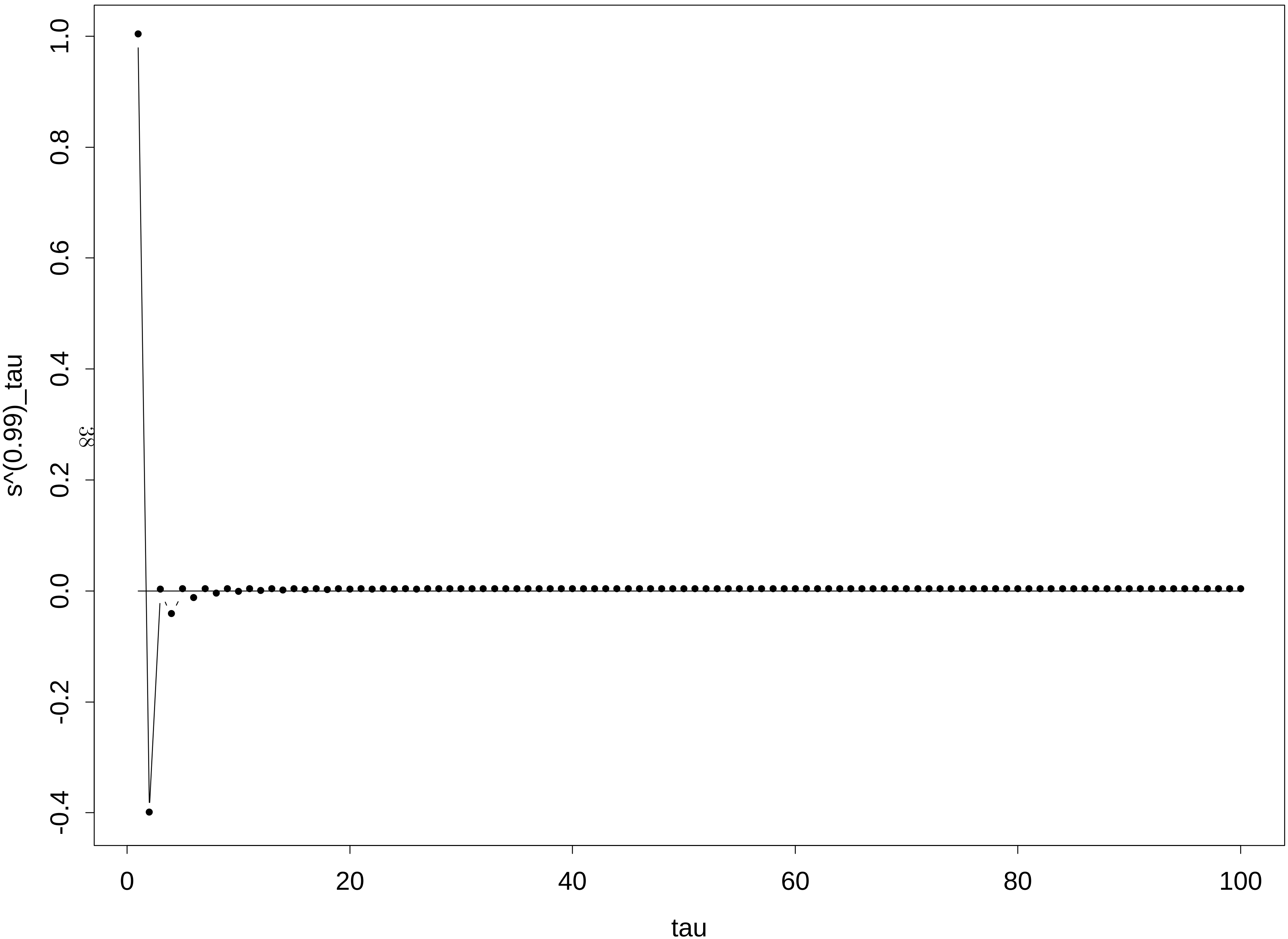
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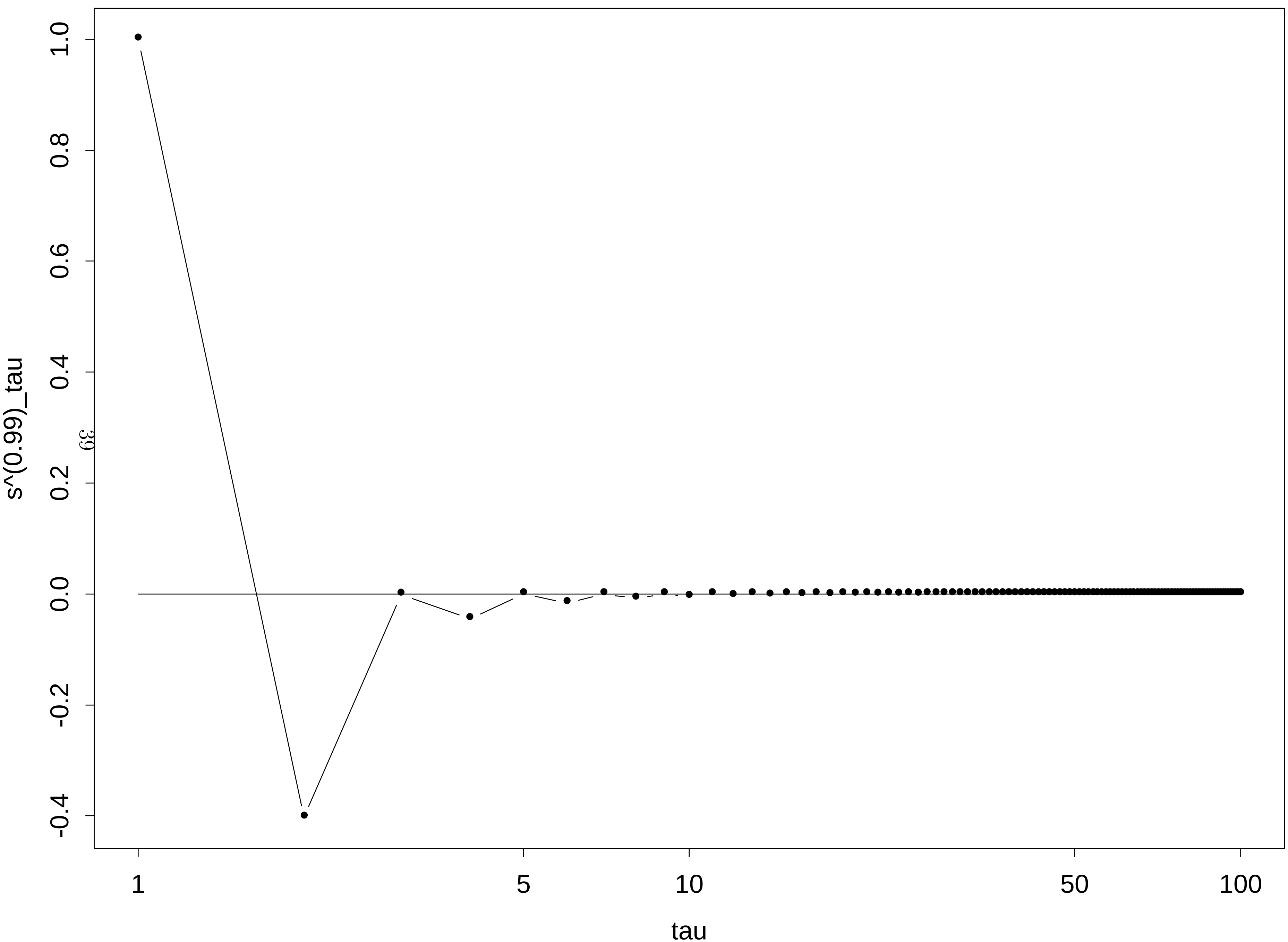
80
100
