5 Renewal Theory

Examples
Definition 5.1 Let \( \{X_n, n = 1, 2, \ldots\} \) be a sequence of non-negative i.i.d. r.v.s with cdf \( F \).
Let \( S_n = \sum_{i=1}^{n} X_i \) and set \( S_0 \equiv 0 \) and let \( N(t) = \max\{n|S_n \leq t\} \). Then \( \{N(t), t \geq 0\} \) is a renewal process.

Some Observations
Example: St. Petersburg Paradox
Let $X = 2^N$ where $N =$ number of tosses of a fair coin until the first head. What is $E[X]$?
More Observations

Let $N(t) = \max\{n|S_n \leq t\}$.

**Question 1** Is $N(t)$ finite w.p.1?

**Question 2** What is the distribution of $N(t)$?
Proposition 5.1 Let \( \{N(t), t \geq 0\} \) be a renewal process, then \( N(t) < \infty \) for all \( t \geq 0 \) a.s. and we can write \( N(t) = \max \{n : S_n \leq t\} \).

Proposition 5.2 Let \( \{N(t), t \geq 0\} \) be a renewal process with renewal distribution \( F \). Let \( F_n \) be the \( n \)-fold convolution of \( F \) with itself. Then, \( P(N(t) \leq n) = 1 - F_{n+1}(t) \).

The expected value of \( N(t) \), \( E[N(t)] \) is called the renewal function, \( m(t) \).

Question 3 What is \( m(t) \) in terms of \( F \)?
Theorem 5.1

\[ m(t) = \sum_{n=1}^{\infty} F_n(t) \]

and

\[ m(t) < \infty \text{ for all } 0 \leq t < \infty. \]

Theorem 5.2 The renewal function \( m(t) \) uniquely determines the distribution of the renewal process, i.e. \( F \).

Example

Proposition 5.3 \( \lim_{t \to \infty} N(t) = \infty \) w.p.1.

Theorem 5.3

\[ \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{w.p.1.} \]

Proof
Example Consider an $M/G/1$ queue with balking, so that if the server is busy when a customer arrives, the customer departs without receiving service, i.e. the customer is lost. What is the long-run fraction of customers that are lost?
5.1 Stopping Times

Some things to ponder
Definition 5.2 A positive integer-valued random variable \( N \) is called a **stopping time** for the sequence of random variables \( X_1, X_2, \ldots \) if the event \( \{ N = n \} \) is dependent only on \( X_1, X_2, \ldots, X_n \) (i.e. whether or not \( N = n \) can be determined based solely on observing \( X_1, X_2, \ldots, X_n \)). (definition from page 298)

**Example: Gambler’s Ruin**
Each play, I win $1 with probability \( p \) and lose $1 to my opponent with probability \( 1 - p \). I start with a fortune of $1 and my opponent starts with $2. We play until one of us is broke.

Let

\[
X_i = \begin{cases} 
1 & \text{if I win on the } i^{th} \text{ day,} \\
-1 & \text{if I lose on the } i^{th} \text{ day.}
\end{cases}
\]

Let \( N \) = number of plays until “ruin”, i.e. until the end of the game.

**Question** Is \( N \) a stopping time for \( X_1, X_2, \ldots \)?
Theorem 5.4 Wald’s Equation

If $X_1, X_2, ...$ are i.i.d. r.v.s with finite expectation and if $N$ is a stopping time for $X_1, X_2, ...$ and $E[N] < \infty$, then

$$E \sum_{n=1}^{N} X_n = ENEX.$$ 

Proof

Definition 5.3 Let $N$ be a non-negative integer-valued r.v. Then $N$ is a generalized stopping time with respect to the sequence of r.v. $(X_n : n \geq 0)$ if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, ...$ for all $n = 0, 1, ....$

Remark about Wald’s equation and generalized stopping time
Examples revisited

Gambler’s Ruin example continued
5.2 Back to Renewal Theory

Question: Is \( N(t) \) a stopping time for \( X_1, X_2, \ldots \)?

Corollary 5.1 Let \( \{N(t), t \geq 0\} \) be a renewal process. Then

\[
E[S_{N(t)+1}] = E \left[ \sum_{i=1}^{N(t)+1} X_i \right] = \mu(m(t) + 1).
\]

Theorem 5.5 Elementary Renewal Theorem

\[
\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{\mu}.
\]

Proof
Definition 5.4 A non-negative r.v. $X$ is lattice if there exists a constant $d > 0$ such that
\[
\sum_{n=0}^{\infty} P(X = nd) = 1,
\]
i.e. all possible values of $X$ are non-negative integer multiples of $d$. The largest such constant $D$ is called the period of the r.v.

