Problem definition: We analyze a new model of crowdfunding recently introduced by Bolstr, Localstake, and Startwise. A platform acts as a matchmaker between a firm needing funds and a crowd of investors willing to provide capital. Once the firm is funded, it pays back the investors using revenue sharing contracts, with a pre-specified investment multiple (investors will receive $M \geq 1$ dollars for every dollar invested) and a revenue-sharing proportion, over an investment horizon of uncertain duration. Academic/Practical Relevance: We analyze the revenue-sharing contract approach to crowdfunding, and we assist the firm to determine its optimal contract parameters to maximize its expected net present value, subject to investor participation constraints and platform fees. Methodology: A natural multi-period formulation for the firm’s problem results in an intractable stochastic optimization model, which we approximate using a deterministic model. In the approximation model, we use a cash buffer for dealing with cash-flow uncertainties; we are able to solve the approximation model analytically. Results: Parametrized on real data from Bolstr campaigns, our approximation solutions give a NPV, in the stochastic problem, that is within 0.2% of the simulation-based optimal NPV, for all levels of cash-flow uncertainty. We compare revenue-sharing contracts with equity crowdfunding and observe that the former result in higher NPVs and comparable bankruptcy probabilities. We also compare revenue-sharing contracts with fixed-rate loans, and find that, for most cases considered, revenue-sharing contracts provide a higher NPV and a lower probability of bankruptcy than a fixed-rate loan. We also show that these benefits are more significant for firms with higher levels of cash-flow uncertainty. Managerial Implications: Revenue-sharing contracts are a novel approach to crowdfunding, which we show are superior to other financing models.

Key words: analytics, crowdfunding, revenue-sharing contracts, optimization, simulation

1. Introduction
Crowdfunding is a relatively new approach for firms to raise capital from a crowd of individuals, rather than traditional sources of capital (e.g., banks, venture capitalists, etc.). The crowdfunding industry is already large, and growing fast. In 2013, the industry was estimated to have raised over $5.1 billion worldwide (Noyes 2014). Looking to the future, PricewaterhouseCoopers has approximated that by 2025, crowdfunding will be a $150 billion industry (PricewaterhouseCoopers 2015). A recent Wall Street Journal article (Karabell 2015) provides some insight for this growth: 1) an individual crowdfunding investment can be risky, offering high returns, 2) lenders can diversify their risk by spreading their investments over different crowdfunding campaigns, and 3) due
to recent financial crises, investors have decreased trust in traditional financial investments (e.g., banks, stocks, etc.), so nontraditional lending markets have increased appeal. A recent regulatory change has also reduced the barriers to entry: starting from May 2016, ordinary people are permitted to invest in small firms (U.S. Securities and Exchange Commission 2016); before May 2016, only accredited investors (i.e., those with an annual income of at least $200,000 or a net worth of at least $1 million) could invest (Cowley 2016). This new crowdfunding regulation, which allows firms to raise up to $1 million over a 12-month period, provides a great opportunity for firms to raise their investment targets more easily.

A number of different firms have become rather well known: Kickstarter, Indiegogo, GoFundMe, Kiva, Prosper, Lending Club. These firms have vastly different crowdfunding models driving their businesses. For example, Kickstarter and Indiegogo, perhaps the most well known crowdfunding firms, essentially solicit donations for individuals and firms needing capital. Kiva deals with microloans, targeting low-income entrepreneurs worldwide, that must be repaid to the lender. Prosper and Lending Club operate in the peer-to-peer lending marketplace, where loans have fixed repayment terms. The peer-to-peer lending market was the most dominant alternative finance market in 2015 in the United States, with approximately $25.7 billion raised (Alois 2016).

Our paper is motivated by an emergent model of crowdfunding, pioneered by Bolstr (www.bolstr.com), Localstake (www.localstake.com), and Startwise (www.startwise.com), which targets small and medium-sized firms needing capital; however, this model is not intrinsically limited by firm size, and could be applied to any sized firm. These platforms match a firm needing funding with investors from the crowd, and the firm pays back the investors via revenue-sharing contracts. This more flexible repayment agreement is linked with the financial performance of the firm, allowing variable payments and investment horizons, thus reducing financial stress on the borrower. In contrast, if business goes well, the borrower is obligated to increase the payments, thus reducing the investment horizon, which results in a higher effective interest rate for the investors. Therefore, revenue-sharing contracts intuitively align a firm’s and investors’ incentives, in a way not possible with traditional fixed-rate loans. Notably, many of the loans administered by, for example, Bolstr are repaid well before their estimated investment horizon: according to the Bolstr website, the estimated investment horizons usually range from 2-5 years for loan sizes of $25,000 - $500,000, yet Chase and Company (2015) report on the example of a lobster roll restaurant in Chicago that paid back its loan of $70,000 in seven months. Furthermore, in 2016, Bolstr announced that “An investor who participated at the minimum investment level in every deal would have a portfolio tracking to a 19.18% net Internal Rate of Return.”

The speed of funding in crowdfunding can be very fast, with respect to traditional funding sources. Traditional loans, such as from banks and the U.S. Small Business Administration (SBA),
have low approval rates and typically take 2-3 months to fund. However, revenue-sharing contracts can fund very fast in practice. The following quotes are real subject lines from marketing emails from Bolstr:

"Paddy Wagon Raised $20,000 in Less than 10 Minutes,"

"Dubina Brewing Co. Raised $30,000 in 20 minutes,"

"The Bacon Jams Raised $40,000 in Less than 30 Minutes,"

"Underground Butcher Raised $75,000 in Less Than 1 Hour."

Firms can alternatively raise investments from other online lenders such as Lending Club, Prosper, and Kabbage, but according to the Bolstr website, loans from these lenders have annual percentage rates as high as 80%, whereas revenue-sharing loans have annual percentage rates of 8 – 25%. Therefore, revenue-sharing contracts have advantages over traditional and alternative funding sources such as having flexible payments, fast funding time, and lower equivalent interest rates. In our paper we show that, the firm’s NPV under a revenue-sharing contract is larger than that of equity crowdfunding, and the firm’s probabilities of bankruptcy are comparable. Additionally, in most cases considered, a firm’s Net Present Value (NPV) under a revenue-sharing contract is larger than the NPV of a fixed-rate loan, even when the latter has low annual interest rates. Similarly, a firm’s probability of bankruptcy is lower under a revenue-sharing contract than under a fixed-rate loan. These benefits stem from the flexible nature of the revenue-sharing contract.

The proposed revenue-sharing contract bears some superficial similarity to performance-sensitive debt (e.g., step-up bonds and performance-pricing loans) studied by Manso et al. (2010), where the debt payments depend on the borrower’s performance: the borrower pays higher interest rates during low performance and lower interest rates during high performance. However, this approach has the opposite behaviour of our proposed revenue-sharing contract, since, under the latter, a high performing firm will have high debt payments and will pay off the fixed loan amount early, resulting in a higher effective interest rate. Performance-sensitive debt has been shown to harm both the borrower and investors via earlier borrower default (Manso et al. 2010), whereas our new model can result in positive outcomes for all parties.

In this paper, we provide an analysis of the firm’s multi-period problem of maximizing its expected net present value, subject to platform fees and investor participation constraints, for stochastic revenues and costs. The firm decides how much investment \( Y \geq 0 \) is needed, an investment multiple \( M \geq 1 \) (where investors are guaranteed to be paid back \( M \) times their initial investment), and a revenue-sharing proportion \( \gamma \geq 0 \) (in each period the firm pays all investors a proportion \( \gamma \) of its revenues). These variables induce a stochastic investment horizon \( T \) and a stochastic bankruptcy time \( B \) (possibly infinite). The platform fees consist of an origination percentage \( \alpha \in [0,1] \) (where the firm pays a fee \( \alpha Y \) to the platform at time zero) and a servicing
percentage $\beta \in [0, 1]$ (where in each period the firm pays a $\beta$ percent of all investor revenue payments to the platform). We design a stochastic programming formulation of the firm’s problem, which is a rather intractable model.

Since the stochastic model is difficult to analyze, we derive a deterministic approximation model for it, in which we use a cash buffer to cope with uncertainties. We then solve the approximation problem analytically, which provides generalizable insights. We also solve our stochastic model numerically using Monte Carlo simulation and a grid-based optimization framework, for serially correlated random cash-flows that are parameterized using real data from Bolstr. We then compare the NPVs of the approximate and optimal solutions in the true stochastic model. We conclude that our approximation provides high quality solutions: the worst-case average error of the approximation solution’s NPV, over all feasible Bolstr campaigns for all levels of cash-flow uncertainty, is approximately 0.2%.

Finally, we compare the performance of the proposed revenue-sharing contract with equity crowdfunding and fixed-rate loans, and we identify which type of financing results in a higher NPV or a lower chance of bankruptcy for a firm; we find that the revenue-sharing contract is superior in most cases. We next provide a literature review to appropriately position our paper’s contributions.

1.1. Literature Review
There are many papers that study other crowdfunding models, papers that study various aspects of the interface of operations and finance, as well as a vast literature on revenue-sharing contracts, primarily in the areas of supply chain management and educational financing. In this section, we survey the most relevant results in these three streams. However, to the best of our knowledge, there are no studies investigating crowdfunding via revenue-sharing contracts, except Fatehi and Wagner (2017), which is a book chapter summary of a preliminary version of this research. Thus, our research uniquely lies at the intersection of these two literature streams.

Babich et al. (2017) provided a detailed study on how crowdfunding impacts the financing decisions of entrepreneurs, banks, and venture capital investors. Chen et al. (2017) considered fundraising success in crowdfunding and studied the optimal referring policies by entrepreneurs, empirically and analytically. Chakraborty and Swinney (2016) studied how entrepreneurs can signal their product’s quality to contributors via campaign design in reward-based crowdfunding. Zhang and Liu (2012) studied a data set from Prosper, focusing on a study of herding behavior in microloan markets. Lin and Viswanathan (2015) also analyzed data from Prosper and studied the “home bias” in crowdfunding markets. Iyer et al. (2015) also studied a dataset from Prosper and showed that lenders use standard information along with nonstandard, or soft, information to evaluate borrower creditworthiness. Belleflamme et al. (2014) compared two forms of crowdfunding, profit
sharing and pre-ordering of products, and showed that if the firm’s investment goal is relatively high with respect to the market size, then the firm prefers profit-sharing crowdfunding. Finally, Wei and Lin (2016) study, both theoretically and empirically, the difference between auctions and posted prices on Prosper.com.

Our paper is also relevant to the literature on the interface of operations and finance. In particular, our research is relevant to operational financing. There can be many sources of financing, including traditional bank financing, as well as more novel arrangements that have been attracting attention in the OM literature, such as buyer financing (Deng et al. 2018) or supplier trade credits (Lee et al. 2017) or both (Kouvelis and Zhao 2017). We identify crowdfunding as an additional source of financing, and while we focus on revenue-sharing crowdfunding, we also consider equity crowdfunding.

The structure of our crowdfunding contracts, revenue-sharing, has also been extensively studied in the supply chain management literature. However, the combination with financial aspects is limited. Kouvelis and Zhao (2015) studied contract design for a supply chain with one supplier and one retailer in the presence of financial constraints and bankruptcy costs, and they show that a revenue-sharing contract can still coordinate the supply chain. Kouvelis et al. (2017) showed that revenue-sharing contracts can achieve high efficiency in the presence of cost uncertainty and working capital constraints. Similarly, we show that a firm can benefit significantly from revenue-sharing contracts under stochastic cash flows.

Revenue sharing contracts have also appeared in the education field, as a novel approach to funding students. Nerlove (1975) studied an income-contingent loan program for the financing of education, which was originally proposed by Friedman and Kuznets (1945) for professional education and by Friedman (1955) for vocational education. One example of such a loan program is the Yale Tuition Postponement Option, which started in 1971, but was discontinued in 1978 (Ladine 2001). Another example is the Pay-it-Forward plan in Oregon: in 2013, state legislators proposed a program in which students could attend public colleges without paying tuition, and in return they would pay 3% percent of their future income to the state for several years after graduation (Palacios and Kelly 2014). Similarly, the Back-a-Boiler program at Purdue University, which started in Fall 2016 and has already raised $2.2 million, provides funds to undergraduate students to finance their education through an Income Share Agreement in which students agree to pay a percentage of their future income over a standard payment term (Purdue 2016). The results and insights of our paper can also be applied to these student loan agreements.

1.2. Contributions

The contributions of our paper are as follows:
1. We are the first, to our knowledge, to study a new emergent model of crowdfunding pioneered by Bolstr, Localstake, and Startwise. Furthermore, our models are parameterized using real data from 56 Bolstr campaigns.

2. We study a firm’s expected NPV maximization problem under a revenue-sharing contract, which is an intractable stochastic model. To overcome the technical difficulties, we design a tractable deterministic approximation model, in which we use a cash buffer to cope with cash-flow uncertainties. We solve the approximation model analytically, which provides qualitative insights. Numerical experiments, calibrated on real Bolstr data, indicate that the approximation solutions, when inserted into the true stochastic model, result in an expected NPV that is within 0.2% of the true optimal NPV, on average. Furthermore, the approximation solutions result in almost the same bankruptcy probabilities as the optimal stochastic solutions.

3. Our results provide managerial guidelines for the firm; for instance:

(a) A firm can attain a higher NPV and a comparable probability of bankruptcy under a revenue-sharing contract than under equity crowdfunding. We show that the NPV benefit of revenue-sharing increases as the firm’s cashflow volatility increases.

(b) A firm can attain a higher NPV and a lower probability of bankruptcy under a revenue-sharing contract than under a more traditional fixed-rate loan. We also show that these benefits are more significant for firms with higher levels of cash-flow uncertainty. Intuitively, these benefits are due to the more flexible nature of the revenue-sharing contract.

(c) As cash-flow uncertainty increases, the optimal investment amount increases, the revenue-sharing percentage decreases, resulting in a stochastically larger investment horizon (e.g., larger mean and variance). However, the firm’s maximized NPV is rather insensitive to cash-flow uncertainty. Thus, while the details of the optimal revenue-sharing contract can change considerably with cash-flow uncertainty, the bottom line NPV is rather robust to this uncertainty.

2. Stochastic Model of Firm

In this section we detail our basic stochastic model of a firm, whose revenue and cost in time period $t$, $R_t \geq 0$ and $C_t \geq 0$, are random variables; we model these cashflows using random walks with drift, which is explained in detail in Section 2.2. Many firms on Bolstr, Localstake, and Starwise have raised money to upgrade their existing stores, build a store in a new location, buy new equipment, and create/upgrade their websites. Due to this expansion, the firm has cash-flow shortages for a limited time and therefore needs to raise capital, in the amount of $Y \geq 0$ (e.g. dollars), via an intermediary platform that pairs interested individual investors that are willing to invest. Therefore, the investment $Y$ is a buffer to avoid a negative cash flow.

The investors do not receive equity in the firm, but are paid back, with interest, via a revenue-sharing contract: at the end of each time period $t$, the firm is contractually obligated to pay out to
all investors a proportion $\gamma \geq 0$ of its revenues for that time period. These payments continue until each investor receives a multiple $M \geq 1$ of his/her initial investment, which occurs at time $t = T$ (for all investors); this definition of $M$ is motivated by practical implementations (e.g., Bolstr, Localstake, and Startwise). Note that the investment payments are not fixed, since they depend on firm revenues, which can vary. The contract’s duration is therefore the stochastic stopping time

$$T = \min \left\{ \hat{T} \geq 1 : \sum_{t=1}^{\hat{T}} \gamma R_t \geq MY \right\},$$

which captures the firm’s contractual obligation to pay $\gamma$ percent of its revenue to investors until a total nominal amount of $MY$ has been paid. Time period $t = 0$ is the initialization of the revenue-sharing contract when the firm receives total investment $Y$. The firm must also pay the platform 1) an origination fee of $\alpha \in [0, 1]$ percent of the total amount raised $Y$ at time $t = 0$ and 2) a servicing fee of $\beta \in [0, 1]$ percent of all revenue payments made to investors at times $t > 0$.

We next discuss cash flows, and, for simplicity, we assume the risk-free interest rate is zero (i.e., cash does not earn interest). If, in period $t$, $R_t - C_t < 0$, there is a cash shortfall; however, this can potentially be addressed using an excess of cash from previous periods. Thus, we focus on cumulative cash flows. If there exists $\tau \geq 1$ such that $\sum_{t=1}^{\tau} (R_t - C_t) < 0$, then the firm needs cash in month $\tau$. The firm’s cumulative cash flow at the end of month $\tau = 1, \ldots, T$, during the revenue sharing contract, is $\sum_{t=1}^{\tau} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{\tau} R_t$; if $\tau > T$, after the completion of the contract, the cumulative cash flow is $\sum_{t=1}^{T} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{T} R_t$. These two scenarios can be combined into one expression for the cash flow in period $\tau \geq 1$: $\sum_{t=1}^{\tau} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{\min\{\tau, T\}} R_t$.