tau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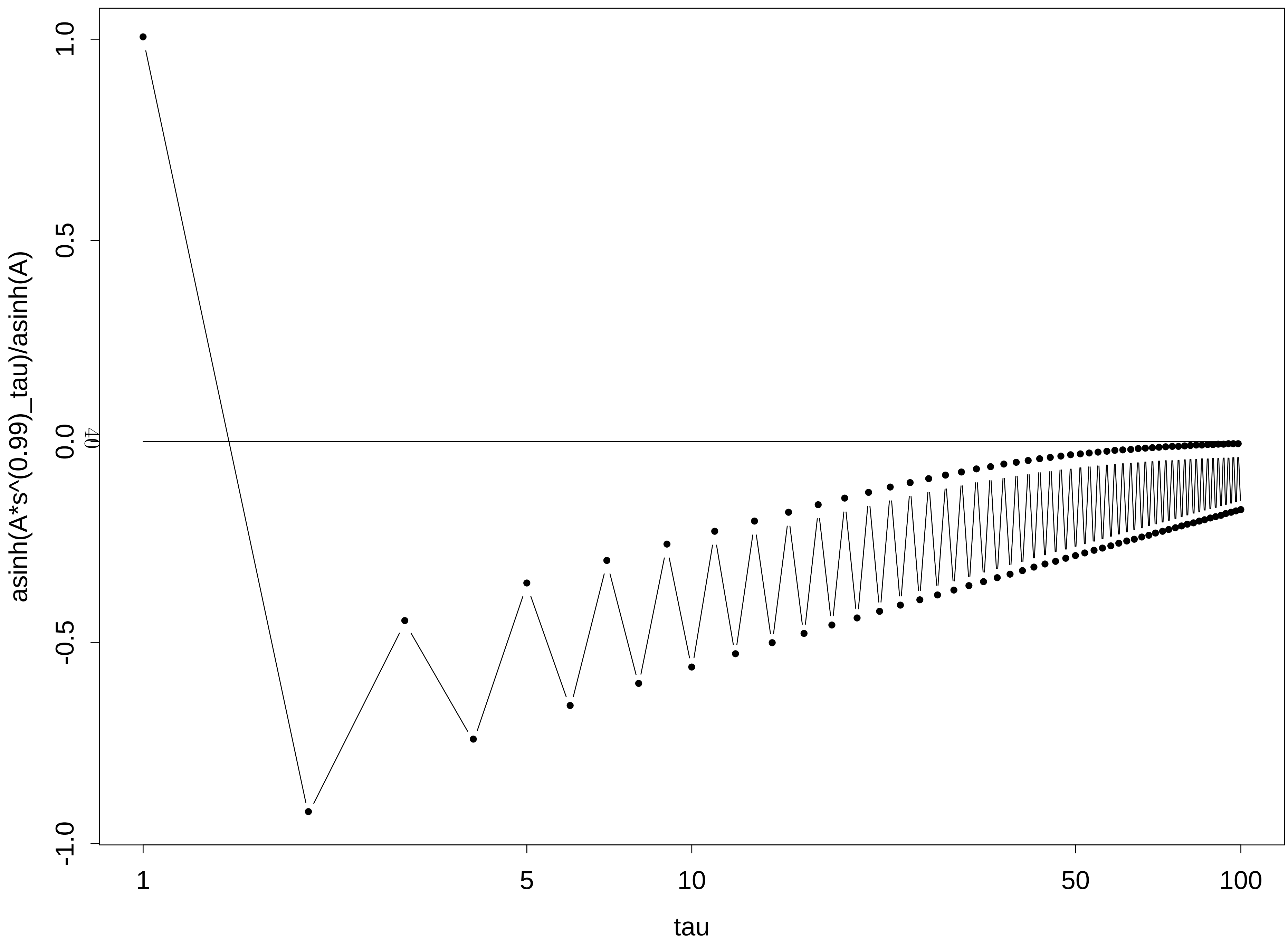
100
tau



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1
