In this lecture we want to provide an introduction to how calculations in pQCD are performed, \textit{i.e.}, describe what sorts of techniques are used. A typical issue is the need to evaluate an integral that is formally infinity, either so that we can sum a series of such “infinities” as in the renormalization of the coupling or so that we can cancel with the infinity with another contribution (usually a “real” contribution canceling with a “virtual” contribution). So let’s begin with a brief interlude about “Dimensional Regularization”, a subject which we have already touched on briefly (recall Appendix B to Lecture 30). We will first look at an integral relevant to the UV divergence in the renormalization of the coupling. Assume that we want to evaluate the generic (logarithmically divergent) integral

\begin{equation}
I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}.
\end{equation}

(33.1)

We proceed by imagining that we can smoothly vary the number of space-time dimensions from \(d = 4 \to 4-2\varepsilon\), where \(\varepsilon\) is a small parameter that we will eventually set to zero. We will also introduce an arbitrary dimensionfull parameter \(\mu\) so that the integral will keep the same overall dimension even as we vary \(\varepsilon\) (this is how dimensional transmutation creeps in),

\begin{equation}
d^4k \to \left(\mu\right)^{2\varepsilon} d^{4-2\varepsilon}k.
\end{equation}

(33.2)

Next consider a Wick rotation to Euclidean space, \textit{i.e.}, a space with the same sign in the metric for all directions (think \(k^0 \to ik^0\) but ignore the \(i\) for now),

\begin{equation}
\int \frac{d^4k}{(2\pi)^4} \to \left(\mu\right)^{2\varepsilon} \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \to \left(\mu\right)^{2\varepsilon} \int \frac{d\Omega_{4-2\varepsilon}}{(2\pi)^{4-2\varepsilon}} \int dk_E k_E^{3-2\varepsilon}.
\end{equation}

(33.3)

We can do the angular integral (or look it up) to find

\begin{equation}
\int \frac{d\Omega_{4-2\varepsilon}}{(2\pi)^{4-2\varepsilon}} = \frac{2}{(4\pi)^{2-\varepsilon}} \frac{1}{\Gamma(2-\varepsilon)}.
\end{equation}

(33.4)
The remaining integral has the form of a (by now familiar) beta function,

\[
\left(\mu^{2\varepsilon}\right) \int_{0}^{\infty} dk \frac{k_{E}^{3 - 2\varepsilon}}{(k_{E}^{2} + m^{2})^{2}} = \frac{1}{2(m^{2})^{2\varepsilon}} \int_{0}^{1} dz z^{1 - \varepsilon} (1 - z)^{\varepsilon - 1} = \frac{1}{2} \left(\frac{\mu}{m}\right)^{2\varepsilon} \frac{\Gamma(\varepsilon) \Gamma(2 - \varepsilon)}{\Gamma(2)}. \tag{33.5}
\]

Using the relations \(\Gamma(1 + z) = z\Gamma(z); \Gamma'(1) = -\gamma_{E} = -0.5772\ldots\), we can finally write

\[
I = \frac{\Gamma(\varepsilon)}{(4\pi)^{2\varepsilon}} \left(\frac{\mu}{m}\right)^{2\varepsilon} \to \frac{1}{(4\pi)^{2}} \left[2\ln\left(\frac{\mu}{m}\right) + \frac{1}{\varepsilon} - \gamma_{E} + \ln(4\pi) + O(\varepsilon)\right]. \tag{33.6}
\]

This brings us to an important issue. In arranging for the subtraction of infinities there is an ambiguity about the remaining finite bit. The above expression will arise all the time in pQCD and, when we arrange to subtract the divergent \(1/\varepsilon\) bit, we could choose to also subtract the finite \(\gamma_{E}\) and \(\ln(4\pi)\) terms at the same time. (Note that you can hide nearly anything in infinity!) This particular choice of what finite bits to also subtract is called the \(\overline{\text{MS}}\) Scheme. It is the most commonly used choice of how much of the finite part to take with the infinity (although other schemes have been used). Physical quantities cannot depend on the scheme but theoretically defined quantities can and do depend on the choice of scheme.