Due to the stochasticity of the firm’s cash flows, the firm can potentially go bankrupt, for any combination of contractual parameters $(Y, M, \gamma)$. Babich and Tang (2016) and Uhrig-Homburg (2005) model bankruptcy as occurring during the first period where the firm is cash-flow negative; we adopt this approach. Letting $B$ denote the time period where the firm goes bankrupt, we model $B$ as a stochastic stopping time (that depends on the stochastic stopping time $T$):

$$B = \min \left\{ \hat{B} \geq 1 : \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{\min\{\hat{B}, T\}} R_t < 0 \right\}. \quad (2)$$

The platform and investors receive their payments at the end of each month if the borrower is not bankrupt; thus, the revenue sharing contract is in effect for $\tau = 1, \ldots, \min\{B, T\}$. As costs and revenues are stochastic, a risk-neutral firm wants to maximize the firm’s expected NPV

$$E \left[ \sum_{t=1}^{\min\{B, T\}} \frac{R_t - C_t}{(1 + r_t)^t} \right] - (\beta + 1)\gamma E \left[ \sum_{t=1}^{\min\{B, T\}} \frac{R_t}{(1 + r_t)^t} \right] + (1 - \alpha)Y. \quad (3)$$
where \( \gamma E \left[ \sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t} \right] \) is the expected NPV of all payments made to investors, 
\( \beta \gamma E \left[ \sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t} \right] \) is the expected NPV of all servicing fees paid to the platform, \( \alpha Y \) is the origination fee paid to the platform at time \( t = 0 \), and, in period \( t \), risk is quantified via imputed discount rates \( r_t \) (Arrow and Lind 1978). We assume these discount rates are known. According to Pratt et al. (2014), small-sized firms can estimate their NPV by using the cost of capital as the discount rate. Alternatively, Wei and Lin (2016) assume that firms determine their discount rate according to the lowest interest rate offered to them from other financial institutions.

Finally, we consider the investors. We assume that there is a large pool of investors and the target investment is raised by \( n \) investors, where investor \( i \) invests an amount \( y_i \in (0,Y] \), and \( \sum_{i=1}^{n} y_i = Y \). We model the investor participation constraints as
\[
E \left[ \min\{B,T\} \sum_{t=1}^{\gamma Y R_t (1 + \delta_t)^t} y_i - \frac{y_i}{Y} \right] \geq A_i, \ i = 1,...,n,
\]
where \( \frac{y_i}{Y} \) is the fraction of the total payment \( \gamma R_t \) investor \( i \) receives in period \( t \), \( \delta_t \) is the discount rate of investors in period \( t \), the left-hand-side is the investor's expected NPV, and \( A_i \geq 0 \) is the target rate of return, in NPV terms, for investor \( i \). In other words, \( A_i \) captures investor \( i \)'s opportunity cost of alternative investments. We simplify the constraints in Expression (4) to the following single constraint
\[
\sum_{i=1}^{\min\{B,T\}} \frac{\gamma Y R_t (1 + \delta_t)^t}{Y (1 + \delta_t)^t} \geq \max_{1 \leq i \leq n} \left\{ \frac{A_i}{y_i} \right\} + 1.
\]

The firm is able to select the investment amount \( Y \), the multiple \( M \), and the revenue-sharing proportion \( \gamma \) in order to maximize its expected NPV. The above analysis results in the firm’s problem:
\[
\begin{align*}
Z_F &= \max_{Y,M,\gamma} E \left[ \sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t} \right] - (\beta + 1) \gamma E \left[ \sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t} \right] + (1-\alpha)Y \\
&\text{s.t. } T = \min \left\{ \hat{T} \geq 1 : \sum_{t=1}^{\hat{T}} \gamma R_t \geq MY \right\} \quad \text{(contractual obligation)} \\
&\quad B = \min \left\{ \hat{B} \geq 1 : \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1-\alpha)Y - (\beta + 1) \gamma \sum_{t=1}^{\min\{B,T\}} R_t < 0 \right\} \quad \text{(bankruptcy definition)} \\
&\quad E \left[ \sum_{t=1}^{\min\{B,T\}} \frac{\gamma Y R_t (1 + \delta_t)^t}{Y (1 + \delta_t)^t} \right] \geq \max_{1 \leq i \leq n} \left\{ \frac{A_i}{y_i} \right\} + 1 \quad \text{(investor participation)}
\end{align*}
\]

Table 2, in the appendix, summarizes the main problem parameters and firm variables.
2.1. Model Parameterization Using Data from Bolstr.com

We have collected cost and revenue projections from 56 campaigns on the Bolstr platform, and performed regression analyses on them. The $R^2$ values for the cost regressions ranged from 0.595 to 0.995, with a mean of 0.905 and a standard deviation of 0.079. The $R^2$ values for the revenue regressions ranged from 0.226 to 0.995, with a mean of 0.922 and a standard deviation of 0.065. Therefore, the Bolstr data suggest that linear models of cost and revenue projections, in expectation, are reasonable assumptions.

We denote $a$ and $b$ as the intercept and slope of a generic revenue regression line, respectively; similarly, we denote $c$ and $d$ as the intercept and slope of an arbitrary cost regression line, respectively. Letting $E[R_t]$ and $E[C_t]$ denote the expected revenue and cost in period $t$, respectively, we assign

$$E[R_t] = a + bt \quad \text{and} \quad E[C_t] = c + dt. \quad (7)$$

Many, but not all, of our results will utilize the linearity of cash flows.

2.2. Modeling Cashflows

Motivated by the cash flow models in Dechow et al. (1998), we generate revenues ($R_1, R_2, \ldots$) and costs ($C_1, C_2, \ldots$) using random walk processes:

$$R_t = R_{t-1} + Z^r_t \quad \text{and} \quad C_t = C_{t-1} + Z^c_t, \quad t \geq 1, \quad (8)$$

where $R_0 = a$ and $C_0 = c$ are given in Equation (7). The $Z^r_t$ are independent normal random variables with common mean $\mu^r = b$, where $b$ is given by Equation (7), and standard deviation $\sigma^r = \mu^r / k$, where $k$ is a tunable parameter. Similarly, $Z^c_t$ are independent normal random variables with common mean $\mu^c = d$, where $d$ is given by Equation (7), and standard deviation $\sigma^c = \mu^c / k$.

The means can be easily calculated: $E[R_t] = a + bt$ and $E[C_t] = c + dt$, in agreement with Equation (7). Furthermore, the random walk model exhibits serial correlation: it is straightforward to show that, for $s < t$, $\text{cov}(R_s, R_t) = s(\sigma^r)^2$ and $\text{cov}(C_s, C_t) = s(\sigma^c)^2$. Note that Brownian motion is a limit of a random walk process (Kac 1947). Therefore, our proposed random walk models for revenues and costs can be considered as approximate Brownian motion processes with drifts $\mu^r$ and $\mu^c$ and volatilities $\sigma^r$ and $\sigma^c$, respectively (Ross 1996, Sigman 2006).

2.3. Analysis Roadmap

We found an analytical solution to Problem (6) to be intractable. In the next section, we derive an approximation for the stochastic problem. The approximation problem is a deterministic relaxation where $\sigma^r = \sigma^c = 0$, but we add a cash-flow buffer to the bankruptcy definition to deal with cash-flow uncertainties in the stochastic model; in this case, we are able to derive analytical solutions that
provide generalizable insights. These results are provided in Section 3. To evaluate the quality of our approximation, we also solve Problem (6) for real Bolstr data, using Monte Carlo simulation, to determine the stochastic stopping times \((B,T)\) and expectations, combined with numerical optimization on a fine grid of \((Y,M,\gamma)\) space. We utilize random walks with drift for the cash flows, where \(\sigma^r = \mu^r/k\) and \(\sigma^c = \mu^c/k\) for various values of \(k\); this analysis can be found in Section 4. Finally, in Sections 5–6, we show that the revenue-sharing contract compares favorably with equity crowdfunding and fixed-rate loans, respectively.

### 3. Deterministic Approximation to Stochastic Model

In this section, we consider a deterministic approximation to Problem (6) where the cash flows \(R_t\) and \(C_t\) are known exactly. This simplification results in the conversion of the bankruptcy time \(B\) and investment duration \(T\) into parameters, rather than random variables. We assume, given that cash flows are known exactly, the firm desires to avoid bankruptcy. Thus, reversing the condition for bankruptcy in Equation (2), we introduce constraints that require the firm to be cash-flow positive above \(\theta \geq 0\), for all time periods \(\tau \geq 1\), where \(\theta\) is a cash buffer to account for cash-flow uncertainties in the stochastic problem:

\[
\sum_{t=1}^{\tau} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{\min(\tau,T)} R_t \geq \theta, \quad \tau \geq 1. \tag{9}
\]

Similarly, in project management, Long and Ohsato (2008) developed a deterministic schedule and used a project buffer for dealing with resource uncertainty. They determined the size of the buffer numerically; similarly, in Section 4.2 we determine the size of the cash-buffer \(\theta\) numerically as a function of problem data. We show that as the uncertainty of revenues and costs increases, \(\theta\) should increase to make the deterministic approximation solution feasible for the stochastic problem and provide a high quality approximation for the stochastic problem.

These constraints imply that \(B = \infty\) as the firm will never go bankrupt. In Section 4 we show, via computational experiments, that the optimal solution to the stochastic model in Problem (6) induces a low probability of firm bankruptcy over feasible Bolstr campaigns, which suggests that the constraints in (9) are unlikely to eliminate the optimal solution to the stochastic model; in Section 4.2 we evaluate the approximation quality of the deterministic model developed in this section for Problem (6), with very encouraging results.

Next, since the variables \(Y, M,\) and \(\gamma\) are continuous, we assume that the definition of \(T\) in Equation (1) holds exactly and deterministically:

\[
\sum_{t=1}^{T} \gamma R_t = MY. \tag{10}
\]
These simplifications result in a deterministic approximation to Problem (6), parameterized by the investment duration $T$:

$$
\hat{z}_F(T) = \max_{Y, M; \gamma} \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1 + r_t) Y} - (\beta + 1) \gamma \sum_{t=1}^{\infty} \frac{R_t}{(1 + r_t) Y} + (1 - \alpha) Y
$$

s.t. \[
\sum_{t=1}^{\infty} \gamma R_t = M Y \quad \text{(contractual obligation)}
\]

\[
\sum_{t=1}^{\infty} (R_t - C_t) + (1 - \alpha) Y - (\beta + 1) \gamma \sum_{t=1}^{\min(\tau, T)} R_t \geq \theta, \quad \tau \geq 1 \quad \text{(cash-flow constraints)}
\]

\[
\sum_{t=1}^{\infty} \gamma R_t \geq \max_{1 \leq i \leq n} \left\{ \frac{A_i}{y_i} \right\} + 1 \quad \text{(investor participation)}
\]

$$
Y, \gamma \geq 0, M \geq 1.
$$

(11)

### 3.1. Analysis for fixed $T \in \mathbb{N}$

In this subsection, we solve Model (11) for a fixed $T \in \mathbb{N}$. To begin our analysis, we point out some simplifications. First, we let $\hat{A} = \max_{1 \leq i \leq n} \left\{ \frac{A_i}{y_i} \right\}$ to simplify the exposition. Second, the contractual obligation constraint can be used to solve for $\gamma = \frac{MY}{\sum_{t=1}^{\infty} R_t}$, and $\gamma$ is eliminated as a variable. In the subsequent analysis, it will be convenient to define the set

$$
X \equiv \left\{ \tau \in \mathbb{N} : \sum_{t=1}^{\tau} (R_t - C_t) < \theta \right\},
$$

which indexes all the time periods where the firm, without any investment, has cash flow below $\theta$. It is also convenient to define the parameters $Z_r(T)$, which depend only on firm problem data and $T$:

$$
Z_r(T) \equiv (\hat{A} + 1)(\beta + 1) \sum_{t=1}^{T} \frac{\min\{\tau, T\} R_t}{(1 + \delta_t)(1 + r_t)} - (1 - \alpha).
$$

The resulting model has the following solution.

**Proposition 1.** Problem (11), with $T \in \mathbb{N}$ fixed and $X \neq \emptyset$, is feasible if and only if $Z_r(T) < 0$, $\forall \tau \in X$ and $\max_{r \in X} \left\{ \frac{\sum_{t=1}^{\tau} (R_t - C_t) - \theta}{Z_r(T)} \right\} \leq \min_{r \in X} \left\{ \frac{\sum_{t=1}^{\tau} (R_t - C_t) - \theta}{Z_r(T)} \right\}$, and has the optimal solution $M^*(T) = \frac{(\hat{A} + 1)\sum_{t=1}^{T} \frac{R_t}{(1 + r_t)}}{\sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)}}$ and $\gamma^*(T) = \frac{(\hat{A} + 1)Y^*(T)}{\sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)}}$.

where

- if $(\beta + 1)(\hat{A} + 1)\frac{\sum_{t=1}^{\tau} \frac{R_t}{(1 + r_t)}}{\sum_{t=1}^{\tau} \frac{R_t}{(1 + \delta_t)}} - (1 - \alpha) \geq 0$, then $Y^*(T) = \max_{r \in X} \left\{ \frac{\sum_{t=1}^{\tau} (R_t - C_t) - \theta}{Z_r(T)} \right\}$; alternatively, $X = \emptyset$, $Y^* = 0$, $\gamma^* = 0$, and $M^* = 1$ for all $T \in \mathbb{N}$.

- if $(\beta + 1)(\hat{A} + 1)\frac{\sum_{t=1}^{\tau} \frac{R_t}{(1 + r_t)}}{\sum_{t=1}^{\tau} \frac{R_t}{(1 + \delta_t)}} - (1 - \alpha) < 0$, then $Y^*(T) = \min_{r \in X} \left\{ \frac{\sum_{t=1}^{\tau} (R_t - C_t) - \theta}{Z_r(T)} \right\}$; alternatively, if $X = \emptyset$, then $Y^*(T) = \min_{r \in X} \left\{ \frac{\sum_{t=1}^{\tau} (R_t - C_t) - \theta}{Z_r(T)} \right\}$ for all $T \in \mathbb{N}$.

The maximized NPV is

$$
\hat{z}_F(T) = \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1 + r_t)} - \left( (\beta + 1)(\hat{A} + 1)\frac{\sum_{t=1}^{T} \frac{R_t}{(1 + r_t)}}{\sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)}} - (1 - \alpha) \right) Y^*(T).
$$
Note that Proposition 1 only requires that the cash flows $R_t$ and $C_t$ be deterministic, but does not require them to be linear. The feasibility constraints in Proposition 1 can be interpreted as financial conditions where the firm can eventually survive on its own; as an example where this is not possible, consider the case where $C_t > R_t$ for all $t$. In particular, $Z_\tau(T) < 0$, $\forall \tau \in X$ ensures that the firm is cash-flow positive for all periods $\tau \in X$, due to the investment $Y^*(T)$, and the condition $Y^*(T) \leq \min_{\tau \in X, Z_\tau(T) > 0} \left\{ \frac{\sum_{t=1}^T (R_t - C_t) - \theta}{Z_\tau(T)} \right\}$ ensures that the firm can afford to pay back $M^*(T)Y^*(T)$ to investors and $\beta M^*(T)Y^*(T)$ to the platform.

If the firm’s discount rates $r_t$ are not too large, relative to the investors’ discount rates $\delta_t$, and $\beta + 1)(\hat{A} + 1)\frac{\sum_{t=1}^T R_t}{\sum_{t=1}^T (1+\delta_t)} - (1-\alpha) \geq 0$ holds, then we see that, intuitively, if $X = \emptyset$, then the firm does not need any investment and $Y^*(T) = 0$. Alternatively, if $X \neq \emptyset$ and the feasibility conditions are satisfied, then $\left( (\beta + 1)(\hat{A} + 1)\frac{\sum_{t=1}^T R_t}{\sum_{t=1}^T (1+\delta_t)} - (1-\alpha) \right) Y^*(T)$ can be interpreted as the firm’s cost for avoiding bankruptcy.

Alternatively, if $(\beta + 1)(\hat{A} + 1)\frac{\sum_{t=1}^T R_t}{\sum_{t=1}^T (1+\delta_t)} - (1-\alpha) < 0$, then the firm’s discount rates $r_t$ are relatively larger than the investors’ discount rates $\delta_t$. Therefore the firm’s gain from the $(1-\alpha)Y$ investment at time zero is greater than the firm’s NPV of future payments to the investors and the platform. As a result, the firm benefits by raising a larger investment.

In Figure 1 we plot the average of the objective function value $\hat{z}_F(T)$ and the average of the optimal variables $(Y^*(T), M^*(T), \gamma^*(T))$ from Proposition 1 as a function of $T$, over feasible Bolstr campaigns, for the following parameter set: For each campaign we let linear revenues $R_t = E[R_t]$ and costs $C_t = E[C_t]$, where $E[R_t]$ and $E[C_t]$ are given in Equation (7) and only use the given campaign’s data, $\theta = 0$, $\alpha = 0.05$ and $\beta = 0.01$ (per a Bolstr memorandum), $\hat{A} = 0.1$ (i.e., a 10% NPV return for investors), and $r_t = \delta_t = 0.01$, $\forall t$ (the discount rate per period, typically a month). These results are useful in the sequel for interpreting the results for the stochastic problem where the level of variability in costs and revenues is small.