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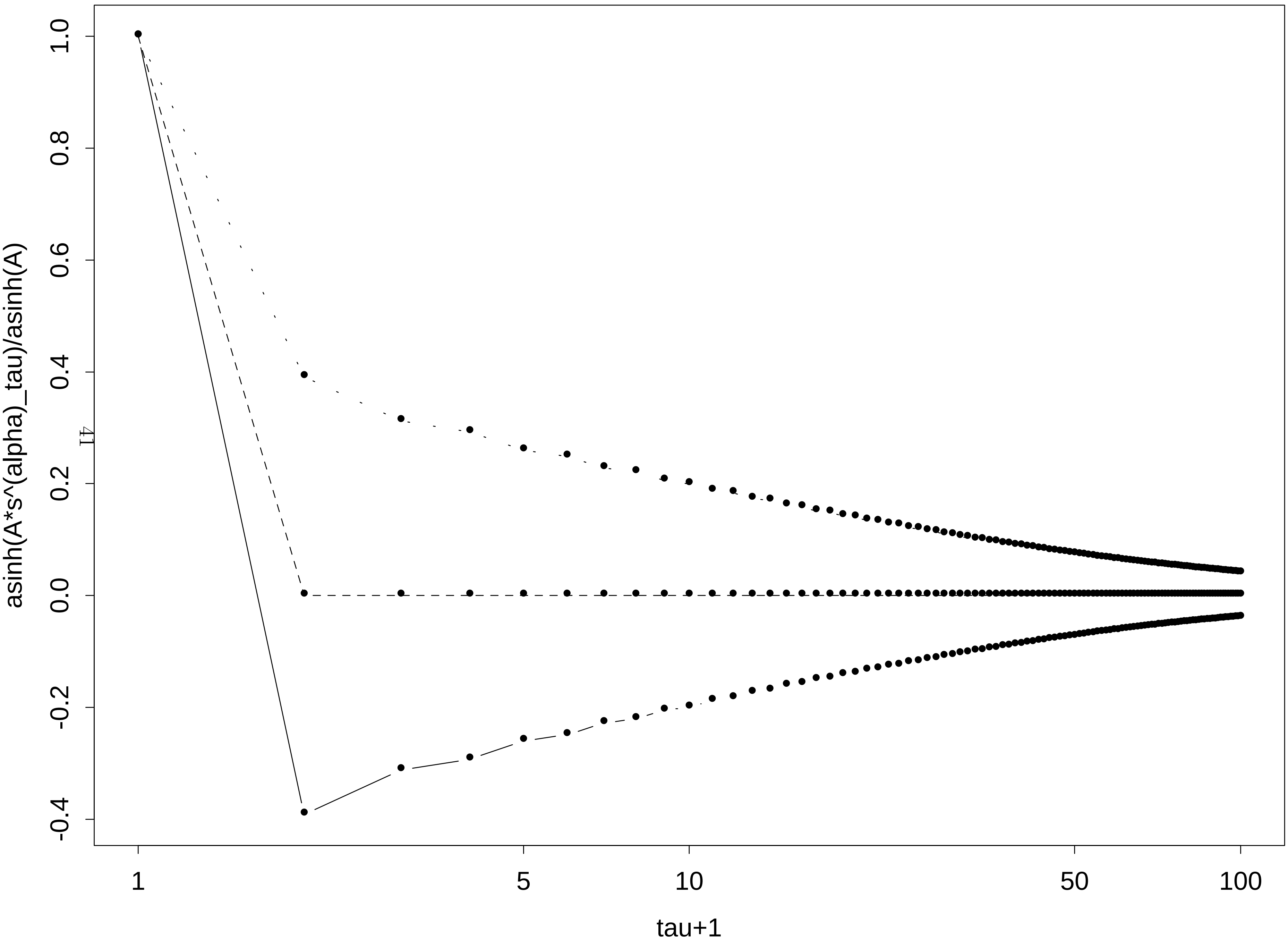
10
50
100
tau