Let us now look at a specific example of a perturbative calculation. Consider again the total electron-positron annihilation cross section to hadrons as in the last lecture. At lowest order (LO) we have

\[
\sigma_{0} = \frac{4\pi\alpha^{2}}{3Q^{2}} \cdot 3 \sum_{f} e_{f}^{2}. \tag{33.7}
\]

At the next-to-leading-order (NLO) we proceed to look at the real emission process applying dimensional regularization (or Dim-Reg for short). Recall that the corresponding Feynman diagrams look like the following.
The real emission contribution to the cross section looks like (note that, since we must use the matrix element squared evaluated in 4-2ε dimensions, various traces are no longer 4 but 4-2ε)

\[ \sigma^{qg}(\epsilon) = \sigma_0 H(\epsilon) \int dx_1 dx_2 \frac{C_F \alpha_s(\mu)}{2\pi} \times \]
\[ \frac{(1-\epsilon)(x_1^2 + x_2^2) - \epsilon (1-x_1 - x_2)}{(1-x_1)^{1+\epsilon}(1-x_2)^{1+\epsilon}(2-x_1 - x_2)^\epsilon - 2\epsilon} \]
\[ = \sigma_0 H(\epsilon) \frac{C_F \alpha_s(\mu)}{2\pi} \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right], \quad (33.8) \]
\[ H(\epsilon) = \frac{3(1-\epsilon)^2}{(3-2\epsilon)(2-2\epsilon)} = 1 + \mathcal{O}(\epsilon). \]

As expected, this expression approaches the one we had in the previous lecture in the limit \( \epsilon \to 0 \).

The corresponding NLO virtual diagram is

![Virtual Diagram](image)

and the virtual contribution to the cross section contribution is

\[ \sigma^{qg(\epsilon)}(\epsilon) = \sigma_0 H(\epsilon) \frac{C_F \alpha_s(\mu)}{2\pi} \left[ \frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right], \quad (33.9) \]

Note the relative signs of the potentially singular terms compared to the previous equations!

If we now sum the two expressions in Eqs.(33.8) and (33.9), we see that the divergent terms (i.e., the inverse powers of \( \epsilon \)) cancel. Defining, as usual, \( R = \sigma(e^+e^-\to \text{hadrons})/\sigma(e^+e^-\to \mu^+\mu^-) \), we can set \( \epsilon \to 0 \) and find the following result to first nontrivial order in the strong coupling.
\[ R = 3 \sum_{f} e_{f}^{2} \left[ 1 + \frac{3C_{F} \alpha_{s}(\mu)}{4\pi} + \mathcal{O}(\alpha_{s}^{2}) \right] \]
\[ = 3 \sum_{f} e_{f}^{2} \left[ 1 + \frac{\alpha_{s}(\mu)}{\pi} + \mathcal{O}(\alpha_{s}^{2}) \right], \tag{33.10} \]

After performing a bit more work we find at (even) higher order that we can define the quantity

\[ K \equiv \frac{R}{3 \sum_{f} e_{f}^{2}} - 1 = \frac{\alpha_{s}(\mu)}{\pi} + \left\{ 1.4092 + 1.9167 \ln \left( \frac{\mu^{2}}{Q^{2}} \right) \right\}^{2} \left( \frac{\alpha_{s}(\mu)}{\pi} \right) \]
\[ + \left\{ -12.805 + 7.8179 \ln \left( \frac{\mu^{2}}{Q^{2}} \right) + 3.674 \ln^{2} \left( \frac{\mu^{2}}{Q^{2}} \right) \right\} \left( \frac{\alpha_{s}(\mu)}{\pi} \right)^{3} + \mathcal{O} \left( \left( \frac{\alpha_{s}(\mu)}{\pi} \right)^{4} \right) \tag{33.11} \]

This result exhibits several features that are typical of pQCD results –

- physical quantities are \( \mu \) independent, but (fixed order) pQCD results exhibit explicit \( \ln(\mu^{2}/Q^{2}) \) terms (besides the running of the coupling),

- including higher order contributions reduces the dependence on the unphysical parameter \( \mu \),

- the \( \mu \) dependence is an artifact of the truncation of the perturbative expansion,

- at order \( \alpha_{s}^{n} \) the residual \( \ln(\mu^{2}) \) dependence is of order \( \alpha_{s}^{n+1} \) \( (i.e., \ the \ first \ term \ not \ included \ in \ the \ truncated \ series) \) and provides some measure of that contribution.
The quantity $K$ at various orders (1, 2 or 3) in perturbation theory is exhibited in the figure to the right illustrating the various features just listed (LO = Leading Order, NLO = Next to Leading Order, etc.). The x-axis corresponds to a varying value of the scale $\mu$, varied by changing the exponent (p) of 2 in the expression $\mu = 2^p Q$.