The plot on the top left of Figure 1 suggests that $\hat{z}_F(T)$ converges rather quickly to an asymptote. In the next section, for linear cash-flows, we derive conditions for which $\sup_T \hat{z}_F(T)$ is attained when $T \to \infty$. However, our numerical results suggest that relatively small values of $T$, say $T \in \{18, \ldots, 30\}$, suffice to attain almost all the potential value of $\sup_T \hat{z}_F(T)$. In the top right plot, we observe that $Y^*(T)$ decreases rather quickly to an asymptote as well. In the bottom left plot we see that $M^*(T)$ is increasing in $T$ and in the bottom right plot we see that $\gamma^*(T)$ is decreasing in $T$, which intuitively align with the increased investment duration $T$. We point out that our model was only feasible for $T \geq 12$ for all feasible campaigns for $\theta = 0$; in the next section, for linear cash flows, we provide a rigorous proof that Problem (11) can only be feasible for large enough $T$. 
Figure 1 Illustration of Proposition 1 for feasible Bolstr campaigns with linear costs and revenues.

3.2. The Optimal Investment Horizon $T^*$

In the previous section, we fixed $T \in \mathbb{N}$ and solved Problem (11); we provided closed-form expressions in Proposition 1 for $\gamma^*(T)$, $M^*(T)$, and $Y^*(T)$ for feasible problems. In this section, we find the optimal investment horizon $T^*$. Proposition 1 indicates that Model (11) becomes

$$
\max_{T \in \mathbb{N}} \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1+r_t)^t} - \left( (\beta + 1)(A + 1) \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} (1+r_t)^t} \right) Y^*(T) \\
\text{s.t. } Z_{\tau}(T) < 0, \ \tau \in X \\
\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_{\tau}(T)} \right\} \leq \min_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_{\tau}(T)} \right\},
$$

(14)

where $Y^*(T)$ is defined in Proposition 1. It is convenient to define the term $R^\infty \triangleq \sum_{t=1}^{\infty} \frac{R_t}{(1+\delta_t)^t}$, which we assume is finite, in order to present the next set of results, which are valid for large $T$ and characterize the structure of Problem (14).

**Lemma 1.** If $(A + 1)(\beta + 1) \sum_{t=1}^{\tau} R_t/R^\infty < (1-\alpha)$ for all $\tau \in X$, then $Z_\tau(T) < 0$ for all $\tau \in X$.

**Lemma 2.** For $\tau \in X$ and $Z_\tau(T) < 0$, $\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_{\tau}(T)} \right\}$ is strictly decreasing in $T$. 
LEMMA 3. For \( \tau \notin X \) and \( Z_\tau(T) > 0 \), \( \min_{r \in X} \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \) is strictly increasing in \( T \).

The next lemma builds upon Lemmas 2–3 to characterize the objective function of Problem (14).

LEMMA 4. If \( r_t = r \geq \delta = \delta_t, \forall t \) or \( r_t = \delta_t, \forall t \), then the objective function of Problem (14) is increasing in \( T \), for large \( T \).

Lemmas 2 and 3 suggest that \( \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} \leq \min_{r \in X} \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \) is attained asymptotically. However, this is not always possible. Consider the following example.

EXAMPLE 1. Let the revenues \( R_t = 10 \) for \( t \geq 1 \) and the costs are \( C_1 = 15 \), \( C_2 = 5 \), and \( C_t = 10 \) for \( t \geq 3 \), and \( \theta = 0 \). The set \( X = \{1\} \) and, for feasible values of \( (A, \alpha, \beta, \delta_t) \), \( \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} > 0 \) for all feasible \( T \). In contrast, \( \sum_{t=1}^{\tau}(R_t - C_t) = 0 \) for all \( \tau \notin X \), implying \( \min_{r \in X} \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} = 0 \).

In the next subsection, we see that an assumption of linear cash flows resolves the issue encountered in the above counterexample, and Problem (14) is solvable analytically. Recall that the linearity of revenues and costs is supported by real data from Bolstr, as explained in Section 2.1.

3.2.1. Linear Firm Costs and Revenues In this subsection, we consider Problem (14) when the costs and revenues are linear: \( R_t = a + bt \) and \( C_t = c + dt \). We could, for instance, parameterize \((a, b, c, d)\) using the regression analysis on Bolstr data. The set \( X \), for linear costs and revenues, simplifies to

\[
X = \left\{ \tau \in \mathbb{N} : (a - c)\tau + (b - d)\frac{\tau(\tau + 1)}{2} < \theta \right\}. \tag{15}
\]

The following function is useful for the subsequent analysis and results:

\[
Z_{\tau}(T) = \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} = \frac{(a - c)\tau + (b - d)\frac{\tau(\tau + 1)}{2} - \theta}{\eta(T) \left( a\tau + b\frac{(\tau + 1)}{2} \right) - (1 - \alpha)}, \tag{16}
\]

where \( \eta(T) \triangleq \frac{(A + 1)(\beta + 1)}{\sum_{t=1}^{\tau}(R_t - C_t) - \theta} > 0 \). This function allows us to write \( \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} = \max_{\tau \in X} f(\tau) \) and \( \min_{r \in X} \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} = \min_{\tau \in X} f(\tau) \), where \( \max f(\tau) \) is a lower bound on \( Y^*(T) \) to allow cash-flow positivity at or above \( \theta \) and \( \min f(\tau) \) is an upper bound on \( Y^*(T) \) to ensure that the firm can afford to pay back \( M^*(T)Y^*(T) \) to investors and \( \beta M^*(T)Y^*(T) \) to the platform. Our next two results characterize the cases where the set \( X \) is unbounded or empty, which results in infeasible and trivial firm optimization problems, respectively.

LEMMA 5. If 1) \( b < d \) or 2) \( b = d \) and \( a < c + \theta \), then Problem (14) is infeasible.

Under the conditions of Lemma 5, the firm will eventually go bankrupt for all values of \((Y, M, \gamma)\).
Lemma 6. If 1) $b > d$ and $(b - d) \geq (c - a) + \theta$ and $(\beta + 1)(\hat{A} + 1) \sum_{t=1}^{T} \frac{R_t}{(1+\delta)^t} - (1 - \alpha) \geq 0$ or 2) $b = d$ and $a \geq c + \theta$, then $Y^*(T) = 0$ for all $T \in \mathbb{N}$ in Problem (11).

Under either of the conditions of Lemma 6, the firm does not need any outside funding and is cash-flow positive at or above $\theta$ for all periods $\tau \geq 1$.

Another set of conditions where the firm is cash-flow positive at or above $\theta$ for all periods $\tau \geq 1$ is $b > d$ and $b - d \geq (c - a) + \theta$ and $(\beta + 1)(\hat{A} + 1) \sum_{t=1}^{T} \frac{R_t}{(1+\delta)^t} - (1 - \alpha) < 0$. In the next lemma, we show that under these conditions, if feasible, the firm borrows money from investors although it is already cash-flow positive at or above $\theta$. The reason is that, under these conditions, the firm’s gain from raising an investment at time $t = 0$ is larger than the NPV of its future monthly payments to the investors and the platform, due to its high discount rates.

Lemma 7. If $b > d$ and $b - d \geq (c - a) + \theta$ and $(\beta + 1)(\hat{A} + 1) \sum_{t=1}^{T} \frac{R_t}{(1+\delta)^t} - (1 - \alpha) < 0$, then for $\theta \leq \theta_L(T)$, where $\theta_L(T)$ is a function of $T$ and problem parameters, we have for all $T \in \mathbb{N}$:

- if $cb - da \geq 0$, then $Y^*(T) = \min \{ f([\tau^{**}]), f([\tau^{**}]) \}$, where $\tau^{**}$ is real and equal to

$$\tau^{**} = \frac{(1 - \alpha)(b - d) - \eta(T)b + \sqrt{((1 - \alpha)(b - d) - \eta(T)b)^2 - \eta(T)(cb - da)((2(c - a) - (b - d))(1 - \alpha) + \eta(T)(2a + b))}}{\eta(T)(cb - da)}.$$  

- if $cb - da < 0$, then $Y^*(T) = (b - d)/\eta(T)b$.

We have characterized all combinations of $(a, b, c, d)$, except the case where $b > d$ and $b - d < (c - a) + \theta$. We break this case into two sub-cases: i) $b > d$ and $b - d < (c - a) + \theta$ and $cb > da$, and ii) $b > d$ and $b - d < (c - a) + \theta$ and $cb \leq da$. These cases result in Problem (14) being feasible with a non-trivial solution. In particular, Case i includes firms that have cash-flow shortages for a limited time, but have expectations of positive cash flows in the future. Case ii includes firms
that have positive cash flows, but they are below \( \theta \), and the firm receives investment to increase its cash flow to at least \( \theta \). Recall that \( \theta \) is a cash buffer in the deterministic approximation problem to account for cash-flow uncertainties in the stochastic problem. We begin by characterizing the set \( X \) for both these cases.

**Lemma 8.** If \( b > d \) and \( b - d < (c - a) + \theta \), then \( X = \{ \tau \in \mathbb{N} : 1 \leq \tau < \frac{(c-a-b-d)\theta + \sqrt{(c-a-b-d)^2 + 2\theta(b-d)}}{b-d} \} \).

From the set \( X \) it is clear that as \( \theta \) increases, the number of periods where the firm has a cash flow below \( \theta \) increases. The following lemma indicates that the problem is not feasible for \( \theta \) larger than a threshold.

**Lemma 9.** The first constraint in Problem (14), \( Z_\tau(T) < 0 \) for all \( \tau \in X \), is equivalent to the conditions \( \theta \leq \tilde{\theta}(T) \) and \( (ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1-\alpha)b/\eta(T)} \geq 0 \), where \( \tilde{\theta}(T) \) is a function of \( T \) and problem parameters.

Note that the left-hand side of the second condition in Lemma 9 is increasing in \( T \), albeit asymptotically, due to \( \eta(T) \). Therefore, if it is possible for the first constraint of Problem (14) to hold, it will be feasible for a large enough \( T \) and a small enough \( \theta \). We next analyze Cases i and ii, building upon the condition in Lemma 9, to address the second constraint in Problem (14).

**Case i:** \( b > d \) and \( b - d < (c - a) + \theta \) and \( cb > da \). The next three lemmas characterize the second constraint in Problem (14) for Case i: \( b > d \) and \( b - d < (c - a) + \theta \) and \( cb > da \), assuming the feasibility conditions of the first constraint in Lemma 9 hold.

**Lemma 10.** If \( b > d \), \( cb > da \), and \( (ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1-\alpha)b/\eta(T)} \geq 0 \), then for \( \theta \leq \tilde{\theta}(T) \) and \( b - d < (c - a) + \theta \), where \( \tilde{\theta}(T) \) is a function of \( T \) and problem parameters:

a) If \( \theta > \frac{(b-d)-2(c-a)(1-\alpha)}{\eta(T)(2a+b)} \), then \( \max_{\tau \in X} f(\tau) = \max\{f(\lceil \tau^{*} \rceil), f(\lfloor \tau^{*} \rfloor)\} \), where \( \tau^{*} \) is real and equal to

\[
\tau^{*} = \frac{(1-\alpha)(b-d) - \eta(T)\theta b - \sqrt{((1-\alpha)(b-d) - \eta(T)\theta b)^2 - \eta(T)(cb-da)((2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b))}}{(cb-da)\eta(T)}.
\]

b) Otherwise, \( \max_{\tau \in X} f(\tau) = \frac{a + b - c - d - \theta}{\eta(T)(a+b) - (1-\alpha)} \).

**Lemma 11.** If \( b > d \), \( cb > da \), and \( (ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1-\alpha)b/\eta(T)} \geq 0 \), then for \( \theta \leq \tilde{\theta}(T) \) and \( b - d < (c - a) + \theta \), where \( \tilde{\theta}(T) \) is a function of \( T \) and problem parameters, we have

\[
\min_{\tau \in X} \{f(\tau)\} = \min\{f(\lceil \tau^{**} \rceil), f(\lfloor \tau^{**} \rfloor)\}, \text{ where } \tau^{**} \text{ is real and equal to}
\]

\[
\tau^{**} = \frac{(1-\alpha)(b-d) - \eta(T)\theta b + \sqrt{((1-\alpha)(b-d) - \eta(T)\theta b)^2 - \eta(T)(cb-da)((2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b))}}{(cb-da)\eta(T)}.
\]
LEMMA 12. If \( b > d \), \( cb > da \), and \((ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0\), then \( \exists \, \bar{\theta}(T) \) such that for \( \theta \leq \bar{\theta}(T) \) and \( b - d < (c - a) + \theta \), where \( \bar{\theta}(T) \) is a function of \( T \) and problem parameters, the inequality \( \max_{\tau \in \mathcal{X}} \{ f(\tau) \} \leq \min_{Z_{\tau(T)} \geq 0} \{ f(\tau) \} \) holds for \( T \) large enough.

Lemmas 10 – 12 prove that, for linear cash flows and Case i, if \( \theta \) is small enough and \( T \) is large enough, Problem (14) is feasible. We next accomplish the same task for Case ii.

**Case ii:** \( b > d \) and \( b - d < (c - a) + \theta \) and \( cb \leq da \). The next three lemmas characterize the second constraint in Problem (14) for Case ii: \( b > d \) and \( b - d < (c - a) + \theta \) and \( cb \leq da \), assuming the feasibility conditions of the first constraint in Lemma 9 hold.

LEMMA 13. If \( b > d \) and \( cb \leq da \), and \((ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0\), then for \( \theta \leq \bar{\theta}(T) \) and \( b - d < (c - a) + \theta \), where \( \bar{\theta}(T) \) is a function of \( T \) and problem parameters, we have \( \max_{\tau \in \mathcal{X}} f(\tau) = \frac{a + b - c - d - \theta}{\eta(T)(a + b) - (1 - \alpha)} \).

LEMMA 14. If \( b > d \) and \( cb \leq da \), and \((ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0\), then for \( \theta \leq \bar{\theta}(T) \) and \( b - d < (c - a) + \theta \), where \( \bar{\theta}(T) \) is a function of \( T \) and problem parameters, we have \( \min_{Z_{\tau(T)} \geq 0} \{ f(\tau) \} = (b - d)/\eta(T)b \).

Note that from \( b > d \) and \( cb \leq da \), we conclude \( a \geq c \). Conditions \( b > d \) and \( a \geq c \) represent a firm that has higher revenues than costs in all periods, with the revenue growth larger than that of cost; however, the firm’s cash flow is not above \( \theta \) in all periods, which drives the need for investment.

LEMMA 15. If \( b > d \) and \( cb \leq da \), and \((ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0\), then \( \exists \, \bar{\theta}(T) \) such that for \( \theta \leq \bar{\theta}(T) \) and \( b - d < (c - a) + \theta \), where \( \bar{\theta}(T) \) is a function of \( T \) and problem parameters, the inequality \( \max_{\tau \in \mathcal{X}} \{ f(\tau) \} \leq \min_{Z_{\tau(T)} \geq 0} \{ f(\tau) \} \) holds.

The conclusion of Lemmas 12 and 15, \( \max_{\tau \in \mathcal{X}} \{ f(\tau) \} \leq \min_{Z_{\tau(T)} \geq 0} \{ f(\tau) \} \), guarantees that the firm is cash-flow positive at or above \( \theta \) in all months for Cases i and ii, respectively, which is only possible if \( \theta \) is small enough. Lemmas 2 – 3 prove that the inequality \( \max_{\tau \in \mathcal{X}} \{ f(\tau) \} \leq \min_{Z_{\tau(T)} \geq 0} \{ f(\tau) \} \), if feasible, is feasible for \( T \) large enough. Thus, Problem (14), under linear cash flows and Cases i and ii, is feasible for \( T \) large enough and \( \theta \) small enough. The results in Lemmas 7, 10, and 13 provide closed-form solutions for \( Y^*(T) \) under all non-trivial cases and Proposition 1 provides closed-form solutions for \( M^*(T) \), and \( \gamma^*(T) \) as a function of \( Y^*(T) \). We found that \( T = 120 \) worked well to generate high-quality approximations for problems parameterized by real Bolstr data. These optimal variables for Problem (14) are used as approximate solutions for the stochastic model in Problem (6), whose quality we explore in the next section.