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0.4
0.2
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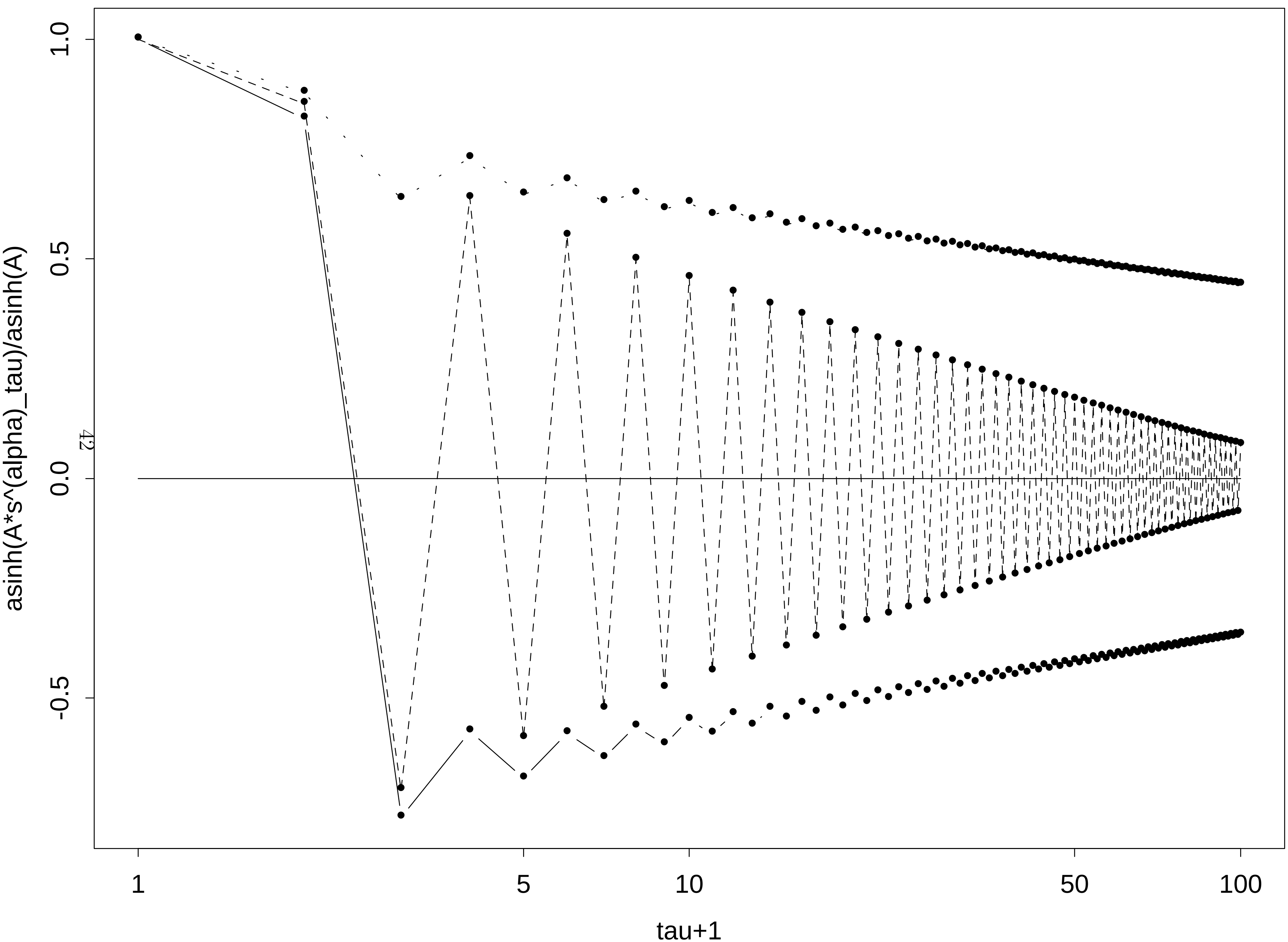
-0.4
1
5
10
50