The lessons we should learn from this exercise are

- there is a large improvement (i.e., reduction in the $\mu$ dependence) in going LO $\rightarrow$ LO + NLO,
- at lowest order the behavior is monotonic in $\mu$,
- at higher order there is a stable region around $p = 0, \mu \sim Q$ (the “natural” scale),
  and
- the sign of the $\mu \rightarrow 0$ divergence oscillates order by order.

Since there typically is some residual dependence on $\mu$ at any fixed order, the question arises – how do we choose a specific value for $\mu$ in order to provide the best estimate of the “true” value of the quantity calculated? There are various prescriptions for this choice appearing in the literature:

- PMS (Principle of Minimal Sensitivity), i.e., choose $\mu$ so that $\left. \frac{dK}{d\mu} \right|_{\mu_{\text{PMS}}} = 0$, and
- FAC (Fastest Apparent Convergence), i.e., choose $\mu$ so that $K^{(1)}(\mu_{\text{FAC}}) = K^{(2)}(\mu_{\text{FAC}})$, and
- BLM – i.e., choose $\mu$ so that we absorb all $n_f$ dependence into $\alpha_s$.

But it is important to note that strictly speaking there is no right answer! The residual dependence on $\mu$ in the “physical range” (e.g., $p = 0\pm1$ in the figure above) is simply a reminder of, and a (crude) measure of the uncertainty due to the uncalculated higher order contributions! The specific choice of $\mu$ cannot remove this uncertainty.
Now let us return to the issue of DIS (Deeply Inelastic ep Scattering) and see how Parton Distribution Functions are changed in the context of QCD. The first change to consider is the inclusion of real gluon emission (taking massless partons). The corresponding diagrams look like

where (as usual) singularities can arise when the internal propagators (k and k’) go on-shell \( i.e., \) collinear and soft gluon emission. In the appropriate (light-cone) gauge, the divergent contribution in the middle diagram can be written (the “location” of the divergence is gauge dependent, in the appropriate light-cone gauge only the middle graph is singular)

\[
\hat{F}_2 \bigg|_{\text{Div}} = e_q^2 \frac{\alpha_s}{2\pi} x \hat{P}_{qq}(x) \int_0^2 \frac{d|k^2|}{|k^2|},
\]  

(33.12)

We note that the \(|k^2|\) integral goes all the way up to the kinematic boundary, \( i.e., \) it is not cutoff at a fixed (small) value as assumed by the parton model (so expect some differences), and the \(|k^2|\) integral is singular at the lower limit. We will control this infrared divergence with a cutoff \( \kappa^2 \) for now. This “long distance” behavior is controlled by “confinement” in real life. The coefficient of this (collinear) singularity is multiplied by a characteristic function of the quark’s momentum fraction \( x \) – the so-called “splitting function” \( \hat{P}_{qq} \) that tells us how the longitudinal momentum is shared, \( i.e., \) describes the quark \( (1) \rightarrow \) quark \( (x) + \) gluon \( (1 - x) \) vertex in the collinear limit,

\[
\hat{P}_{qq}(x) = C_F \frac{1 + x^2}{1 - x}. \]  

(33.13)

Note the (extra) singularity in the limit \( x \rightarrow 1 \) corresponding to the emission of a collinear and soft gluon. To learn a bit more about how this structure arises and how to interpret it, we consider some of the details of the calculation (see, \( e.g., \) Chapter 4 in the QCD text by Ellis, \textit{et al.}). We choose the following vectors for the incident...
quark, light-like gauge fixing vector and virtual photon (these are slightly different choices from what we made earlier and the mass factor has been scaled out):  

\[ p^\mu = (P, 0, 0, P); \quad n^\mu = \left( \frac{1}{2P}, 0, 0, -\frac{1}{2P} \right); \quad q^\mu = \nu n^\mu + q_T^\mu \]  

(33.14)

where the vector \( q_T^\mu \) is a “transverse” vector typical of such relativistic calculations,

\[ q_T^\mu = (0, q_T, 0). \]  

(33.15)

With these definitions we have

\[ p^2 = n^2 = q_T \cdot n = q_T \cdot p = 0; \quad n \cdot p = 1; \quad q \cdot p = \nu \]  

(33.16)

\[ q_T^2 = -q^2 = Q^2; \quad x = Q^2/2\nu. \]