To complete the analysis of the deterministic problem, we now collect all these results to solve Problem (14) for the cases where \( r_t = r \geq \delta = \delta_t, \forall t \) or \( r_t = \delta_t, \forall t \). Proposition 1, and Lemmas 7, 10,
and 13 for linear cash flows, provide closed-form solutions for the optimal \( \gamma^*(T), M^*(T), \) and \( Y^*(T), \) assuming model feasibility and a fixed \( T \in \mathbb{N} \). Lemma 4 indicates that the objective of Problem (14) is strictly increasing in \( T \). Lemmas 1 – 3 prove that Problem (14), if feasible, is feasible for large enough \( T \). We also note that \( \lim_{T \rightarrow \infty} \eta(T) = \lim_{T \rightarrow \infty} \frac{\hat{A} + 1}{(\hat{A} + 1)} = (\hat{A} + 1)/(\hat{A} + 1) \). Lemmas 11 – 12 and 14 – 15 prove, for Cases i and ii, respectively, that for linear cash flows and large enough \( T \), the model is feasible. Together, these results solve Problem (14) for linear costs and revenues, which we summarize in the next propositions. Specifically, in Proposition 2 we characterize the optimal solutions for \( (\beta + 1)(\hat{A} + 1)/R^\infty \) for \( \theta \leq \lim_{T \rightarrow \infty} \theta(T) \), and \( (ad - bc) + (b - d)\sqrt{(a + b)^2 + 2(1 - \alpha)b}\frac{R^\infty}{(\hat{A} + 1)(\beta + 1)} \geq 0 \), then

**Proposition 2.**

For \( (\beta + 1)(\hat{A} + 1)/R^\infty \) and \( (ad - bc) + (b - d)\sqrt{(a + b)^2 + 2(1 - \alpha)b}\frac{R^\infty}{(\hat{A} + 1)(\beta + 1)} \geq 0 \), then \( T^* = \infty \), \( M^* = \infty \), \( Y^* = \max\{f(\tau^*), f(\tau^{**})\}, \gamma^* = \frac{(\hat{A} + 1)Y^*}{R^\infty} \), where

\[
\tau^* = \frac{(1 - \alpha)(b - d) - \theta b(\hat{A} + 1)(\beta + 1)/R^\infty}{(cb - da)(\hat{A} + 1)(\beta + 1)/R^\infty}
\]

\[
\sqrt{\left((1 - \alpha)(b - d) - \theta b(\hat{A} + 1)(\beta + 1)/R^\infty\right)^2 - (\hat{A} + 1)(\beta + 1)/R^\infty(cb - da)\left((2(c - a) - (b - d))(1 - \alpha) + (\hat{A} + 1)(\beta + 1)/R^\infty\theta(2a + b)\right)}
\]

Otherwise, \( T^* = \infty \), \( M^* = \infty \), \( Y^* = \frac{a + b + c - d - \theta}{(\hat{A} + 1)(\beta + 1)(\alpha + b)/R^\infty - (1 - \alpha)} \), \( \gamma^* = \frac{(\hat{A} + 1)Y^*}{R^\infty} \).

The firm’s maximized NPV is \( \hat{z}_P = \sum_{t=1}^{\infty} \frac{a - c + (b - d)l_t}{(1 + \gamma_t)^t} - (\beta + 1)(\hat{A} + 1)/R^\infty \). \( \tau^{**} \) is real and equal to

\[
\tau^{**} = \frac{(1 - \alpha)(b - d) - \theta b(\hat{A} + 1)(\beta + 1)/R^\infty}{(cb - da)(\hat{A} + 1)(\beta + 1)/R^\infty}
\]

\[
\sqrt{\left((1 - \alpha)(b - d) - \theta b(\hat{A} + 1)(\beta + 1)/R^\infty\right)^2 - (\hat{A} + 1)(\beta + 1)/R^\infty(cb - da)\left((2(c - a) - (b - d))(1 - \alpha) + (\hat{A} + 1)(\beta + 1)/R^\infty\theta(2a + b)\right)}
\]

and \( T^* = \infty \), \( M^* = \infty \), \( \gamma^* = \frac{(\hat{A} + 1)Y^*}{R^\infty} \). The firm’s maximized NPV is \( \hat{z}_P = \sum_{t=1}^{\infty} \frac{a - c + (b - d)l_t}{(1 + \gamma_t)^t} - (\beta + 1)(\hat{A} + 1)/R^\infty \).
From Propositions 2 and 3, we see that if \( r_t = r \geq \delta = \delta_t, \forall t \) or \( r_t = \delta_t, \forall t \), then \( Y^* \) and \( \gamma^* \) are finite, but \( M^* \) and \( T^* \) are infinite. This solution corresponds to a financial perpetuity (an annuity with no termination) with non-fixed payments that are a function of firm revenues. Note that perpetuities are common in modern business (e.g., anyone can purchase a perpetuity through Bank of America’s Merrill Edge brokerage). More prominent examples include LeBron James, a four-time NBA MVP, and soccer star David Beckham having lifetime contracts with Nike and Adidas, respectively (Novy-Williams 2015). Furthermore, Gerber et al. (2012) provides a study of three crowdfunding platforms and showed that many individuals and firms use crowdfunding to make direct and long term connections with investors; a perpetuity precisely achieves a long-term connection.

Our perpetuity solution can also be interpreted as a pseudo type of equity crowdfunding. Ross et al. (2002) explains that the present value of stock is equivalent to the discounted present value of all future dividends, which are typically a share of profits (page 198 of O’Sullivan and Sheffrin (2007)). If we replace profits with revenues, we obtain our perpetuity contract. In Section 5 we formally study the difference between our model and an equity crowdfunding model that shares profits, rather than revenues, and we show that the revenue-sharing contract is superior.

4. Analysis of Stochastic Model

4.1. Simulation-based Numerical Optimization of Stochastic Model

In order to solve Problem (6), we let revenues \( R_t \) and costs \( C_t \) of each of the 56 Bolstr campaigns follow the random walk processes in (8). We set the highest allowable values of \( \sigma^r \) and \( \sigma^c \) as \( \mu^r / 3 \) and \( \mu^c / 3 \), respectively, so that revenues and costs are non-negative with high probability. More specifically, we consider \( \sigma^r = \mu^r / k \) and \( \sigma^c = \mu^c / k \) for \( k \in \{3, 4, 5, 6, 7, 8, 9, 10, 15\} \), along with the additional deterministic case of \( k \to \infty \).

In our base parameter set, we let \( r_t = \delta_t = 0.01 \), for all \( t \), \( \alpha = 0.05 \), \( \beta = 0.01 \), and \( \hat{A} = 0.1 \) (i.e., a 10% NPV return for investors); note that \( r \) is the discount rate per period, which is typically a month and that the choice of \( \alpha = 0.05 \) and \( \beta = 0.01 \) is supported by a Bolstr memorandum. Many online platforms such as Bolstr, LendingClub, and Prosper charge 1% servicing fee for collecting and processing payments. These online lenders usually charge borrowers origination fees of typically 1% – 8%.

We approximate an infinite horizon by considering \( t \in \{1, \ldots, N\} \) where \( N = 1000 \); we selected this value of \( N \) so that \( N > \max\{B, T\} \) holds with high probability; furthermore, there is no evidence that revenue-sharing contracts at Bolstr and Localstake last longer than 1000 months, and as pointed out earlier most Bolstr campaigns last 2-5 years. For analyzing Problem (6) numerically, we discretized the \((Y, M, \gamma)\) space. In particular, we considered values of \( Y \in \{0, \Delta Y, 2\Delta Y, \ldots, \overline{Y}\} \),
M \in \{1 + \Delta M, 1 + 2\Delta M, \ldots, \bar{M}\}, \quad \text{and} \quad \gamma \in \{0, \Delta \gamma, 2\Delta \gamma, \ldots, \bar{\gamma}\}, \quad \text{where} \quad (\Delta Y, \Delta M, \Delta \gamma, \bar{Y}, \bar{M}, \bar{\gamma}) = (5000, 0.25, 0.01, 2000000, 3, 1) \quad \text{were chosen to balance computational time and solution quality.}

For each campaign, we used Monte Carlo simulation to generate \(m = 1000\) realizations of the \((R_1, \ldots, R_N)\) and \((C_1, \ldots, C_N)\) vectors, which allowed us to generate \(m\) realizations of the \(T\) and \(B\) random variables for each \((Y, M, \gamma)\) tuple in the discretized set. Then, for each variable tuple, averaging over the \(m\) trials, we estimate \(E\left(\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r)^t}\right)\), \(E\left(\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r)^t}\right)\), and \(E\left(\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+b)^t}\right)\), which allowed us to evaluate the feasibility of the variable tuple. Finally, we evaluated the objective function for each feasible \((Y, M, \gamma)\) tuple and chose the one that maximizes the objective function as the optimal solution. Depending on the standard deviations, 90-95% of the 56 Bolstr campaigns were feasible.

4.2. Evaluation of Deterministic Approximation in Section 3

In this subsection, we evaluate the value of the approximate Problem (14) with respect to that of the stochastic Problem (6), for \(T\) large enough. For the numerical results presented below, we consider \(T = 120\), though the performance is insensitive for larger \(T\). In Section 4.3 we show that if we use the approximation solutions in the true stochastic model, the expected investment horizon \(T\) is close to the choice of \(T = 120\). The main parameters that we vary for each campaign are the standard deviations \(\sigma^r\) and \(\sigma^c\); all other problem parameters are listed above. As mentioned previously, we consider \(\sigma^r \in [0, \mu^r/3]\) and \(\sigma^c \in [0, \mu^c/3]\), so that revenues and costs are non-negative with high probability.

For each level of variability, we find the optimal \(\theta \in [0, 2M]\) which makes the approximation solutions \((Y^*(\theta), M^*(\theta), \gamma^*(\theta))\) feasible for Problem (6) and results in the minimum approximation error for the stochastic problem. We denote this value of \(\theta\) by \(\theta^*\) in the sequel. In the left panel of Figure 3 we show that as cash-flow variability increases, \(\theta^*\) increases to provide a larger cash buffer to absorb the additional variability.

The right panel of Figure 3 shows the percentage of feasible campaigns for Problem (6) that are also feasible for Problem (11), for \(\theta^*\), and can therefore be approximated via the approximation Problem (14). As the level of cash-flow uncertainty decreases, the approximation Problem (14) provides an approximate solution for Problem (6) for almost all campaigns.

For evaluating the quality of the approximation, we input the approximation solutions \((Y(\theta^*), M(\theta^*), \gamma(\theta^*))\) into Model (6) and compare the firm’s expected NPV under this solution with the true optimal expected NPV \(z_F\), calculated numerically. In the left panel of Figure 4, we present the average error, over all feasible Bolstr campaigns, as a function of the standard deviations \(\sigma^r\) and \(\sigma^c\). The length of each bar above and below the average value is equal to the standard deviation of the error over feasible campaigns. We observe that for the highest level of variability,
Figure 3 Left: The average of $\theta^*$ over feasible campaigns. Right: Percentage of feasible campaigns for Problem (6) which are feasible for Problem (14).

$\sigma_r = \mu_r / 3$ and $\sigma_c = \mu_c / 3$, the average error is 0.2% with a standard deviation of 0.4%. As variability decreases, the mean and standard deviation of the errors decrease such that for $k \to \infty$ the mean and standard deviation of the errors are 0.003% and 0.01%, respectively. For $k = 6$ ($\sigma_r = \mu_r / 6$ and $\sigma_c = \mu_c / 6$) the average error is within 0.03%. Consequently, we expect that the approximation quality is acceptable for reasonable levels of variability encountered in practice, and we conclude that the approximation problem provides tractable solutions for the intractable stochastic problem.

Figure 4 Left: Evaluation of the approximation quality of Problem (14) for stochastic Problem (6). Right: $z_F$ over feasible campaigns.

Finally, the average values of $z_F$, as a function of cash-flow variability, are shown in the right panel of Figure 4. As variability decreases, $z_F$ is increasing rather significantly for higher values of cash-flow variability and very slowly for smaller values of variability. In other words, firms
can increase their maximized expected NPV significantly by slightly decreasing their cash-flow uncertainty in very uncertain environments. However, when cash-flow uncertainty is not significant, the maximized expected NPV is slowly decreasing in the standard deviations $\sigma^r$ and $\sigma^c$, and is rather robust to their precise values.

4.3. Estimations of the $T$ and $B$ Distributions

In this subsection, we analyze the distributions of $B$, the firm’s stochastic bankruptcy time, and $T$, the stochastic duration of the contract, for feasible campaigns under both the stochastic and approximation solutions. We present results for two qualitatively different campaigns that are feasible for all $k \geq 3$. One campaign, Campaign 1, has a relatively high bankruptcy probability for $k = 3$ (as representative of a risky firm) and the other campaign, Campaign 2, has a zero bankruptcy probability for $k = 3$ (as representative of a risk-less firm). The estimated probabilities $P(B < \infty)$ are given in Table 1, for the optimal and approximate solutions, respectively, as a function of $k$. We also evaluated the estimated probabilities $P(B < T)$, and they are identical to those in Table 1. We see that, as cash-flow variability decreases, the firm’s bankruptcy probability decreases for Campaign 1 (and Campaign 2’s probability remains at zero) for both optimal and approximate variables; furthermore, for a given $k$, the probabilities are mostly identical across the two sets of variables, providing further evidence of the quality of the approximation.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$P(B &lt; \infty)$ for Campaign 1</th>
<th>$P(B &lt; \infty)$ for Campaign 2</th>
<th>$P(B &lt; \infty)$ for Campaign 1</th>
<th>$P(B &lt; \infty)$ for Campaign 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.061</td>
<td>0</td>
<td>0.059</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.008</td>
<td>0</td>
<td>0.005</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.001</td>
<td>0</td>
<td>0.002</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.001</td>
<td>0</td>
<td>0.001</td>
<td>0</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 5 shows the average of $P(B < \infty)$ over feasible Bolstr campaigns under both stochastic and approximation solutions. The approximation solutions result in almost the same bankruptcy probabilities as the stochastic optimal solutions, for feasible Bolstr campaigns over different levels of cash-flow variability.

In the left plots of Figures 6 and 7, we provide the estimated distributions of $T$ for $k \in \{3, 4, 5, \infty\}$ for Campaigns 1–2, respectively, for Problem (6) under the stochastic optimal solutions. Visually, we see that the mean and standard deviation of $T$ for the stochastic optimal solutions are decreasing in $k$ (we confirmed this numerically). In other words, as $k$ increases, the cash-flow standard deviations $\sigma^r$ and $\sigma^c$ decrease, resulting in a problem with less uncertainty, which requires less...
investment \( Y^* \), and can be paid back earlier with more certainty about the investment duration. Furthermore, when \( k \to \infty \), resulting in \( \sigma^r = \sigma^c = 0 \), our stochastic model returns a deterministic investment duration.

In the right plots of Figures 6 and 7, we see the estimated distributions of \( T \) for \( k \in \{3, 4, 5, \infty\} \) for Campaigns 1–2, respectively, for Problem (6) under the approximation solutions. Similar to the left plots, the standard deviation of \( T \) is decreasing in \( k \). However, the mean of the investment horizon is approximately the same for different levels of variability, and close to \( T = 120 \), which we set earlier for the approximation problem. Therefore if we use the approximation solution for setting the contract terms \( (Y, M, \gamma) \), for \( T \) large enough, then the expected investment horizon is close to \( T \) for all firms with different levels of variability.

Figure 5  The average of \( P(B < \infty) \) over feasible campaigns.

![Figure 5](image)

Figure 6  Estimated distributions of \( T \) in the stochastic problem for \( (Y^*, M^*, \gamma^*) \) (left) and for \( (Y(\theta^*), M(\theta^*), \gamma(\theta^*)) \) (right), for Campaign 1.

![Figure 6](image)
Finally, since we observed that the firm can complete the repayment of the revenue-sharing contract within our horizon of $N = 1000$ months, this value of $N$ is unlikely a limiting factor of our numerical analysis. We also expect that it is unlikely that the firm will go bankrupt at some time $t > N$, under the less restrictive financial conditions due to the absence of repayment requirements, and we believe our estimates of the probability of firm bankruptcy are not limited by the value of $N = 1000$.

4.4. Sensitivity Analysis for Investors’ Opportunity Cost

In this subsection, we perform sensitivity analysis for Problem (6) with respect to investors’ opportunity cost $\hat{A}$. Figure 8 shows the average of $z_F$ over the feasible Bolstr campaigns, for different values of $\hat{A}$, as a function of $k$. In this subsection we are analyzing the firm’s problem for the cases where investors’ opportunity cost is high, and specifically we are analyzing the optimal solutions under a revenue-sharing contract with $\hat{A} \in \{0.1, 0.2, 0.3, 0.4\}$.

As shown in Figure 8, as investors’ opportunity cost increases, the firm’s maximized NPV decreases. However, the firm’s maximized NPV is rather robust with respect to $\hat{A}$ under a flexible revenue-sharing contract. Therefore, firms can benefit from flexible revenue-sharing contracts even with investors with moderately high opportunity costs, and the benefit increases as cash-flow variability decreases.

5. Equity Crowdfunding

In this section, we analyze an equity crowdfunding investment, and compare it with our revenue-sharing contract. According to pages 33-34 of Bradford (2012), investors purchase equity from the firm and receive returns in the form of profit-sharing. Similarly, Ross et al. (2002) explains that the present value of equity is equal to the discounted present value of all future dividends, which are typically shares of profit (O’Sullivan and Sheffrin 2007).
As in the analyses of previous sections, we assume that the firm raises \( Y \) from \( n \) investors such that \( Y = \sum_{i=1}^{n} y_i \). Investors, in exchange for their equity investments, collectively receive \( \eta \) percent of the firm’s monthly profit indefinitely, divided among the investors proportionally to their individual investment amounts.