100
tau+1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10
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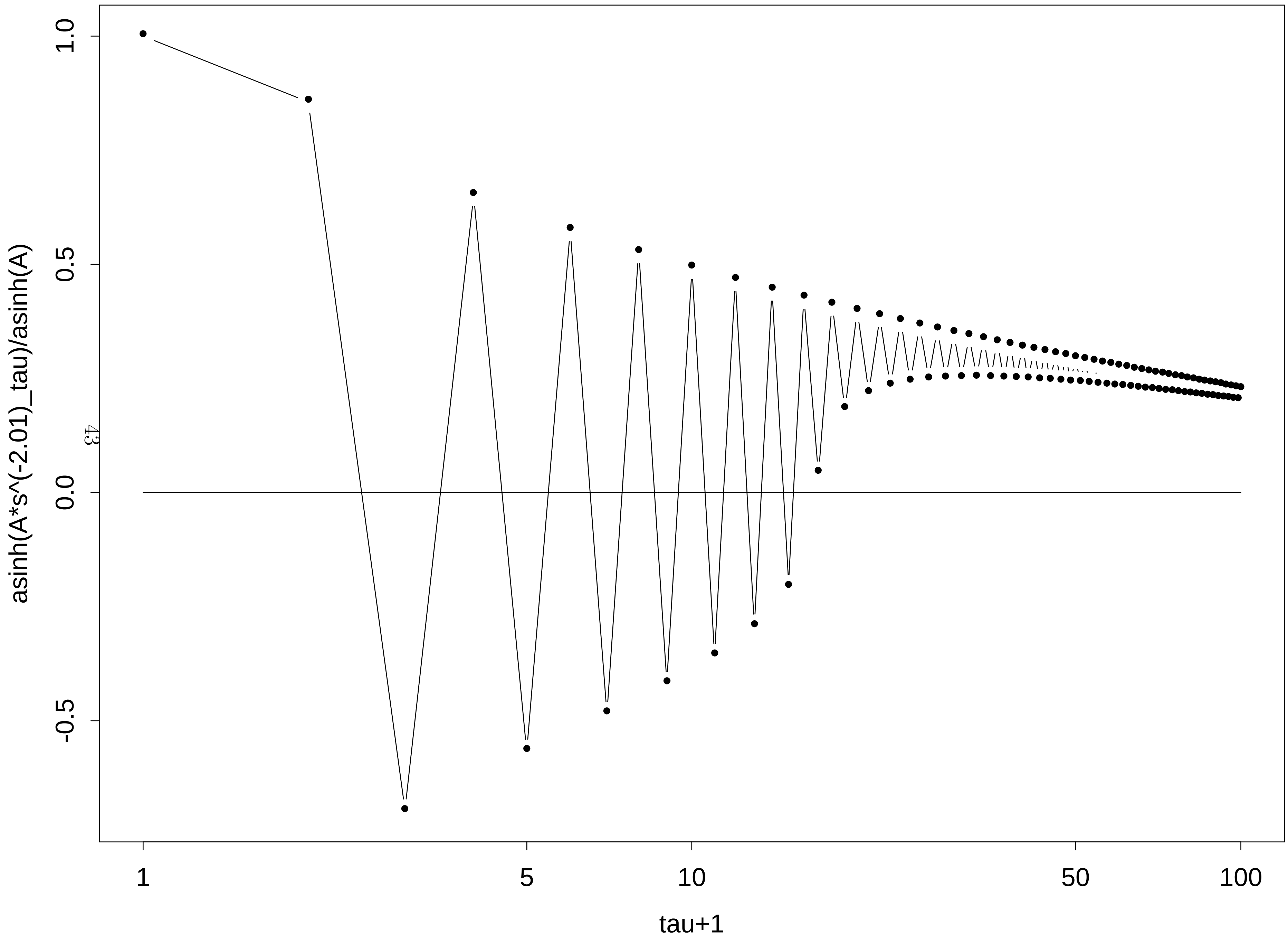