If the emitted gluon has momentum \( r^\mu \) and polarization \( \epsilon^\mu \), we require (conserved current and gauge choice)

\[ \epsilon \cdot r = \epsilon \cdot n = 0. \]  

(33.17)

The momentum of the internal quark leg can be expressed in terms of components along \( p^\mu, n^\mu \) and a transverse vector \( k_T^\mu \) (similar to \( q_T \))

\[ k^\mu = \xi p^\mu + \frac{k_T^2 - |k|^2}{2\xi} n^\mu + k_T^\mu; \quad d^4k = \frac{d\xi}{2\xi} dk^2 dk^2 k_T. \]  

(33.18)

Note that this internal quark momentum is necessarily spacelike, \( k^2 < 0 \) (in our metric). The appropriately summed, averaged (spin and color) and projected matrix element is

\[ \frac{1}{4\pi} n^\alpha n^\beta \bar{\Sigma} |M|_\alpha^2 = \frac{8e_q^2 \alpha_s}{|k|^2} \xi \hat{P}(\xi) \]  

(33.19)

with \( \hat{P}(\xi) = \xi \) as defined above in Eq.(33.13). The 2-body phase space in these variables is (note that we choose to express this phase space in terms of the internal quark line in order to more easily identify the inherent singularities)
\[ d\Phi_2 = \frac{1}{16\nu\pi^2} \int d\xi dk^2 dk_T^2 d\theta \delta \left( k_T^2 - (1 - \xi) |k^2| \right) \times \delta \left( \xi - x - \frac{|k^2| + 2q_T \cdot k}{2\nu} \right). \tag{33.20} \]

The \( \delta \) functions in this expression put the outgoing \( q \) and \( g \) on-shell and, in the scaling limit, give the expected factor of \( \delta (\xi - x) \). Performing all of the integrals \((0 < \theta < \pi)\) including \( d\xi \) yields the result above in Eq. (33.12),

\[ \hat{F}_2 \bigg|_{\text{div}} = e_q^2 \frac{\alpha_s}{2\pi} \int d\xi \delta (\xi - x) \xi \hat{P}(\xi) \int_0^{2\nu} \frac{d|k^2|}{|k^2|}. \tag{33.21} \]

We can change the upper limit of the \( k^2 \) integration to the more familiar \( Q^2 \) at the expense of adding a non-divergent term proportional to \( \ln x \). This term is absorbed into the finite (no collinear singularity) term to be described below. We can think of the divergent term as describing the (collinearly) divergent piece of the scattering of a quark of momentum fraction 1 by a photon of momentum (squared) \( Q^2 \), where the photon is actually absorbed by a quark of momentum fraction \( \xi \) after the emission of the gluon.

When we put in the IR cutoff \( \kappa^2 \) and include all of the diagrams above (in standard form and including the non-divergent contributions), we obtain a revised form of the quark contribution to \( F_2 \), i.e., including the NLO pQCD correction,

\[ \hat{F}_2^{\text{NLO}}(x, Q^2) = e_q^2 x \left[ \delta (1 - x) + \frac{\alpha_s}{2\pi} \left( \hat{P}_{qq}(x) \ln \left( \frac{Q^2}{\kappa^2} \right) + C_q(x) \right) \right]. \tag{33.22} \]

In this expression both the splitting function \( \hat{P}_{qq}(x) \) and the non-singular bit \( C_q(x) \) are calculable functions in pQCD.

At this stage in the calculation we can already see that the prediction of a scaling \( F_2 \), which was characteristic of the naïve parton model, is broken by \( \ln(Q^2) \) terms in QCD. Further, just as we did for the logarithms that appeared in the renormalization of the coupling, we will need to sum the logarithms that appear here!
Next we should consider the contributions of the virtual graphs that contribute in this order in pQCD (through their interference with the LO diagram above), which will serve to regulate the “soft” singularity ($x \to 1$).