As in our previous analyses, the investors’ discount rate in period \( t \) is \( \delta_t \), and investor \( i \)’s expected NPV is given by \( E \left[ \sum_{t=1}^{B} \frac{y_i}{Y} \frac{\eta \max\{R_t - C_t, 0\}}{(1 + \delta_t)^t} \right] \), where \( \max\{R_t - C_t, 0\} \) captures profit sharing. Thus, the investors’ participation constraints are \( E \left[ \sum_{t=1}^{B} \frac{y_i}{Y} \frac{\eta \max\{R_t - C_t, 0\}}{(1 + \delta_t)^t} \right] - y_i \geq A_i, \ i = 1, \ldots, n \Leftrightarrow E \left[ \sum_{t=1}^{B} \frac{\eta}{Y} \frac{\max\{R_t - C_t, 0\}}{(1 + \delta_t)^t} \right] \geq \hat{A} + 1 \). As in our previous analyses, we assume that the platform charges an origination fee \( \alpha \) and a servicing fee \( \beta \). The firm’s discount rate in period \( t \) is again \( r_t \). The expected NPV of the firm after the crowdfunding investment (from the firm’s perspective) is \( V \triangleq E \left[ \sum_{t=1}^{B} \frac{R_t - C_t}{(1 + r_t)^t} \right] + (1 - \alpha)Y - (\beta + 1)\eta E \left[ \sum_{t=1}^{B} \frac{\max\{R_t - C_t, 0\}}{(1 + \delta_t)^t} \right] \). As a result of their investment, investors collectively own a \( \frac{\eta}{Y} \) proportion of the company’s value, and the remaining \( (1 - \frac{\eta}{Y}) \) proportion belongs to the firm. The expected NPV of the firm’s remaining ownership is given by \( (1 - \frac{\eta}{Y}) V \), which results in the firm’s maximization problem being defined as

\[
z_E = \max_{Y, \eta} E \left[ \sum_{t=1}^{B} \frac{R_t - C_t}{(1 + r_t)^t} \right] - \alpha Y - (\beta + 1)\eta E \left[ \sum_{t=1}^{B} \frac{\max\{R_t - C_t, 0\}}{(1 + r_t)^t} \right] \\
\text{s.t. } B = \min \left\{ \hat{B} \geq 1: \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\eta \sum_{t=1}^{\hat{B}} \max\{R_t - C_t, 0\} < 0 \right\} \\
E \left[ \sum_{t=1}^{B} \frac{\eta}{Y} \frac{\max\{R_t - C_t, 0\}}{(1 + \delta_t)^t} \right] \geq \hat{A} + 1 \\
Y, \eta \geq 0; \quad (17)
\]

for simplicity, we assume that investors never sell their equity and only benefit from the dividends of profit sharing.
Figure 9 presents the results of numerical comparisons between revenue-sharing and equity crowdfunding contracts, where investors’ opportunity cost $\hat{A} = 0.1$ and $r_t = \delta_t = 0.01, \forall t$. The left plot of Figure 9 presents the ratios $z_E/z_F$, where $z_F$ ($z_E$) is the maximized NPV of revenue-sharing (equity) crowdfunding, averaged over feasible Bolstr campaigns, as a function of the volatility of cash flows. Since these ratios are less than one, they can be interpreted as the percentages of the revenue-sharing NPV that are attained by the equity crowdfunding contract. We conclude that, if feasible, a firm can attain a higher NPV under a revenue-sharing contract than an equity crowdfunding contract. However, we note that 4 out of 56 campaigns were feasible for high levels of uncertainty under equity crowdfunding, but not under revenue-sharing crowdfunding. The right plot of Figure 9 presents the firm’s bankruptcy probabilities for both revenue-sharing and equity crowdfunding, again averaged over feasible Bolstr campaigns, as a function of the volatility of cash flows. We observe that the firm’s bankruptcy probabilities under revenue-sharing contracts are within 0.1% of its bankruptcy probabilities under equity crowdfunding contracts. Furthermore, for low and medium cash-flow volatility levels, revenue-sharing contracts have lower bankruptcy probabilities. In summary, similar bankruptcy probabilities and higher NPVs highlight the superiority of our proposed revenue-sharing contract over equity crowdfunding contracts.

\section*{6. Fixed-Rate Loans}

In this section we analyze the performance of a standard fixed-rate loan, offered by other traditional and alternative lenders, to serve as a benchmark for the revenue-sharing contract. We assume these lenders offer fixed-rate loans with flexible payment terms: The firm can decide on the investment amount $Y$ (e.g. dollars) and the duration $D$ (e.g. months) for a loan with a fixed interest rate of $s$ per period (e.g. month). For simplicity in our analysis, we consider $s$ to be constant with respect to
$D$, and we show that, even for this conservative interest rate structure, revenue-sharing contracts result in higher NPVs for firms.

The loan payment per period is a standard amortization and is equal to $\frac{sY}{1 - (1 + s)^{-D}}$ (Stoft 2002). We assume the lender charges the borrower an origination fee of $w$ percent of $Y$. The firm’s maximization problem can be written as:

$$
\begin{align*}
z_L &= \max_{Y,D} E \left[ \sum_{t=1}^{B} \frac{R_t - C_t}{(1 + r_t)^t} \right] - E \left[ \sum_{t=1}^{\min\{B,D\}} \frac{sY}{(1 - (1 + s)^{-D})(1 + r_t)^t} \right] + (1 - w)Y \\
\text{s.t. } &B = \min\left\{ \hat{B} \geq 1 : \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1 - w)Y - \sum_{t=1}^{\min\{\hat{B},D\}} \frac{sY}{1 - (1 + s)^{-D}} < 0 \right\} \text{ (bankruptcy definition)} \\
&Y, D \geq 0.
\end{align*}
$$

(18)

Next, we solve the above problem numerically for $Y$ and $D$. For this purpose, we need a base parameter set for the monthly interest rate $s$. For a conservative comparison, we consider $w = 0$. Interest rates and eligibility requirements vary across lenders. Banks usually have more strict eligibility requirements but they offer relatively lower interest rates than some online lenders who have less stringent eligibility requirements. We consider monthly interest rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ for the numerical analysis.

We performed the numerical analysis for $D \in [1, 120]$ months and $Y \in [0, 2000000]$. We selected the upper bound of $D$ to be 120 to account for long term loans offered by the US Small Business Administration (SBA) and banks that give more flexibility to firms. The upper bound of $Y$ is chosen so that it is consistent with the numerical analysis for Problem (6). Figure 10 presents the firm’s maximized NPV under loans with monthly fixed-rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ as a ratio to the firm’s maximized NPV under revenue-sharing contracts, with investors’ opportunity costs $\hat{A} \in \{0.1, 0.2\}$. The ratios in Figure 10 are interpreted as percentages of the revenue-sharing NPV that are attained by the fixed-rate loans. Therefore, a ratio less than one indicates that the firm’s maximized NPV is higher under a revenue-sharing contract and a ratio greater than one indicates that the firm’s maximized NPV is higher under a fixed-rate loan.

The firm’s maximized NPV under a revenue-sharing contract with $\hat{A} \in \{0.1, 0.2\}$ is higher than loans with monthly fixed-rates as small as $s = 0.07/12$. The firm’s benefit from a flexible revenue-sharing contract is more significant when cash-flow variability is high. For a revenue-sharing contract with $\hat{A} = \{0.1, 0.2\}$, the firm benefits from loans with a monthly fixed-rate $s = 0.03/12$, but not loans with higher rates, especially for higher levels of cash-flow uncertainty. However, most loans offered to firms, especially small to medium-sized firms, have relatively high interest rates. This underscores the importance of flexible payments that are possible under revenue-sharing contracts. Additionally, most low-rate loans, such as those offered by the SBA, have a processing time
of 2–3 months. In contrast, according to marketing emails from Bolstr, firms can raise investments under revenue-sharing contracts within hours.

Figure 11 shows the average bankruptcy probabilities over feasible Bolstr campaigns under loans with monthly fixed-rates \( s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\} \) and under revenue-sharing contracts with investors’ opportunity costs \( \hat{A} \in \{0.1, 0.2\} \).

The bankruptcy probability for a firm under a revenue-sharing contract with \( \hat{A} \leq 0.2 \) is smaller than that of loans with monthly rates as small as \( s = 0.14/12 \). The highest bankruptcy probability is for a loan with monthly rate \( s = 0.21/12 \) and this again underscores the importance of flexible revenue-sharing contracts, especially for firms with higher levels of cash-flow variability.

We conclude that firms benefit significantly more from revenue-sharing contracts than traditional fixed-rate loans, if investors’ opportunity costs are not too large or if loans have non-negligible interest rates. These results are of great importance to small firms which usually have higher
levels of cash-flow uncertainty and need investment; according to a report by the SBA (U.S. Small Business Administration 2016), 73% of small firms used some type of financing in 2015 – 2016.

7. Conclusion
In this paper we analyzed an emergent model of crowdfunding in which a firm borrows capital and then pays back investors via revenue sharing contracts. Specifically, the firm pays the investors a percentage of its revenues monthly until a predetermined investment multiple is paid, over an uncertain investment horizon.

This paper is the first, to our knowledge, that studies this new model of crowdfunding. This model is facilitated by a platform (e.g., Bolstr, Localstake, or Startwise) that matches investors with a firm needing capital. This new model helps firms in need of investment to survive and thrive with a flexible contract whose terms depend on the firm’s performance. Indeed, when these contracts are used optimally, we provide evidence that the likelihood of firm bankruptcy is small, even for highly variable cash flows, due to the flexible monthly payments facilitated by the contract. If the revenue performance of the firm goes well, then the monthly payments to investors increase, which results in higher effective interest rates for investors. If revenue performance is poor, payments are lowered to reduce financial stress on the firm. We use real data from 56 Bolstr campaigns to motivate and calibrate our analytical models, and to parameterize our numerical studies.

The first part of our paper formulates a stochastic programming model of the firm’s expected NPV maximization problem with the contract parameters as variables, for given values of the platform’s origination and servicing fees, and investors’ opportunity costs; unfortunately, this model is difficult to analyze. Therefore, in the second part of our paper, we formulate a deterministic approximation that we solve analytically. In the approximation problem, we use a cash buffer for dealing with cash-flow uncertainties. In the third part, we evaluate the quality of the approximation solutions for the main stochastic model, over feasible Bolstr campaigns, for different levels of cash-flow uncertainty. We see that the worst-case average error over the campaigns is approximately 0.2%. Therefore, we conclude that the approximation problem provides high quality solutions for the intractable stochastic problem.

In the final part of our paper, we compare revenue-sharing contracts with equity crowdfunding and fixed-rate loans. We find that, for most cases considered, revenue-sharing contracts provide a higher NPV and a lower probability of bankruptcy than equity crowdfunding or a fixed-rate loan. We also show that these benefits are more significant for firms with higher levels of cash-flow uncertainty.

Acknowledgments
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References


Appendix A: Table of Notation

| $R_t$ | stochastic revenue in period $t$ |
| $C_t$ | stochastic cost in period $t$ |
| $r_t$ | firm’s discount rate for cash flows in period $t$ |
| $\delta_t$ | investors’ discount rate for cash flows in period $t$ |
| $y_t$ | investor $i$’s investment amount |
| $A_i$ | investor $i$’s opportunity cost |
| $n$ | number of investors |
| $\alpha$ | platform’s origination fee (% of $Y$) |
| $\beta$ | platform’s servicing fee (% of investors’ payments) |
| $Y$ | firm variable representing total investment amount |
| $\gamma$ | firm variable representing revenue-share rate |
| $M$ | firm variable representing guaranteed investment multiple |
| $T$ | stochastic stopping time representing investment horizon |
| $B$ | stochastic stopping time representing occurrence of bankruptcy |

Appendix B: Proofs

Proof of Proposition 1. Using $\gamma = \frac{MY}{\sum_{t=1}^{\infty} R_t}$, we first substitute out $\gamma$. We then solve the problem in stages. We first fix $Y$ and $T$, and solve for the optimal $M^*(Y,T)$; Problem (11) simplifies to

$$\tilde{z}_P(Y,T) = \max_M \left( 1 - (\beta + 1) \frac{MY}{\sum_{t=1}^{\infty} R_t} \sum_{t=1}^{T} \frac{R_t}{(1 + r_t)^t} + \sum_{t=T+1}^{\infty} \frac{R_t}{(1 + r_t)^t} - \sum_{t=1}^{\infty} \frac{C_t}{(1 + r_t)^t} \right) \frac{1}{1 - (\beta + 1)\sum_{t=1}^{T} \frac{R_t}{(1 + r_t)^t}},$$

s.t. \( \sum_{t=1}^{T} (R_t - C_t) - \theta - (1 - \alpha)Y \geq M, \quad \tau \geq 1 \) (19)

where $\sum_{t=1}^{T} \frac{R_t}{(1 + r_t)^t} > 1$, which implies $M \geq 1$ is redundant. Problem (19), if feasible, has the solution $M^*(Y,T) = (\hat{A} + 1) \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T+1} R_t (1 + r_t)^t}$.

We next solve for the optimal $Y^*(T)$, which is the solution to the following problem

$$\tilde{z}_P(T) = \max_{Y \geq 0} \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1 + r_t)^t} - \left( \beta + 1 \right)(\hat{A} + 1) \frac{\sum_{t=1}^{T} \frac{R_t}{(1 + r_t)^t}}{(1 + r_t)^t} - \left( 1 - \alpha \right)Y \leq \sum_{t=1}^{\tau} (R_t - C_t) - \theta, \quad \tau \geq 1,$$

(20)

the objective is obtained by inserting the optimal $M^*(Y,T)$ into the objective of Problem (19); the constraints are derived by 1) making sure the lower bound on $M$ is smaller than the upper bound on $M$, making Problem (19) feasible, and 2) inserting the optimal $M^*(Y,T)$ into the resulting constraints.

Using Equation (13), the constraints of Problem (20) become $Z_v(T)Y \leq \sum_{t=1}^{T_\tau} (R_t - C_t) - \theta$, for $\tau \geq 1$. If $\tau \in X$ (i.e., $\sum_{t=1}^{T_\tau} (R_t - C_t) < \theta$), then, since $Y \geq 0$, for the feasibility of Problem (20), we require $Z_v(T) < 0$. If $\tau \notin X$ (i.e., $\sum_{t=1}^{T_\tau} (R_t - C_t) \geq \theta$), then we consider two cases: 1) if $Z_v(T) \leq 0$, the constraint $Z_v(T)Y \leq \sum_{t=1}^{T_\tau} (R_t - C_t) - \theta$, for $\tau \geq 1$.
\[\sum_{t=1}^{\tau}(R_t - C_t) - \theta \text{ is trivially true; } 2) \text{ if } Z_\tau(T) > 0, \text{ then } Y \leq \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}. \] Thus, Problem (20) can be written as

\[
\hat{z}_\tau(T) = \max_{Y \geq 0} \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1 + \delta_t)^t} - \left((\beta + 1)(\hat{A} + 1) \sum_{t=1}^{\tau} \frac{R_t}{(1 + \delta_t)^t} - (1 - \alpha)\right) Y
\]

s.t. \( Y \geq \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} \), \( Y \leq \min_{Z_{\tau}(T) > 0} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}. \)

Problem (21) is feasible if and only if \( Z_\tau(T) < 0 \) for all \( \tau \in X \) and

\[
\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} \leq \min_{Z_{\tau}(T) > 0} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\};
\]

If \((\beta + 1)(\hat{A} + 1) \sum_{t=1}^{\tau} \frac{R_t}{(1 + \delta_t)^t} - (1 - \alpha) \geq 0\), then the objective function is a decreasing function of \( Y \) and if feasible, the solution is \( Y^*(T) = \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} \) if \( X \neq \emptyset \) and \( Y^*(T) = 0 \) if \( X = \emptyset \). Otherwise if

\[
(\beta + 1)(\hat{A} + 1) \sum_{t=1}^{\tau} \frac{R_t}{(1 + \delta_t)^t} - (1 - \alpha) < 0,
\]

then the objective function is an increasing function of \( Y \) and if feasible, the solution is \( Y^*(T) = \min_{Z_{\tau}(T) > 0} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}. \)

Proof of Lemma 1. For \( T > \tau \), the solution is

\[
Y^*(T) = \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\},
\]

which is strictly decreasing in \( T \). The lemma’s condition \((\hat{A} + 1)(\beta + 1) \sum_{t=1}^{\tau} \frac{R_t}{(1 + \delta_t)^t} - (1 - \alpha) < 0\) implies that there exists \( T_X \) where, for \( T > T_X \), \( Z_\tau(T) < 0 \) for all \( \tau \in X \).

Proof of Lemma 2. We first show that \( T > \tau \), for all \( \tau \in X \). For a contradiction, suppose that \( \exists \tau_0 \in X \) such that \( T \leq \tau_0 \). The cash-flow constraint of Problem (11) for \( \tau = \tau_0 \) is

\[
\sum_{t=1}^{\tau_0} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{T} R_t \geq \theta.
\]

Using the contractual obligation constraint of Problem (11), \( \gamma \sum_{t=1}^{T} R_t = MY \), this inequality implies that

\[
MY \leq \sum_{t=1}^{\tau_0} (R_t - C_t) - \theta + (1 - \alpha)Y \leq \frac{(1 - \alpha)Y}{(\beta + 1)} \leq Y,
\]

which contradicts the constraint \( M \geq 1 \). Thus, \( T > \tau \) for all \( \tau \in X \).