100
tau+1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100
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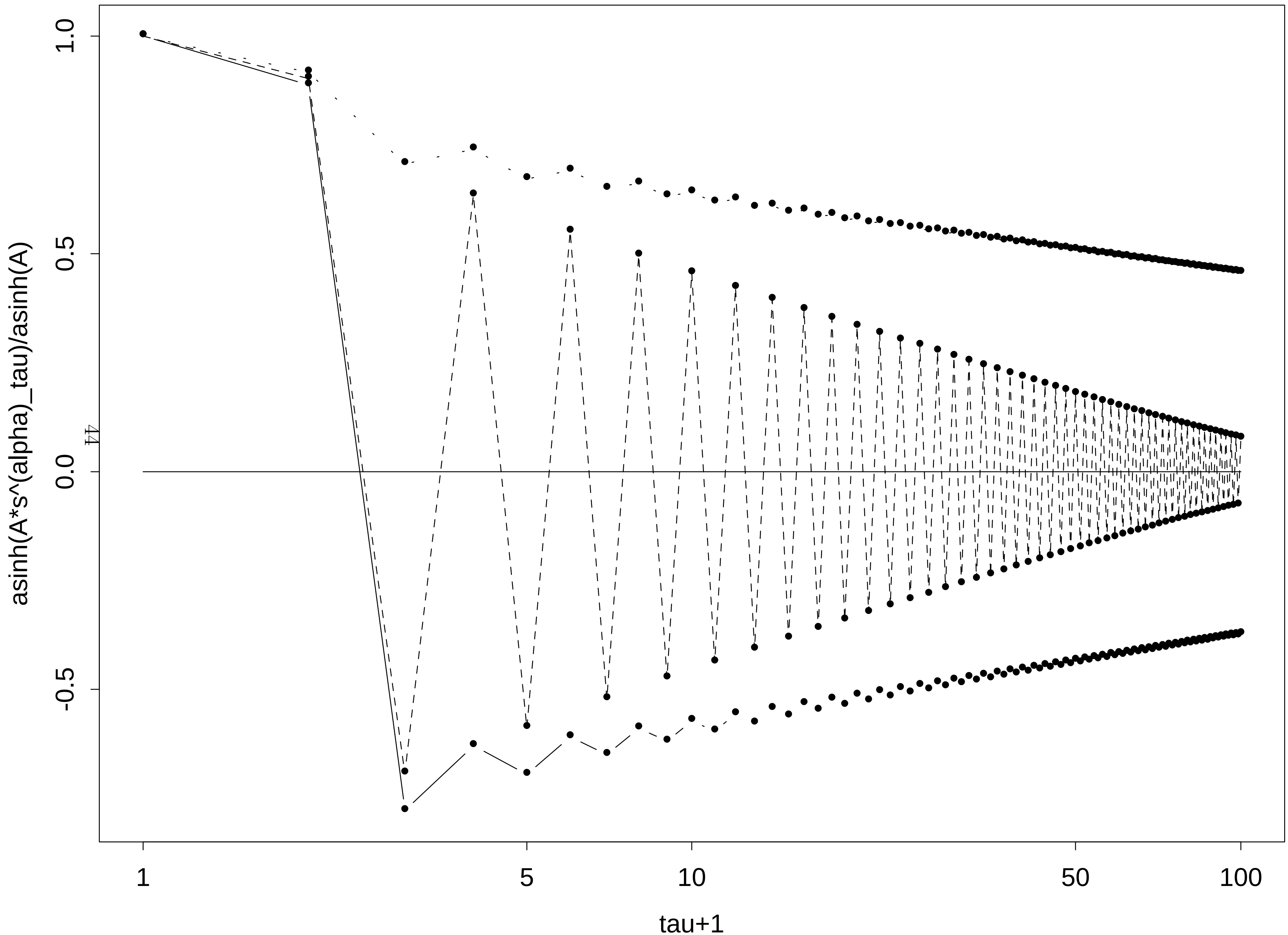
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10
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