In the language of the real emission variables these virtual contributions are proportional to $\delta((p+q)^2)$ and contribute only for $x = 1$, i.e., contribute to $\delta(1-x)$ terms. By defining a new sort of function, the “+$” function (thought of like the $\delta$ function – only defined inside an integral), we obtain the “full” splitting function at this order in pQCD, which has the attractive feature that quark (baryon) number is conserved (you should check this), independent of $Q^2$

$$\hat{P}_{qq}(x) \to P_{qq}(x) = C_F \left[ \frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right]$$

$$= C_F \left( \frac{1+x^2}{1-x} \right)_+. \quad (33.23)$$

The “+$” function introduced here is defined by ($f(x)$ is a general integrable function)

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx \frac{f(x) - f(1)}{(1-x)}. \quad (33.24)$$

With care taken below for the process $g \to q\bar{q}$, the virtual contributions lead to the a replacement of the form

$$\hat{P}(x) \to \hat{P}(x)_+ = P(x). \quad (33.25)$$

Now we are prepared to consider how these first order corrections change our interpretation of the full quark distributions. As we did in the naïve quark model we want to convolute $\hat{F}_i$ with the “bare” quark distributions to obtain the “full” $F_i$, from which we can again identify the QCD corrected quark distribution functions. For that
purpose it is useful to redefine \( \hat{F}_x \) so that it describes an incoming quark of momentum fraction \( \xi \) and a momentum fraction \( \xi' \) after the gluon emission. We can express this in the form (with an explicit upper index to remind us that this form includes just the first non-trivial order in pQCD)

\[
\hat{F}_{2,q}^{NLO} (x, \xi, Q^2) = e_q^2 \int d\xi' \delta(\xi' - x) \left[ \xi' \delta(\xi - \xi') + \frac{\alpha_s}{2\pi} \frac{\xi'}{\xi} \left( P_{qq} \left( \frac{\xi'}{\xi} \right) \ln \left( \frac{Q^2}{\kappa^2} \right) + C_q \left( \frac{\xi'}{\xi} \right) \right) \right] \\
+ \frac{\alpha_s}{2\pi} \frac{\xi'}{\xi} \left( P_{qq} \left( \frac{\xi'}{\xi} \right) \ln \left( \frac{Q^2}{\kappa^2} \right) + C_q \left( \frac{\xi'}{\xi} \right) \right) \\
= e_q^2 x \delta(\xi - x) + \frac{\alpha_s}{2\pi} \frac{1}{\xi} \left( P_{qq} \left( \frac{x}{\xi} \right) \ln \left( \frac{Q^2}{\kappa^2} \right) + C_q \left( \frac{x}{\xi} \right) \right). 
\]  

(33.26)

Convoluting this expression with the bare quark distributions we find the quark contribution to the full \( F_2 \) at orders \( \alpha_s^0 \) and \( \alpha_s^1 \) is now (recall that \( x \) here is \( x_{bj} \), while \( \xi \) is playing the role of \( x_F \) of the last lecture)

\[
F_{2,q} (x, Q^2) = \sum_{q,\bar{q}} \int d\xi \hat{F}_{2,q}^{NLO} (x, \xi, Q^2) q_0 (\xi) \\
= \sum_{q,\bar{q}} e_q^2 x \left[ q_0 (x) + \frac{\alpha_s}{2\pi} \frac{x}{\xi} q_0 (\xi) \left( P_{qq} \left( \frac{x}{\xi} \right) \ln \left( \frac{Q^2}{\kappa^2} \right) + C_q \left( \frac{x}{\xi} \right) \right) \right]. 
\]  

(33.27)

Note that, since the splitting function \( P(z) \) vanishes for \( z > 1 \), the limits on the \( \xi \) integral are as indicated. The idea is that, since the momentum fraction after the gluon emission is always smaller than that before, we must have \( \xi > x \).

To proceed we introduce a factorization scale \( \mu \), which is the analog of the renormalization scale (typically also called \( \mu \)) that we used in our study of the running coupling. There are really two theoretical scales \( \mu_{\text{ren}} \) (the UV or renormalization scale) and \( \mu_{\text{fact}} \) (the collinear or factorization scale), although they are often set equal to each other in practice. We can now formally split the logarithm term into 2 pieces,

\[
\ln \left( \frac{Q^2}{\kappa^2} \right) = \ln \left( \frac{Q^2}{\mu^2} \right) + \ln \left( \frac{\mu^2}{\kappa^2} \right). 
\]  