Noting that, for \( \tau \in X \), \( \sum_{t=1}^{\tau}(R_t - C_t) - \theta < 0 \) and \( Z_\tau(T) < 0 \), we expand the expression \( \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \) using positive expressions:

\[
\sum_{t=1}^{\tau}(R_t - C_t) - \theta = \sum_{t=1}^{\tau}(C_t - R_t) + \theta = \sum_{t=1}^{\tau}(C_t - R_t) + \theta = \sum_{t=1}^{\tau}(C_t - R_t) + \theta,
\]

where the last equality is due to \( T > \tau \), for all \( \tau \in X \). Consequently, as \( T \) is increased, \( \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \) strictly decreases since \( R_t > 0 \). Since all its arguments are strictly decreasing in \( T \), so is the maximization problem

\[
\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}. \]

Proof of Lemma 3. For any \( \tau \notin X \) and \( T \geq \tau \), the minimization argument

\[
\sum_{t=1}^{T}(R_t - C_t) - \theta Z_{\tau}(T) = \sum_{t=1}^{T}(R_t - C_t) - \theta (\hat{A} + 1)(\beta + 1) - (1 - \alpha) \sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)^t} - (1 - \alpha)
\]

is strictly increasing in \( T \). Note that if \( Z_{\tau}(T) \) drops to or below zero, that argument is eliminated from consideration; this is only possible if \((\hat{A} + 1)(\beta + 1)\sum_{t=1}^{T} \frac{R_t}{R_{\infty}} < (1 - \alpha)\) (this condition, in Lemma 1, is for \( \tau \in X \)). Thus, for large enough \( T \), each minimization argument is strictly increasing in \( T \), and we may conclude that \( \min_{\tau \in X} \left\{ \sum_{t=1}^{T}(R_t - C_t) - \theta \right\} \) is also strictly increasing in \( T \). \( \square \)

Proof of Lemma 4. First we prove the lemma for \( r_t = \delta_t, \forall t \): According to Proposition 1, the objective function of Problem (14) is simplified to

\[
\hat{z}_P(T) = \max_{Y \geq 0} \sum_{t=1}^{T} \frac{R_t - C_t}{(1 + r_t)^t} - \left( (\beta + 1)(\hat{A} + 1) - (1 - \alpha) \right) \max_{\tau \in X} \left\{ \sum_{t=1}^{T}(R_t - C_t) - \theta \right\}.
\]

As proved in Lemma 2, \( \max_{\tau \in X} \left\{ \sum_{t=1}^{T}(R_t - C_t) - \theta \right\} \) is decreasing in \( T \) and so the objective function is increasing in \( T \), which completes the proof for \( r_t = \delta_t, \forall t \).

We next prove the lemma for \( r_t = \delta_t, \forall t \) for the cases where 1) \( (\beta + 1)(\hat{A} + 1)\sum_{t=1}^{T} \frac{R_t}{R_{\infty}} - (1 - \alpha) \geq 0 \) and 2) \((\beta + 1)(\hat{A} + 1)\sum_{t=1}^{T} \frac{R_t}{R_{\infty}} - (1 - \alpha) < 0 \) separately. First consider case 1: as shown in Proposition 1, the objective function under this case is simplified to

\[
\max_{Y \geq 0} \sum_{t=1}^{T} \frac{R_t - C_t}{(1 + r_t)^t} - \left( (\beta + 1)(\hat{A} + 1) - (1 - \alpha) \right) \max_{\tau \in X} \left\{ \sum_{t=1}^{T}(R_t - C_t) - \theta \right\}.
\]

The derivative of the objective function with respect to \( T \) is

\[
\frac{d\hat{z}_P(T)}{dT} = - \left( (\beta + 1)(\hat{A} + 1) \sum_{t=1}^{T} \frac{R_t}{(1 + r_t)^t} - (1 - \alpha) \right) \frac{d}{dT} \max_{\tau \in X} \left\{ \sum_{t=1}^{T}(R_t - C_t) - \theta \right\}.
\]

If we show \( \sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)^t} \) is non-increasing in \( T \), then since \( \max_{\tau \in X} \left\{ \sum_{t=1}^{T}(R_t - C_t) - \theta \right\} \) is positive and decreasing in \( T \) and \((\beta + 1)(\hat{A} + 1)\sum_{t=1}^{T} \frac{R_t}{R_{\infty}} - (1 - \alpha) \geq 0 \), we conclude the objective function is increasing in \( T \).

To show \( \sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)^t} \) is non-increasing in \( T \), we calculate differences:

\[
\frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} (1 + \delta_t)^t} = \frac{\sum_{t=1}^{T-1} R_t}{\sum_{t=1}^{T-1} (1 + \delta_t)^t} + \frac{R_t}{\sum_{t=1}^{T-1} (1 + \delta_t)^t} \sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)^t} - \frac{R_t}{\sum_{t=1}^{T-1} (1 + \delta_t)^t} \sum_{t=1}^{T-1} \frac{R_t}{(1 + \delta_t)^t} = \left( \sum_{t=1}^{T-1} \frac{R_t}{(1 + \delta_t)^t} + \frac{R_T}{(1 + \delta_t)^T} \right) \sum_{t=1}^{T-1} \frac{R_t}{(1 + \delta_t)^t} - \left( \sum_{t=1}^{T-1} \frac{R_t}{(1 + \delta_t)^t} + \frac{R_T}{(1 + \delta_t)^T} \right) \sum_{t=1}^{T-1} \frac{R_t}{(1 + \delta_t)^t}.
\]
the problem is infeasible for $b$
and the last inequality above holds due to $\tau \geq \delta$, which completes the proof for case 1. Now consider case 2:

as shown in Proposition 1, the objective function under this case is simplified to

$$\max_{Y \geq 0} \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1+r)^t} - \left( \beta + 1 \right) \left( \sum_{t=1}^{T} \frac{R_t}{(1+r)^t} \right) - (1 - \alpha)$$

The derivative of the objective function with respect to $T$ is

$$\frac{d\hat{z}_P(T)}{dT} = -\left( \beta + 1 \right) \left( \sum_{t=1}^{T} \frac{R_t}{(1+r)^t} \right) - (1 - \alpha) \cdot \frac{d}{dT} \left( \beta + 1 \right) \left( \sum_{t=1}^{T} \frac{R_t}{(1+r)^t} \right)$$

As $\min_{x \in X_{\tau(T) > 0}} \left\{ \sum_{t=1}^{\tau(T)} (R_t - C_t) - \theta \right\}$ is positive and increasing in $T$ and $\left( \beta + 1 \right) \left( \sum_{t=1}^{T} \frac{R_t}{(1+r)^t} \right) - (1 - \alpha)$ is negative and non-increasing in $T$, as proved above, then we conclude the objective function is increasing in $T$, which completes the proof for case 2. □

**Proof of Lemma 5.** If i) $b < d$ or ii) $b = d$ and $a < c$, then $\sum_{t=1}^{T} (R_t - C_t) - \theta = (a-c)\tau + (b-d)\left(\frac{\tau+1}{2}\right) - \theta$ is unbounded from below, implying that, in either case, $\exists \tau_0$ such that $\tau \in X$ for all $\tau > \tau_0$. For any finite $T \in \mathbb{N}$, pick a $\tau \in X$ with $\tau_T > T$; this implies

$$Z_{\tau}(T) = (A + 1)(\beta + 1) \left( \sum_{t=1}^{T} \frac{(a+b)(\tau+1)}{(1+r)^t} \right) - (1 - \alpha).$$

Since $T$ was chosen arbitrarily, we can select it so that $Z_{\tau}(T) \geq 0$, since $\sum_{t=1}^{T} (a+b)(\tau+1)$ is quadratic in $T$ and $\sum_{t=1}^{T} (a+b)(\tau+1)$ grows sub-quadratically. This proves Problem (14) is infeasible, since the first constraint is violated, which implies that Problem (11) is also infeasible.

For iii) $b = d$ and $c \leq a < c + \theta$, $f(\tau)$ can be simplified to $f(\tau) = \frac{(a-c)\tau - \theta}{\eta(\tau)(a+r+\frac{\tau+1}{2}) - (1-\alpha)}$. Note that $\lim_{\tau \to \infty} f(\tau) = 0$. Therefore $\min_{\eta(\tau)(a+r+\frac{\tau+1}{2}) - (1-\alpha)} \{ f(\tau) \} \leq 0$ and as a result from the second constraint in Problem (14), $0 \leq \max_{\tau \in X} \{ f(\tau) \} \leq \min_{\tau \in X_{\tau(T) > 0}} \{ f(\tau) \} \leq 0$, we conclude $Y^* = 0$. However $Y^* = 0$ is infeasible since for $b = d$ and $c \leq a < c + \theta$, the firm is cash-flow below $\theta$ in the first month: $R_1 - C_1 = (a+b) - (c+d) = a - c \leq \theta$. Therefore the problem is infeasible for $b = d$ and $c \leq a < c + \theta$. □

**Proof of Lemma 6.** We first consider the case where $b > d$, $(b-d) \geq (c-a) + \theta$, and $(\beta + 1)(A + 1) \left( \sum_{t=1}^{T} \frac{R_t}{(1+r)^t} \right) - (1 - \alpha) \geq 0$. Condition $b > d$ implies $\sum_{t=1}^{T} (R_t - C_t) - \theta = (a-c)\tau + (b-d)\left(\frac{\tau+1}{2}\right) - \theta \leq F(\tau)$ is
quadratic and convex in \( \tau \), and is minimized when \( \tau^* = \frac{2(c-a)-(b-d)}{2(b-d)} \) with value \( F(\tau^*) = \frac{-2(c-a)-(b-d)^2}{8(b-d)} - \theta < 0 \). If \( \tau^* < 1 \) and \( F(1) \geq 0 \), then \( X \) is empty; these conditions are equivalent to \( (b-d) > \frac{2}{3}(c-a) \) and \( (b-d) \geq (c-a) + \theta \) and the latter constraint dominates (for both \( c \geq a \) and \( c < a \), constraint \( (b-d) > \frac{2}{3}(c-a) \) is trivially true). Since set \( X \) is empty for \( b > d \) and \( (b-d) \geq (c-a) + \theta \), according to Proposition 1, we conclude \( Y^*(T) = 0 \) for \( b > d \), \( (b-d) \geq (c-a) + \theta \), and \( (\beta + 1)(A + 1) \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} (1 + \alpha)^t} \) is at least \( (1 - \alpha) \geq 0 \).

Next consider the case where \( b = d \), \( a \geq c + \theta \), and \( (\beta + 1)(\hat{A} + 1) \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} (1 + \alpha)^t} \) is at least \( (1 - \alpha) \geq 0 \). Under conditions \( b = d \) and \( a \geq c + \theta \), \( \sum_{t=1}^{T} R_t - C_t = (a - c) T - \theta \) is linear, and \( X \) is empty, which according to Proposition 1, we conclude \( Y^*(T) = 0 \) for \( b = d \), \( a \geq c + \theta \), and \( (\beta + 1)(\hat{A} + 1) \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} (1 + \alpha)^t} \) is at least \( (1 - \alpha) \geq 0 \).

Finally, consider the case where \( b = d \), \( a \geq c + \theta \), and \( (\beta + 1)(\hat{A} + 1) \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} (1 + \alpha)^t} \) is at least \( (1 - \alpha) < 0 \).

According to Proposition 1, under the condition \( (\beta + 1)(\hat{A} + 1) \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} (1 + \alpha)^t} \) is at least \( (1 - \alpha) < 0 \) we have \( Y^*(T) = \min_{X \in X} f(\tau) \). As shown earlier, under conditions \( b = d \) and \( a \geq c + \theta \), set \( X \) is empty which results in \( Y^*(T) = \min_{X \in X} f(\tau) \). Note that \( Z_\tau(T) = \eta(T) \left( a \tau + b \frac{\tau + 1}{2} \right) - (1 - \alpha) \), where \( \eta(T) = \frac{\hat{A} + 1}{\sum_{t=1}^{T} (1 + \alpha)^t} \frac{R_t}{b} > 0 \), is a convex function of \( \tau \); the roots of this quadratic are \( -\frac{(a + \frac{1}{2})\eta(T) + \sqrt{(a + \frac{1}{2})^2\eta(T)^2 + 2b\eta(T)(1 - \alpha)}}{b\eta(T)} \).

By inspection, the smaller root is negative and the larger root is positive. This implies that \( \{ \tau \in \mathbb{N} : Z_\tau(T) > 0 \} = \left\{ \tau \in \mathbb{N} : \tau > -\frac{(a + \frac{1}{2})\eta(T) + \sqrt{(a + \frac{1}{2})^2\eta(T)^2 + 2b\eta(T)(1 - \alpha)}}{b\eta(T)} \right\} \). Thus, \( Y^*(T) = \min_{X \in X} f(\tau) = \min_{X \in X} \left( -\frac{(a - c)\tau - \theta}{\eta(T) \left( a \tau + b \frac{\tau + 1}{2} \right) - (1 - \alpha)} \right) \) is continuous and has a unique minimum at \( \tau \to \infty \).

For simplicity and tractability in the analysis, we solve the problem for \( \theta \leq \theta_L(T) \) where \( \theta_L(T) \) is given by

\[
\theta_L(T) = 2b(1 - \alpha)(b - d)\eta(T) + (cb - da)(2a + b) - \sqrt{((cb - da)(2a + b))^2 + 8(1 - \alpha)b(cb - da)^2/\eta(T)}}
\]

Note that \( \theta \) should be chosen such that the approximation solutions from Problem (11) are feasible for Problem (6), and also give high quality approximations for Problem (6). In all the experiments with real data from Bolstr, \( \theta \leq \theta_L(T) \) suffices for high quality approximations. In extensive numerical experiments \( \theta_L \) is positive. In the next lemma we show \( \theta_L(T) \) is non-negative for \( b > d \) and \( b - d < c - a \) which indicate the firm that has cash-flow shortages.

**Lemma 16.** \( \theta_L(T) \) is non-negative for \( b > d \) and \( b - d < c - a \) \( \forall T \geq 1 \).

**Proof of Lemma 16.** Define

\[
\Delta(\theta) = (\eta(T)\theta b)^2 - \eta(T)(cb - da)(2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b))
\]

It can be verified that \( \Delta(\theta) \) is a quadratic convex function of \( \theta \) in which \( \theta_L(T) \) is its smaller root and its larger root is given by

\[
\theta_H(T) = 2b(1 - \alpha)(b - d)\eta(T) + (cb - da)(2a + b) + \sqrt{((cb - da)(2a + b))^2 + 8(1 - \alpha)b(cb - da)^2/\eta(T)}}
\]
As $b > d$ at feasibility, it can be shown by inspection that $\theta y(T) > 0$. If we show $\Delta(\theta) \geq 0$, then since $\Delta(\theta)$ is convex with the larger root being positive, we conclude the smaller root, $\theta_y(T)$, is non-negative. Therefore we next show $\Delta(0) = (1-\alpha)^2(b-d)^2 - \eta(T)(1-\alpha)(cb-da)(2(c-a) - (b-d)) = (1-\alpha)(b-d)^2 \left( (1-\alpha) - \frac{\eta(T)(c-a)(2(c-a) - (b-d))}{(b-d)^2} \right)$ is non-negative. From $F(\tau) = 0$, we know:

$$F(\tau) = 0 \iff (a-c)\tau + (b-d)\tau(\tau + 1)/2 = 0 \iff \tau(\tau + 1)/2 = \frac{\theta + (c-a)\tau}{(b-d)}.$$  

Additionally from $Z_*(T) < 0 \forall \tau \in X$, we conclude $Z_*(T) \leq 0$ which results in

$$Z_*(T) = \eta(T)(a\tau + b\tau(\tau + 1)/2) \leq (1-\alpha).$$

Next we replace $\tau(\tau + 1)/2 = \frac{\theta + (c-a)\tau}{(b-d)}$ in the above inequality and this results in

$$\eta(T) \left( a\tau + \frac{\theta + (c-a)\tau}{(b-d)} \right) \leq (1-\alpha) \iff \eta(T) \left( \frac{cb-da}{(b-d)} \frac{(b-d)}{\tau} + \frac{b\theta}{(b-d)} \right) \leq (1-\alpha) \iff \eta(T) \left( \frac{cb-da}{b-d} \left( \frac{c-a}{b-d} \frac{b\theta}{(b-d)} \right) + \sqrt{\left( \frac{c-a}{b-d} \frac{b\theta}{(b-d)} \right)^2 + 2\theta(b-d)} \right) \leq (1-\alpha)$$

Assuming the problem is feasible for $\theta \geq 0$, we next replace $\theta = 0$ in the above inequality:

$$\eta(T) \left( \frac{cb-da}{b-d} \left( \frac{c-a}{b-d} \right) + \sqrt{\left( \frac{c-a}{b-d} \right)^2 + 2\theta b-d} \right) \leq (1-\alpha) \iff \Delta(0) \geq 0$$

and this completes the proof. $\square$

**Proof of Lemma 7.** According to Proposition 1, for $b > d$, $b-d \geq (c-a) + \theta$, and $(\beta + 1)(\hat{A} + 1)$, we have $Y^*(T) = \min_{\tau \in X, \tau(\tau) > 0} f(\tau)$. As shown in the proof of Proposition 6, for $b > d$ and $b-d \geq (c-a) + \theta$, set $X$ is empty. Thus, we obtain $Y^*(T) = \min_{\tau \in X, \tau(\tau) > 0} f(\tau)$. The derivative of $f(\tau)$ is

$$\frac{df(\tau)}{d\tau} = 2 \frac{(cb-da)\eta(T)\tau^2 - 2((1-\alpha)(b-d) - \eta(T)\theta)\tau + (2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b)}{(\eta(T)\theta^2 + (2a+b)\eta(T)\tau - 2(1-\alpha))^2}. $$

We next prove the proposition for the following two cases: 1) $cb-da \geq 0$ and 2) $cb-da < 0$.