Examples

Theorem 5.6 Blackwell’s Theorem

1. If $F$ is non-lattice, then
\[
\lim_{t \to \infty} m(t + a) - m(t) = \frac{a}{\mu}.
\]

2. If $F$ is lattice with period $d$, then for $k = 1, 2, ...$
\[
\lim_{n \to \infty} m(nd + kd) - m(nd) = \frac{kd}{\mu},
\]
For $k = 1$, this implies that
\[
\lim_{n \to \infty} E[\text{number of renewals at time } nd] = \frac{d}{\mu}.
\]
Proof

Theorem 5.7 **Key Renewal Theorem**

If $F$ is not lattice and $h(t)$ is a directly Riemann integrable function, then

$$
\lim_{t \to \infty} \int_{0}^{t} h(t-x)dm(x) = \frac{1}{\mu} \int_{0}^{\infty} h(t)dt.
$$

A sufficient condition for $h(t)$ to be directly Riemann integrable is that

1. $h(t) \geq 0$ for all $t \geq 0$
2. $h(t)$ is nonincreasing in $t$
3. $\int_{0}^{\infty} h(t)dt < \infty$

Definition 5.5 An equation of the form

$$
g(t) = h(t) + \int_{0}^{t} g(t-x)dF(x),
$$

for $t > 0$ is called a **renewal type equation**.
Proposition 5.4 \textit{A renewal type equation has solution}

\[ g(t) = h(t) + \int_0^t h(t - x)dm(x), \]

where \( m(x) = \sum_{n=1}^{\infty} F_n(x) \), and \( F_n(x) \) is the convolution of \( F(x) \) with itself \( n \) times.

\textbf{Proof}

Theorem 5.8 \textit{The Basic Renewal Theorem}

\textit{If} \( F \) \textit{is not lattice and} \( h(t) \) \textit{is directly Riemann integrable and} \( g(t) \) \textit{satisfies the renewal type equation, then}

\[ \lim_{t \to \infty} g(t) = \frac{1}{\mu} \int_0^\infty h(t)dt. \]

\textbf{Proof}

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Comment: Finding $\lim_{t \to \infty} g(t)$

Theorem 5.9 *Alternating Renewal Process*

Consider a system that can be in one of two states, called “on” and “off.” Initially it is on for a random time $X_1$, followed by being off for a random time $Y_1$. Is is then on for $X_2$ and off for $Y_2$, etc. Suppose the $X_i$’s are i.i.d. r.v.s with c.d.f $F$ and the $Y_i$’s are i.i.d. r.v.s with c.d.f $G$. Although $X_i$ is allowed to be dependent on $Y_i$, the pairs $(X_i, Y_i), i = 1, 2, \ldots$ are i.i.d. Let $p(t)$ be the probability that the system is on at time $t$. Then if $E[X_1 + Y_1] < \infty$ and $X_1 + Y_1$ is not lattice, then

$$
\lim_{t \to \infty} p(t) = \frac{E[X_1]}{E[X_1] + E[Y_1]}
$$

Proof
Application of Alternating Renewal Process Theorem to M/G/1 queue
Application of Alternating Renewal Process Theorem to Battery Replacement
**Definition 5.6** Consider a renewal process. Let $A(t) = t - S_{N(t)}$, then $A(t)$ is referred to as the **age of the renewal process** at time $t$. Let $Y(t) = S_{N(t)+1} - t$, then $Y(t)$ is referred to as the **excess (or residual) life of the renewal process** at time $t$. Define $X(t) = X_{N(t)+1}$ then $X(t) = A(t) + Y(t)$. Note that $X(t)$ does not generally have the same distribution as $X_1$.

**Proposition 5.5** If the inter-event distribution of a renewal process is not lattice and $\mu < \infty$ then

$$\lim_{t \to \infty} P(Y(t) \leq x) = \lim_{t \to \infty} P(A(t) \leq x) = \frac{1}{\mu} \int_0^x 1 - F(y)\,dy.$$ 

Examples and Comments
Proposition 5.6  If the inter-event distribution of a renewal process is not lattice and and 
\( E[X_1^2] < \infty \) then 
\[
\lim_{t \to \infty} E[Y(t)] = \lim_{t \to \infty} E[A(t)] = \frac{E[X_1^2]}{2\mu}.
\]

Proposition 5.7  If the inter-event distribution of a renewal process is not lattice and \( \mu < \infty \) then 
\[
\lim_{t \to \infty} P(X(t) \leq x) = \frac{1}{\mu} \int_0^x ydF(y),
\]
and 
\[
\lim_{t \to \infty} E[X(t)] = \frac{E[X_1^2]}{\mu}.
\]