(33.28)
We will use the factorization scale to “absorb” (i.e., factor) the leading collinear singularities (for $|k^2| < \mu^2$) into the bare distribution (to all orders), i.e., we sum the terms with the $\ln(\mu^2/k^2)$ factors to all orders in perturbation theory and include them by definition in the “renormalized” distribution. Once this resummation has been performed, we can formally set the cutoff $\kappa^2$ to zero while implicitly assuming that the factorized quark distribution is well behaved in this limit. Of course, we have no prediction for its value, but instead must determine it experimentally (similar to the running coupling). The long distance physics is now all in the factorized/renormalized distribution, which is defined by an integral differential equation as discussed in the next lecture. Thus we can express the quark contribution to deep inelastic scattering (where we now include the renormalization scale in the argument of the strong coupling and set the renormalization and factorization scales to be equal) in the form

$$
F_{2,q}(x, Q^2) = x \sum_{q, \bar{q}} e_q^2 \int_x^{\xi} \frac{d \xi}{\xi} q(\xi, \mu^2) \left[ \delta \left( 1 - \frac{x}{\xi} \right) + \frac{\alpha_s(\mu^2)}{2\pi} \left\{ P_{qq} \left( \frac{x}{\xi} \right) \ln \left( \frac{Q^2}{\mu^2} \right) + \mathcal{C}_q \left( \frac{x}{\xi} \right) \right\} + \ldots \right],
$$

(33.29)

where $q(\xi, \mu^2)$ is the new, improved quark distribution. The $\ldots$ in the square bracket is to remind us that we can systematically calculate and include (finite) higher order terms in this expansion in powers of $\alpha_s$. Note that, since we have factorized formally infinite terms from the logarithm into the new distribution function $q(\xi, \mu^2)$, there is real ambiguity about what is left of the finite term $C_q$ (the so-called coefficient function). To make this explicit, we have divided the original term into 2 terms,

$$
C_q(z) \equiv \mathcal{C}_q(z) + \tilde{C}_q(z).
$$

(33.30)

By definition we keep the $\mathcal{C}_q$ term in the equation for $F_2$ and we sum the $\tilde{C}_q$ terms along with the divergent logarithms into the renormalized distribution functions (see the next lecture). Thus this term in $F_2$, and the parton distribution function, will depend on the specific choice of factorization scheme, i.e., how much of the original finite piece gets absorbed with the infinity (and also on the UV renormalization scheme discussed earlier). Physical quantities are scheme independent and the result of a theoretical calculation will be scheme independent also, if all parts are performed in the same scheme!
One choice of factorization scheme is the DIS choice. In this case we absorb
everything, \( \overline{C}^{\text{DIS}}_q = 0, \overline{\tilde{C}}^{\text{DIS}}_q = C_q \). While this tends to simplify the discussion of the
DIS process itself, it leads to more complicated expressions for other physical
processes. A more commonly used choice is the \( \overline{\text{MS}} \) scheme mentioned earlier where we “absorb” the \( \gamma_E \) and \( \ln(4\pi) \) terms that always appear with the offending logarithms
(or \( 1/\epsilon \) terms in DimReg).

The above factorization process implies that we can identify the corresponding NLO
version of the quark distribution function (due to quarks as partons) as

\[
q_q(x, \mu^2) = q_0(x) + \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 \frac{d\xi}{\xi} q_0(\xi) \left[ P_{qq}(\frac{x}{\xi}) \ln \left( \frac{\mu^2}{\kappa^2} \right) + \overline{\tilde{C}}_q \left( \frac{x}{\xi} \right) \right] + \ldots, \tag{33.31}
\]

where the \( \ldots \) reminds us that we are summing the leading logarithmic dependence on \( \kappa \) to all orders in the strong coupling. In this expression \( q_0 \) plays a role similar to that played by \( \alpha_s(M) \) used in our earlier discussion of the renormalized coupling (see Eq. (30.35)), i.e., a place to “hide” infinities. We cannot predict the “bare” quantity, but we can measure the full quantity and predict how it varies with the scale \( \mu^2 \).

To summarize (pictorially), we are summing the largest contributions of the emission
of multiple gluons, where we order the multiple emissions by their “size” as
suggested in the figure to the right. The change in “size” of the gluons represents the
strong ordering of the transverse momenta

\[
k_{T,1}^2 << k_{T,2}^2 << \cdots << k_{T,n}^2 \quad \text{(smaller wavelength means larger momentum)}. \]

The limits on the included emissions are provided by the scales \( Q \) (the short distance limit) and \( \kappa \) (the long distance cutoff).
Next we separate contributions above and below the factorization scale $\mu$ as suggested in the second figure.

Finally we factor scales $\kappa$ to $\mu$ into the renormalized distribution leaving the short distance “structure” in the last figure to be explicitly accounted for in pQCD. Only the short distance physics is explicitly included in this perturbative analysis, which is the part we can hope to treat reliably in this way.

We will further explore these ideas in the next lecture. In particular, we must explore the role of gluons as partons.