First consider case 1: $cb-da \geq 0$. The numerator $n(\tau)$ of $\frac{df(\tau)}{d\tau}$ is a convex quadratic due to $cb > da$. Note that, for $b > d$ and $b-d \geq (c-a) + \theta$ set $X$ is empty, the numerator of $f(\tau)$ is increasing and positive for $\tau \in [1, \infty)$. Additionally as shown in the proof of Lemma 6, the denominator of $f(\tau)$, $Z_*(T) = \eta(T) \left( a\tau + \frac{\theta + \tau(\tau+1)}{2} \right) - (1-\alpha)$ is a convex function of $\tau$ and the roots of this quadratic are $\frac{-\left( \tau + \frac{\theta}{2} \right)n(T) \pm \sqrt{\left( \tau + \frac{\theta}{2} \right)n(T) + \eta(T))} {b(\tau)\eta(T)}$. By inspection, the smaller root is negative and the larger root is positive. Thus for $\tau \in \left[ 1, \frac{-\left( \tau + \frac{\theta}{2} \right)n(T) + \sqrt{\left( \tau + \frac{\theta}{2} \right)n(T) + \eta(T))} {b(\tau)\eta(T)} \right]$, the denominator of $f(\tau)$ is negative and increasing and approaches zero from below at $\tau \to \frac{-\left( \tau + \frac{\theta}{2} \right)n(T) + \sqrt{\left( \tau + \frac{\theta}{2} \right)n(T) + \eta(T))} {b(\tau)\eta(T)}$. Therefore as the numerator of
As mentioned earlier, we solve the problem for $X_n$, and the above inequality holds due to $(da - cb)b < 0$, $\forall \theta \leq \theta_L(T)$:

\[-(1 - \alpha)(b - d) + \eta(T)\theta b < 0, \forall \theta \leq \theta_L(T) \iff -(1 - \alpha)(b - d) + \eta(T)\theta_L(T)b < 0 \iff -\eta(T)(da - cb)(2a + b) - \sqrt{(\eta(T)(cb - da)(2a + b))^2 + 8\eta(T)(1 - \alpha)b(cb - da)^2} < 0\]

and the above inequality holds due to $(da - cb) \geq 0$. For $-(1 - \alpha)(b - d) + \eta(T)\theta b < 0, \forall \theta \leq \theta_L(T)$, it can be easily verified that $\tau^\ast$ is negative. We next prove $\tau^\ast$ is negative. If we rearrange the inequality $-(1 - \alpha)(b - d) + \eta(T)\theta b < 0, \forall \theta \leq \theta_L(T)$, we get $\eta(T)(1 - \alpha)(b - d)/b, \forall \theta \leq \theta_L(T)$. Next we show for $\theta_L(T) < (1 - \alpha)(b - d)/b, \forall \theta \leq \theta_L(T)$, the inequality $-(1 - \alpha)(a - c) + \eta(T)\theta a < 0, \forall \theta \leq \theta_L(T)$ holds by replacing $\eta(T)$ with its upper bound $(1 - \alpha)(b - d)/b$ into $-(1 - \alpha)(a - c) + \eta(T)\theta a < 0$:

\[-(1 - \alpha)(a - c) + \eta(T)\theta a < 0, \forall \theta \leq \theta_L(T) \iff -(1 - \alpha)(a - c) + (1 - \alpha)(b - d)/b a < 0 \iff -(1 - \alpha)(ad - cb)/b < 0\]
and the above inequality holds due to \((da - cb) \geq 0\). If \(-(1 - \alpha)(b - d) + \eta(T)\theta b < 0\) and \(-(1 - \alpha)(a - c) + \eta(T)\theta a < 0\), \(\forall \theta \leq \theta_\ell(T)\), then we conclude \(((2c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b) < 0\), \(\forall \theta \leq \theta_\ell(T)\).

From this result, we show below that \(\tau^* \) is negative:

\[
\tau^* \leq 0 \iff -(1 - \alpha)(b - d) + \eta(T)\theta b + \sqrt{\Delta(\theta)} \leq 0
\]

\[
\iff -(1 - \alpha)(b - d) + \eta(T)\theta b + \sqrt{((1 - \alpha)(b - d) - \eta(T)\theta b)^2 - \eta(T)(cb - da)((2c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b)} \leq 0
\]

\[
\iff \sqrt{((1 - \alpha)(b - d) - \eta(T)\theta b)^2 + \eta(T)(da - cb)((2c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b)} \leq (1 - \alpha)(b - d) - \eta(T)\theta b
\]

and the above inequality holds due to \((1 - \alpha)(b - d) - \eta(T)\theta b > 0\), \((da - cb) \geq 0\), and \(((2c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b) < 0\). As a result we conclude \(\tau^* < 0\). As \(n(\tau)\) is concave with negative roots, we conclude \(f(\tau)\) is strictly decreasing in \([0, \infty)\) and as a result \(\min_{\tau \in \mathbb{N}} f(\tau) = \tau(\hat{T}) = (b - d)/\eta(T)b\). □

**Proof of Lemma 8.** If \(b > d\) and \(b - d < (c - a) + \theta\), then \(\sum_{t=1}^r(R_t - C_t) = \tau = (a - c)\tau + (b - d)(t + 1) - \theta \leq F(\tau)\) is a convex quadratic function of \(\tau\), as \(F(0) = -\theta\), the smaller root is negative and the larger root is positive and is equal to

\[
\tau = \frac{(c - a - \frac{b - d}{2}) + \sqrt{(c - a - \frac{b - d}{2})^2 + 2\theta(b - d)}}{b - d}.
\]

Note that as the discriminant is positive, the roots are real. Thus, \(F(\tau) < 0\) for \(1 \leq \tau \leq \tau(\hat{T})\). □

**Proof of Lemma 9.** Recall that \(Z_r(T) = \eta(T)\left(A\tau + B\tau^2 + \frac{1}{2}\right) - (1 - \alpha)\), where \(\eta(T) = \frac{\sum_{t=1}^r(R_t - C_t)}{b(1 + \alpha)} > 0\), is a convex function of \(\tau\); the roots of this quadratic are 

\[
\frac{-(a + \frac{b}{2})\eta(T) + \sqrt{(a + \frac{b}{2})^2\eta(T)^2 + 2\eta(T)(1 - \alpha)\eta(T)}}{b\eta(T)}.
\]

By inspection, the smaller root is negative and the larger root is positive. This implies that \(\{\tau \in \mathbb{N} : Z_r(T) < 0\} = \left\{\tau \in \mathbb{N} : \tau < \frac{-(a + \frac{b}{2})\eta(T) + \sqrt{(a + \frac{b}{2})^2\eta(T)^2 + 2\eta(T)(1 - \alpha)\eta(T)}}{b\eta(T)}\right\}\) and \(\{\tau \in \mathbb{N} : Z_r(T) > 0\} = \left\{\tau \in \mathbb{N} : \tau > \frac{-(a + \frac{b}{2})\eta(T) + \sqrt{(a + \frac{b}{2})^2\eta(T)^2 + 2\eta(T)(1 - \alpha)\eta(T)}}{b\eta(T)}\right\}\). Therefore the condition \(Z_r(T) < 0\) for all \(\tau \in X\), in which \(X = \{\tau \in \mathbb{N} : 0 \leq \tau < \tau(\hat{T})\}\), is equivalent to \(\tau < \frac{-(a + \frac{b}{2})\eta(T) + \sqrt{(a + \frac{b}{2})^2\eta(T)^2 + 2\eta(T)(1 - \alpha)\eta(T)}}{b\eta(T)}\) or equivalently \(\sqrt{(c - a - (b - d)/2)^2 + 2b(b - d)} < (ad - bc) + (b - d)(\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)}) \geq 0\). Under this condition, the inequality can be rearranged as \(\theta \leq \hat{\theta}(T)\) where

\[
\hat{\theta}(T) = \frac{(b - d)^2(a + b/2)^2 + 2(1 - \alpha)b(b - d)^2}{\eta(T)} + (ad - bc)^2 + 2(b - d)(ad - bc)\sqrt{(a + b/2)^2 + \frac{2(1 - \alpha)b}{\eta(T)}} - b^2(c - a - (b - d)/2)^2}{2b^2(b - d)}.
\]

□

According to Lemma 9, the problem is feasible for \(\theta \leq \hat{\theta}(T)\). As a result we consider \(\theta \leq \min\{\hat{\theta}(T), \theta_{\ell}(T)\} \leq \hat{\theta}(T)\) throughout the next set of analysis.

**Proof of Lemma 10.** According to Lemma 9, the first constraint in Problem (14), \(Z_r(T) < 0\) for all \(\tau \in X\), is equivalent to the conditions \(\theta \leq \hat{\theta}(T)\) and \((ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0\). As
noted earlier, we solve the lemma for $\theta \leq \hat{\theta}(T)$ for Case i. Thus we prove this lemma under the conditions $(ad-bc) + (b-d)\sqrt{(a+b/2)^2 + 2(1-\alpha)b/\eta(T)} \geq 0$, $\theta \leq \hat{\theta}(T)$, $b > d$, $b-d < (c-a) + \theta$, and $cb > da$.

The derivative of $f(\tau)$ is

$$\frac{df(\tau)}{d\tau} = 2(cb-da)\eta(T)\tau^2 - 2((1-\alpha)(b-d) - \eta(T)\theta b)\tau + (2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b).$$

The numerator $n(\tau)$ of $\frac{df(\tau)}{d\tau}$ is a convex quadratic due to $cb > da$. We consider the following two cases separately:

Case 1. If $n(0) = (2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b) \geq 0$, or equivalently $\theta \geq \frac{((b-d) - 2(c-a))(1-\alpha)}{\eta(T)(2a+b)}$, then $f(\tau)$ is increasing at $\tau = 0$. In contrast $f(\tau)$ is strictly decreasing in $[\tau, \left\lceil \frac{-(a+b)\eta(T) + \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)} \right\rceil)$.

The reason is as follows: for all $\tau \in [\tau, \left\lceil \frac{-(a+b)\eta(T) + \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)} \right\rceil)$, the numerator of $f(\tau)$ is non-negative and increasing, according to the definition of set $X$. As shown in Lemma 9, the denominator of $f(\tau)$, $Z_\tau(T) = \eta(T)(a + \frac{b}{2}(\tau+1)) - (1-\alpha)$ is a convex function of $\tau$ and the roots of this quadratic are $\frac{-(a+b)\eta(T) + \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)}$ and $\frac{-(a+b)\eta(T) - \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)}$. By inspection, the smaller root is negative and the larger root is positive. Thus for $\tau \in [\tau, \frac{-(a+b)\eta(T) + \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)}]$,

$$f(\tau) \text{ is negative and increasing and approaches zero from below at } \tau \to -\frac{-(a+b)\eta(T) + \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)}.$$ Therefore as the numerator of $f(\tau)$ is non-negative and increasing and the denominator is negative and increasing, $f(\tau)$ is strictly decreasing on this interval, or equivalently $n(\tau) < 0$ for $\tau \in [\tau, \frac{-(a+b)\eta(T) + \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)}]$.

Since $n(\tau)$ is continuous in $[0, \tau]$, these observations imply that there is one point $\tau^*$ in this interval where $\frac{df(\tau^*)}{d\tau} = 0$. Furthermore, $\tau^*$ must be the smaller root of the convex numerator $n(\tau)$, or

$$\tau^* = \frac{(1-\alpha)(b-d) - \eta(T)\theta b - \sqrt{((1-\alpha)(b-d) - \eta(T)\theta b)^2 - \eta(T)(cb-da)((2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b))}}{(cb-da)\eta(T)}.$$

As shown in Lemma 16, $\theta(T)$ is the smaller root of the convex quadratic function $\Delta(\theta)$. Therefore for $\theta \leq \hat{\theta}(T) \leq \theta(T)$ which we are solving the problem for, $\tau^*$ is real.

Consequently, $\frac{df(\tau)}{d\tau}$ is positive for $\tau \in [0, \tau^*)$ and negative for $\tau \in (\tau^*, \tau]$, which implies $f(\tau)$ is unimodal on $[0, \tau]$, with a maximum at $\tau^*$. Therefore, since $X$ is a set of integers, \max_{\tau \in X} \{f(\tau)\} = \max_{\tau \in X} \{f(\tau^*)\}, f(\tau) \text{ is strictly decreasing at } \tau = 0. \text{ As shown above, } f(\tau) \text{ is strictly decreasing on } [0, \tau] \text{ with a maximizer at } \tau = 0: \max_{\tau \in X} \{f(\tau)\} = f(1) = \frac{a + b - c - d - \theta}{\eta(T)(a + b) - (1-\alpha)}.$$

Proof of Lemma 11. From Lemma 9 we conclude that $\{\tau \in \mathbb{N} : \tau \notin X, Z_\tau(T) > 0\} = \{\tau : \tau > -\frac{-(a+b)\eta(T) + \sqrt{(a+b)^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)}\}$.

Next, building upon the proof of Lemma 10, we recall that the numerator of $\frac{df(\tau)}{d\tau}$, evaluated at $\tau = \tau$, is non-positive: $n(\tau) \leq 0$ or equivalently $f(\tau)$ is decreasing at $\tau = \tau$. Alternatively, $\lim_{\tau \to \infty} n(\tau) = \infty$. Since
the numerator is convex, these observations imply that \( f(\tau) \) is unimodal on \([\tau, \infty)\), with a minimum at the larger root of \( n(\tau) \), or

\[
\tau^{**} = \frac{(1-\alpha)(b-d) - \eta(T)\theta b + \sqrt{((1-\alpha)(b-d) - \eta(T)\theta b)^2 - \eta(T)(cb-da)((2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b))}}{(cb-da)\eta(T)}
\]

Note that \( \tau^{**} \) is real for \( \theta \leq \bar{\theta}(T) \), our domain of interest. We finally prove that 

\[
\tau^{**} > \frac{(a+b)}{2}\eta(T) + \sqrt{\left(\frac{(a+b)}{2}\eta(T)\right)^2 + 2bn(T)(1-\alpha)}
\]

which implies that \( \tau^{**} \in \{ \tau \in \mathbb{R} : \tau \notin X, \ Z_\tau(T) > 0 \} \) and 

\[
\min_{\tau \in X, \ Z_\tau(T) > 0} \{ f(\tau) \} = \min \{ f(\lfloor \tau^{**} \rfloor), f(\lceil \tau^{**} \rceil) \}. \]

If \( \tau^{**} \leq \frac{-a + \frac{1}{2}\eta(T) + \sqrt{(a + \frac{1}{2})^2\eta(T)^2 + 2bn(T)(1-\alpha)}}{bn(T)} \), then \( f(\tau) \) is unimodal with a minimizer in \([\tau, -a + \frac{1}{2}\eta(T) + \sqrt{(a + \frac{1}{2})^2\eta(T)^2 + 2bn(T)(1-\alpha)}] \). In particular, this implies \( f(\tau) \) is increasing as \( \tau \searrow -a + \frac{1}{2}\eta(T) + \sqrt{(a + \frac{1}{2})^2\eta(T)^2 + 2bn(T)(1-\alpha)} \); however, this contradicts 

\[
\lim_{\tau \searrow -a + \frac{1}{2}\eta(T) + \sqrt{(a + \frac{1}{2})^2\eta(T)^2 + 2bn(T)(1-\alpha)}} f(\tau) = -\infty, \text{ since, on this domain, the denominator of } f(\tau), \ Z_\tau(T), \text{ is negative and approaches zero from below, and the numerator is positive, by the definition of } X. \]

**Proof of Lemma 12.** We prove this lemma under the conditions of Lemmas 10 and 11. We first show that the left-hand-side of \( \max \{ f(\tau) \} - \min_{\tau \in X, \ Z_\tau(T) > 0} \{ f(\tau) \} \leq 0 \) is increasing in \( \theta \) and the inequality holds at \( \theta = 0 \).

Then we conclude \( \exists \bar{\theta}(T) \) where \( \max \{ f(\tau) \} - \min_{\tau \in X, \ Z_\tau(T) > 0} \{ f(\tau) \} \leq 0 \) holds for \( \theta \leq \bar{\theta}(T) \).

First consider Case 1 in which \( n(0) = (2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b) \geq 0 \), which according to Lemmas 10–11 we have 

\[
\max_{\tau \in X} \{ f(\tau) \} = \min \{ f(\lfloor \tau^{**} \rfloor), f(\lceil \tau^{**} \rceil) \} \]

For ease of exposition, we assume that \( \tau^* \) and \( \tau^{**} \) are integers, so that we need to prove \( f(\tau^*) \leq f(\tau^{**}) \).

