Derivation of elliptical orbits for Newtonian gravitation.

We are interested in a small mass, *m*, orbiting a big mass *M*, and we assume *M* is much larger than *m*, so the C.M.of the system is at the center of the big mass, which is the origin.
By Newton's second law the motion of the small mass is given by the gravitational force exerted on it by the big mass:

$$m\frac{d\vec{v}}{dt} = -\frac{MmG}{r^2}\hat{r}$$

Then using $\frac{d\vec{v}}{dt} = \frac{d\vec{v}}{d\theta}\frac{d\theta}{dt}$ we get $m\frac{d\vec{v}}{d\theta}\frac{d\theta}{dt} = -\frac{MmG}{r^2}\hat{r}$.

Conservation of angular momentum lets us replace $mr^2\frac{d\theta}{dt} = L$ giving

$\frac{d\vec{v}}{d\theta} = -\frac{MmG}{L}\hat{r}$. The unit vector $\hat{\theta}$ points perpendicular to $\hat{r}$ in the direction $\vec{r}$ moves with increasing θ. By drawing these vectors, you should be able to convince yourself that $\hat{r} = -\frac{d\hat{\theta}}{d\theta}$ and then we get the equation $\frac{d\vec{v}}{d\theta} = \frac{MmG}{L}\frac{d\hat{\theta}}{d\theta} \to \frac{d}{d\theta}(\vec{v} - \frac{MmG}{L}\hat{\theta}) = 0$.

And from this we see that the quantity $\vec{v} - \frac{MmG}{L}\hat{\theta}$ is a constant, independent of θ, all around the orbit. This is called "a constant of the motion." Because the vectors in that quantity lie in the plane of the orbit, we can set this constant to a constant vector $\vec{u}$ which is in the plane of the orbit. Thus $\vec{v} = \vec{u} + \frac{MmG}{L}\hat{\theta}$. The magnitude of $\vec{u}$ must describe some property of the orbit, and its direction must relate to how the orbit is oriented. We can pick any direction for $\vec{u}$ in the plane of the orbit, and then see how the orbit that results is oriented. So we pick $\vec{u} = u\hat{j}$ and multiply (dot product) the equation for $\vec{V}$ by $\hat{\theta}$. This selects the component of $\vec{V}$ in the θ direction. That is $\hat{\theta}\bullet\vec{v} = v_\theta$ and also $\hat{\theta}\bullet\hat{j} = \cos\theta$. So this multiplication gives us $v_\theta = u\cos\theta + \frac{MmG}{L}$ But since $\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} = mrv_\theta$ we can substitute $\frac{L}{mr} = u\cos\theta + \frac{MmG}{L}$. This can be rewritten as $\frac{1}{r} = \frac{muL\cos\theta}{L^2} + m\frac{MmG}{L^2}$ so

$$r = \frac{L^2/(mGmM)}{1+(Lu/GmM)\cos\theta}$$

This is the equation for an ellipse with eccentricity *Lu/GmM*, with a focus at the origin, with its major axis on the x-axis. This orientation resulted from the choice of the direction of $\vec{u}$, and the magnitude of $\vec{u}$ must be chosen to get the right eccentricity. Note that this derivation does not work if L=0 exactly, since we divided by L in various places. For more about ellipses, see http://en.wikipedia.org/wiki/Ellipse.

Note that this derivation required the force to be proportional to $1/r^2$ which was put into the first equation. If the force had some other r dependence, the orbits would not be ellipses with a focus at the center of the large mass.