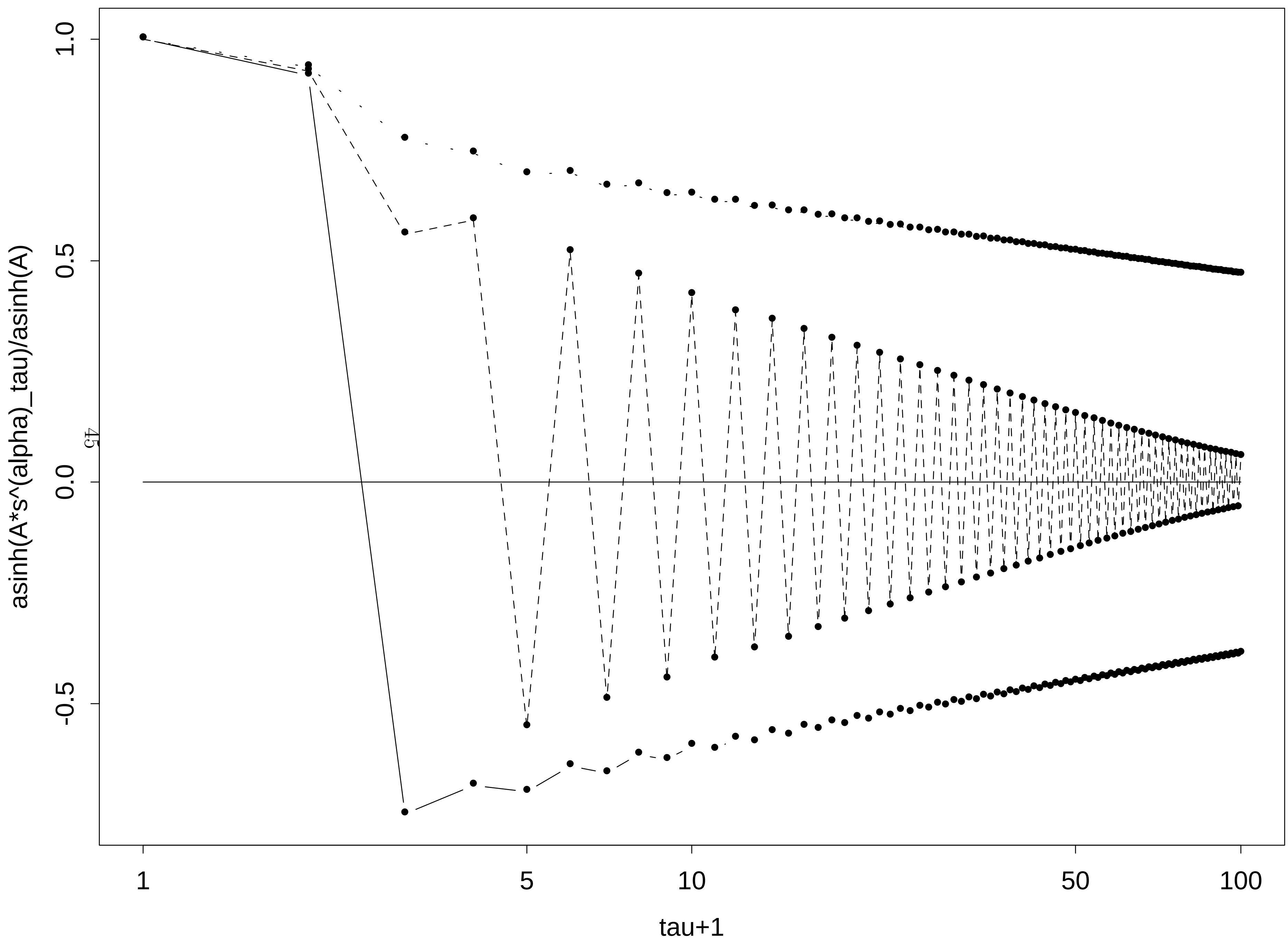
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1
5
10
50
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