Expanding the definition of \( f(\tau) \), we observe

\[
f(\tau^{**}) - f(\tau^*) = \frac{(a-c)\tau^{**} + (b-d)\frac{\tau^{**}(\tau^{**}+1)}{2} - \theta}{Z_{\tau^{**}}(T)} - \frac{(a-c)\tau^* + (b-d)\frac{\tau^*(\tau^*+1)}{2} - \theta}{Z_{\tau^*}(T)}
\]

\[
= \frac{\left((a-c)\tau^{**} + (b-d)\frac{\tau^{**}(\tau^{**}+1)}{2} - \theta\right) - \left((a-c)\tau^* + (b-d)\frac{\tau^*(\tau^*+1)}{2} - \theta\right)}{Z_{\tau^*}(T)Z_{\tau^{**}}(T)}
= \frac{(a-c)\tau^* + (b-d)\frac{\tau^*(\tau^*+1)}{2} - \theta}{Z_{\tau^*}(T)Z_{\tau^{**}}(T)} \left(Z_{\tau^{**}}(T) - \left((a-c)\tau^{**} + (b-d)\frac{\tau^{**}(\tau^{**}+1)}{2} - \theta\right)\right).
\]

From Lemma 10, we know that \( Z_{\tau}(T) < 0 \); similarly, from Lemma 11, we know that \( Z_{\tau^*}(T) > 0 \). Thus, the denominator of Equation (24) is negative. To analyze the numerator of Equation (24), we first consider the expression \( \left((a-c)t + (b-d)\frac{t(t+1)}{2} - \theta\right)Z_{\tau}(T) \), which can be manipulated as follows:

\[
\left((a-c)t + (b-d)\frac{t(t+1)}{2} - \theta\right)Z_{\tau}(T)
= \left((a-c)t + (b-d)\frac{t(t+1)}{2} - \theta\right)\eta(T)\left(a^2 + \tau^*(\tau^*+1)\right) - (1-\alpha)
= \eta(T)b(b-d)\left[(a-c)t^2 + \frac{\eta(T)2a+b(b-d)(1-\alpha)}{4}t^2 - \frac{(b-d)(1-\alpha)}{2}t^2\right]
- \eta(T)b\left(\frac{2(c-a)-(b-d)}{4}\right)t^2 - \eta(T)(2a+b)\left(\frac{2(c-a)-(b-d)}{4}\right)t + (1-\alpha)\left(\frac{2(c-a)-(b-d)}{2}\right)t
- \theta\eta(T)a\left(\frac{\tau^*(\tau^*+1)}{2}\right) - (1-\alpha).
\]
Since \( t^2 \tau^2 = \tau^2 t^2 \) and \( t \tau = \tau t \), we can write
\[
\left( (a - c)t + (b - d) \frac{t(\tau + 1)}{2} - \theta \right) Z, (T) - \left( (a - c)\tau + (b - d) \frac{\tau(\tau + 1)}{2} - \theta \right) Z', (T) = (t - \tau) \left( \frac{2(\tau - \theta)(c - a - \theta)}{4} \right) t \tau - \frac{(b - d)(\tau - \theta)}{2} (t + \tau) + (1 - \alpha) \left( \frac{2(c - a - \theta)(b - d)}{2} \right) + \theta (Z, (T) - Z', (T)) .
\] (25)

Letting \( t = \tau^* \) and \( \tau = \tau^* \), two roots of a quadratic equation, we have that
\[
t + \tau = 2 (1 - \alpha)(b - d - \eta(T)\theta b) \frac{\tau}{c(b - da)\eta(T)} , \quad t \tau = \frac{(2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b) - (b - d)(\tau - \theta)}{\eta(T)(c(b - da))} , \quad \text{and} \quad t - \tau = \frac{2 \sqrt{\Delta} \theta}{\eta(T)(c(b - da))} .
\]

Plugging these expressions into Expression (25), along with \( t = \tau^* \) and \( \tau = \tau^* \), we obtain
\[
\frac{2 \sqrt{\Delta} \theta}{\eta(T)(c(b - da))} \left( (2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(a + b/2) - (b - d)(1 - \alpha) \frac{(b - d)(1 - \alpha) - \eta(T)\theta b}{(c(b - da)\eta(T))} \right) + \theta (Z', (T) - Z''(T)) .
\]

As the denominator of Equation (24) is negative, \( f(\tau^*) - f(\tau^*) \geq 0 \) holds if the numerator of Equation (24) is non-positive or equivalently
\[
\frac{2 \sqrt{\Delta} \theta}{\eta(T)(c(b - da))} \left( (2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(a + b/2) - (b - d)(1 - \alpha) \frac{(b - d)(1 - \alpha) - \eta(T)\theta b}{(c(b - da)\eta(T))} \right) + \theta (Z', (T) - Z''(T)) \leq 0 .
\]

Let
\[
g(\theta) = \frac{2 \sqrt{\Delta} \theta}{\eta(T)(c(b - da))} \left( (2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(a + b/2) - (b - d)(1 - \alpha) \frac{(b - d)(1 - \alpha) - \eta(T)\theta b}{(c(b - da)\eta(T))} \right) ,
\]
and
\[
h(\theta) = \theta(T)(c(b - da)) (Z', (T) - Z''(T)) .
\]

The above inequality is then written as \( g(\theta) + h(\theta) \leq 0 \). We first show \( g(\theta) < 0 \), \( \forall \theta \geq 0 \). The \( g(\theta) \) can be rearranged and written as follows:
\[
g(\theta) = \frac{2 \sqrt{\Delta} \theta}{\eta^2(T)(c(b - da))} \left( - \left( \sqrt{\Delta} \theta \right)^2 - \eta(T)\theta b ((1 - \alpha)(b - d) - \eta(T)\theta b) \right) \leq 0 .
\]

The above inequality holds due to \( (1 - \alpha)(b - d) - \eta(T)\theta b \geq 0 \): As proved in Lemma 10, for Case 1 in which \( n(0) = (2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b) \geq 0 \), we have \( \tau^* \in [0, \tau] \). Therefore from the definition of \( \tau^* \), we conclude \( (1 - \alpha)(b - d) - \eta(T)\theta b \geq 0 \) must hold to have \( \tau^* \geq 0 \). We next show that \( g(\theta) + h(\theta) < 0 \) at \( \theta = 0 \):
\[
g(0) + h(0) = -2 \sqrt{\left( (1 - \alpha)(b - d))^2 - \eta(T)(c(b - da)) ((2(c - a) - (b - d))(1 - \alpha)) \right)^3} \frac{\eta^2(T)(c(b - da))}{\eta^2(T)(c(b - da))} < 0 .
\]

Note that \( \theta = 0 \), if feasible, belongs to this case. The reason is that from the feasibility condition \( b - d < c - a + \theta \) at \( \theta = 0 \) we conclude \( b - d < c - a \) and as a result \( n(0) \geq 0 \), which is the condition of Case 1.

Recall that at feasibility we have \( (Z', (T) - Z''(T)) > 0 \) and as a result \( h(\theta) \geq 0 \). If \( h(\theta) \) is decreasing in \( \theta \), then \( g(\theta) + h(\theta) < 0 \), \( \forall \theta \geq 0 \). Otherwise there exists some \( \theta^*(T) \geq 0 \) where \( g(\theta) + h(\theta) < 0 \) holds for \( \theta \leq \theta^*(T) \leq \theta(\theta) \) and this completes the proof for Case 1.
Next we prove the left-hand-side of $\max \{f(\tau)\} - \min_{\tau \in X} f(\tau) \leq 0$ is increasing in $\theta$ for Case 2, in which $n(0) = (2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b) < 0$. Then we claim, if Case 2 is feasible, there exists some $\mathcal{B}(T) \geq 0$ where $\max_{\tau \in X} f(\tau) - \min_{\tau \in X} f(\tau) \leq 0$ holds for $\theta \leq \mathcal{B}(T) \leq \hat{\theta}(T)$ and this completes the proof for Case 2. According to Lemmas 10–11, under the conditions of Case 2, we have $\max_{\tau \in X} f(\tau) = \frac{a + b - c - d - \theta}{\eta(T)(a + b) - (1 - \alpha)}$ and $\min_{\tau \in X} f(\tau) = \min_{\tau \in X} \{f([\tau^{**}]^T), f([\tau^{**}]^T)\}$. For case of exposition, we assume that $\tau^{**}$ is integer, so $\max_{\tau \in X} f(\tau) \leq \min_{\tau \in X} f(\tau)$ is equivalent to $\theta - ((1 - \alpha) - \eta(T)(a + b)) f(\tau^{**}) - (a + b) + (c + d) \leq 0$. For calculating the derivative of the left-hand-side of the above inequality with respect to $\theta$, first we calculate $\frac{d f(\tau^{**}(\theta), \theta)}{d\theta}$:

$$
\frac{d f(\tau^{**}(\theta), \theta)}{d\theta} = \frac{[(a - c + (b - d)/2 + (b - d)\tau^{**}(\theta)) \frac{d \tau^{**}(\theta)}{d\theta} - 1]}{Z^2_{\tau^{**}(\theta)}(T)} - \frac{[\eta(T)(a \tau^{**}(\theta) + b/2\tau^{**}(\theta)(\tau^{**}(\theta) + 1) - (1 - \alpha)]}{Z^2_{\tau^{**}(\theta)}(T)}.
$$

We know $\tau^{**}(\theta)$ is the root of $n(\tau)$ and therefore $n(\tau^{**}(\theta)) = 0$ or equivalently

$$(cb - da)\eta(T)\tau^{**2}(\theta) - 2((1 - \alpha)(b - d) - \eta(T)\theta b) \tau^{**}(\theta) + (2(c - a) - (b - d))(1 - \alpha) + \eta(T)\theta(2a + b) = 0,$$

which implies $(cb - da)\eta(T)\tau^{**2}(\theta)/2 = ((1 - \alpha)(b - d) - \eta(T)\theta b) \tau^{**}(\theta) - (2(c - a) - (b - d))(1 - \alpha)/2 - \eta(T)\theta(2a + b)/2$. If we replace the left-hand-side of the above equation with its right-hand side into $\frac{d f(\tau^{**}(\theta), \theta)}{d\theta}$, the second term in $\frac{d f(\tau^{**}(\theta), \theta)}{d\theta}$ becomes zero and as a result we get

$$
\frac{d (f(\tau^{**}(\theta), \theta))}{d\theta} = -\frac{\eta(T)(a \tau^{**}(\theta) + b/2\tau^{**}(\theta)(\tau^{**}(\theta) + 1) - (1 - \alpha)}{Z^2_{\tau^{**}(\theta)}(T)} = -\frac{1}{Z_{\tau^{**}(\theta)}(T)}.
$$

Therefore, the derivative of the left-hand-side of the inequality $\theta - ((1 - \alpha) - \eta(T)(a + b)) f(\tau^{**}) - (a + b) + (c + d) \leq 0$, with respect to $\theta$, is

$$
\frac{d (\theta - ((1 - \alpha) - \eta(T)(a + b)) f(\tau^{**}) - (a + b) + (c + d))}{d\theta} = 1 + ((1 - \alpha) - \eta(T)(a + b)) \frac{1}{Z_{\tau^{**}(\theta)}(T)} > 0,
$$

where the above inequality is due to $Z_{\tau^{**}(\theta)}(T) \geq 0$, and $((1 - \alpha) - \eta(T)(a + b)) > 0$ or equivalently $Z_{\tau=1}(T) < 0$, and this completes the proof for Case 2. □

Proof of Lemma 13. According to Lemma 9, the first constraint in Problem (14), $Z_{\tau}(T) < 0$ for all $\tau \in X$, is equivalent to the conditions $\theta \leq \hat{\theta}(T)$ and $(ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0$. As noted earlier, we solve the lemma for $\theta \leq \hat{\theta}(T)$ for Case ii. Thus we prove this lemma under the conditions $(ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0, 0 \leq \hat{\theta}(T), b > d, b - d < (c - a) + \theta$, and $cb \leq da$. Note that for $cb \leq da$, the condition $(ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0$ is always satisfied.
The derivative of $f(\tau)$ is
$$\frac{df(\tau)}{d\tau} = 2\frac{(cb-da)\eta(T)\tau^2 - 2((1-\alpha)(b-d) - \eta(T)\theta b)\tau + (2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b)}{(\eta(T)b\tau^2 + (2a+b)\eta(T)\tau - 2(1-\alpha))^2}.$$ The numerator $n(\tau)$ of $\frac{df(\tau)}{d\tau}$ is a concave quadratic due to $cb \leq da$. Consider the smaller and larger roots of $n(\tau)$, $\tau^{**}$ and $\tau^*$, respectively:
$$\tau^{**} = \frac{-((1-\alpha)(b-d) + \eta(T)\theta b - \sqrt{\Delta(\theta)})}{(da-cb)\eta(T)},$$
and
$$\tau^* = \frac{-((1-\alpha)(b-d) + \eta(T)\theta b + \sqrt{\Delta(\theta)})}{(da-cb)\eta(T)}.$$

As mentioned earlier, we solve the problem for $\theta \leq \tilde{\theta}(T) \leq \theta_L(T)$ which guarantees the discriminant of $n(\tau)$ is non-negative. Next we show $\tau^{**}$ and $\tau^*$ are negative $\forall \theta \leq \theta_L(T)$. First we show $-(1-\alpha)(b-d) + \eta(T)\theta b < 0$, $\forall \theta \leq \theta_L(T)$:
$$-(1-\alpha)(b-d) + \eta(T)\theta b < 0, \forall \theta \leq \theta_L(T) \iff -(1-\alpha)(b-d) + \eta(T)\theta_L(T)b < 0 \iff \quad \quad \quad (26)$$
$$-\eta(T)(da-cb)2a+b - \sqrt{(\eta(T)(cb-da)(2a+b))^2 + 8\eta(T)(1-\alpha)b(cb-da)^2} < 0 \quad \quad (27)$$
and the above inequality holds due to $(da-cb) \geq 0$. For $-(1-\alpha)(b-d) + \eta(T)\theta b < 0$, $\forall \theta \leq \theta_L(T)$, it can be easily verified that $\tau^{**}$ is negative. We next prove $\tau^*$ is negative. If we rearrange the inequality $-(1-\alpha)(b-d) + \eta(T)\theta b < 0$, $\forall \theta \leq \theta_L(T)$, we get $\theta_L(T) < (1-\alpha)(b-d)/b$, $\forall \theta \leq \theta_L(T)$. Next we show for $\theta \eta(T) < (1-\alpha)(b-d)/b$, $\forall \theta \leq \theta_L(T)$, the inequality $-(1-\alpha)(a-c) + \eta(T)\theta a < 0$, $\forall \theta \leq \theta_L(T)$ holds by replacing $\theta \eta(T)$ with its upper bound $(1-\alpha)(b-d)/b$ into $-(1-\alpha)(a-c) + \eta(T)\theta a < 0$:
$$-(1-\alpha)(a-c) + \eta(T)\theta a < 0, \forall \theta \leq \theta_L(T) \iff -(1-\alpha)(a-c) + [(1-\alpha)(b-d)/b] a < 0 \iff -(1-\alpha)(ad-cb)/b < 0$$
and the above inequality holds due to $(da-cb) \geq 0$. If $-(1-\alpha)(b-d) + \eta(T)\theta b < 0$ and $-(1-\alpha)(a-c) + \eta(T)\theta a < 0$, $\forall \theta \leq \theta_L(T)$, then we conclude $((2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b)) < 0$, $\forall \theta \leq \theta_L(T)$.

From this result, we show below that $\tau^*$ is negative:
$$\tau^* \leq 0$$
$$\iff -(1-\alpha)(b-d) + \eta(T)\theta b + \sqrt{\Delta(\theta)} \leq 0$$
$$\iff -(1-\alpha)(b-d) + \eta(T)\theta b + \sqrt{((1-\alpha)(b-d) - \eta(T)\theta b)^2 - \eta(T)(cb-da)((2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b))} \leq 0$$
$$\iff \sqrt{((1-\alpha)(b-d) - \eta(T)\theta b)^2 + \eta(T)(da-cb)(2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b))} \leq (1-\alpha)(b-d) - \eta(T)\theta b$$
and the above inequality holds due to $(1-\alpha)(b-d) - \eta(T)\theta b > 0$, $(da-cb) \geq 0$, and $((2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b)) < 0$. As a result we conclude $\tau^* < 0$. As $n(\tau)$ is concave with negative roots, we conclude $f(\tau)$ is strictly decreasing in $[0,\infty)$ and as a result $\max_{\tau \in X} \{f(\tau)\} = f(1) = \frac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$. \ \Box

**Proof of Lemma 14.** Building upon the proof of Lemma 13, $f(\tau)$ is decreasing in $[0,\infty)$. These observations imply that $\min_{\tau \in X, \eta(T) > 0} \{f(\tau)\} = \lim_{\tau \to \infty} f(\tau) = (b-d)/\eta(T)b$. \ \Box
Proof of Lemma 15. Building upon the proofs of Lemmas 13 and 14, the inequality \( \max_{\tau \in X} \{ f(\tau) \} \leq \min_{\tau \in X, \tau \neq 0} \{ f(\tau) \} \) is equivalent to \( \theta \leq \frac{(1-\alpha)(b-d)+\eta(T)(da-cb)}{bn(T)} \). Therefore the inequality \( \max_{\tau \in X} \{ f(\tau) \} \leq \min_{\tau \neq X} \{ f(\tau) \} \) holds for \( \theta \leq \min \left\{ \tilde{\theta}(T), \frac{(1-\alpha)(b-d)+\eta(T)(da-cb)}{bn(T)} \right\} \), \( b > d \), \( b - d < (c-a) + \theta \), and \( cb \leq da \), and this completes the proof. \( \square \)