Physics 558 – Lecture 30 - Appendix A: Comments on propagators (see, for example, Chapter 4 in Peskin and Schroeder).

We want to discuss how the momentum space propagators of the Feynman rules connect to our understanding of the properties of fields. Consider a scalar field expressed in terms of HO style creation and annihilation operators for momentum eigenstates (plane waves),

\[
\phi(\tilde{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_p e^{-ip \cdot x} + a_p^* e^{ip \cdot x} \right]_{p^0 = E_p = \sqrt{p^2 + m^2}} . \tag{30.A.1}
\]

The propagator to create a (on mass-shell) particle at \( y \) and annihilate one at \( x \), i.e., a propagator that describes the propagation of the particle from \( y \) to \( x \), looks like

\[
D(x - y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle . \tag{30.A.2}
\]

Using the fact that

\[
\langle 0 | a_p a_q^* | 0 \rangle = (2\pi)^3 \delta^3(\tilde{p} - \tilde{q}) , \tag{30.A.3}
\]

we find that

\[
D(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x - y)} |_{p^0 = E_p = \sqrt{p^2 + m^2}} . \tag{30.A.4}
\]

Using this integral form it is possible to verify the expected behavior of this propagator. For a particle sitting still, \( \tilde{x} = \tilde{y} \), \( x^0 - y^0 = t \), the leading behavior, modulo the overall constant, is

\[
D(t) \bigg|_{t \to 0} \sim e^{-imt} , \tag{30.A.5}
\]

while for two points with a purely spatial separation, \( x^0 = y^0 \), \( \tilde{x} - \tilde{y} = \tilde{r} \), we find true damping, again ignoring the overall constant,

\[
D(r) \bigg|_{r \to 0} \sim e^{-mr} . \tag{30.A.6}
\]

We can now connect this configuration space propagator with the corresponding Feynman rule propagator in momentum space. If we transform the latter from momentum space into configuration space, we have
\[ D_F (x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}. \] (30.A.7)

Consider the \( p^0 \) integration, which we shall perform using Cauchy’s integral theorem.

The complex \( p^0 \) plane looks like the figure, where the \( p^0 \) integration is along the real axis. There are poles at
\[ p^0 = \pm \left( \sqrt{p^2 + m^2} - i\varepsilon \right). \]
We must treat two cases: the first where \( x^0 - y^0 > 0 \) and the second where \( x^0 - y^0 < 0 \). In the former we can close the integration contour in the lower half-plane where \( \text{Im} p^0 < 0 \), i.e., where the integrand is exponentially damped, and pick up the contribution of the pole at \( p^0 = + \left( \sqrt{p^2 + m^2} - i\varepsilon \right) \). Since the contour encircles the pole in the clockwise direction, we find

\begin{equation}
D_F (x - y) |_{x^0 - y^0 > 0} = -2\pi i \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\sqrt{p^2 + m^2}} e^{-ip(x-y)} |_{p^0 = \sqrt{p^2 + m^2}} = D(x - y). \end{equation} (30.A.8)

With the opposite time ordering, \( x^0 - y^0 < 0 \), we require \( \text{Im} p^0 > 0 \) to close the contour and pick up the contribution from the other pole, \( p^0 = - \left( \sqrt{p^2 + m^2} - i\varepsilon \right) \), passing around it in the counterclockwise direction. This change in sign of \( p^0 \) at the pole means that we can change the sign in the exponent (recall the integration of the 3-momentum is symmetric) keeping the usual positive definition of \( p^0 \) and find

\begin{equation}
D_F (x - y) |_{x^0 - y^0 < 0} = +2\pi i \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\sqrt{p^2 + m^2}} e^{-ip(x-y)} |_{p^0 = \sqrt{p^2 + m^2}} = D(y - x). \end{equation} (30.A.9)

In summary, the Feynman propagator in configuration space looks like
Thus the momentum space propagator of the Feynman rules provides the desired causal, time-ordered behavior in configuration space (i.e., signals, particles, etc. propagate forward in time).

Let us also consider briefly how this formalism connects to our expectations for the nonrelativistic interaction energy, i.e., the “potential”, provided by the exchange of a scalar particle (the Yukawa potential). Think of two “sources” (particles with large mass) at rest but exchanging scalar particles. In this nonrelativistic limit the energies of the two sources are (approximately) unchanging and \( p^0 \) of the exchange is vanishing small (this is a virtual exchange). Let the coupling be \( g \). The form of the potential (admittedly not thoroughly motivated here) is

\[
V(r) \propto (g)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{0 - \mathbf{p}^2 - m^2 + i\epsilon} e^{i \mathbf{p} \cdot \mathbf{r}}. \tag{30.A.11}
\]

We can evaluate this integral much as we did with the \( p^0 \) integral above. We choose the \( z \) direction to be along \( \mathbf{r} \) and perform the angular integrals to obtain

\[
-g^2 \int \frac{p^2 dp d\cos \theta d\phi}{(2\pi)^3} \frac{1}{p^2 + m^2} e^{ipr \cos \theta} = -g^2 \int \frac{p^2 dp}{(2\pi)^2} \frac{1}{p^2 + m^2} \left( e^{ipr} - e^{-ipr} \right) i pr \tag{30.A.12}
\]

\[
= -g^2 \int_{-\epsilon}^{\epsilon} \frac{p dp}{ir(2\pi)^2} \frac{e^{ipr}}{p^2 + m^2}.
\]

We can treat this final integral as a contour integral. The poles are at \( \pm (m - i\epsilon) \) and, since we can close the contour where \( \text{Im } p^3 > 0 \), we pick up (in a clockwise-direction) the pole in the upper half-plane. Thus the result is
While we have obviously been sloppy about keeping track of signs, this is the correct answer. The exchange of a scalar particle is generically attractive, \textit{i.e.}, moving to smaller values of $r$ lowers the energy. As we will see shortly, the same is not true for the exchange of vector particles. In that case the interaction between like-charged particles is repulsive, \textit{e.g.}, in the scattering of two identical particles via E&M. On the other hand, unlike charged particles, \textit{e.g.}, a particle-antiparticle pair, experience an attractive E&M interaction. The exchange of spin-2 particles, for example gravitons, is again generically attractive.

For the case of massless particle exchange as in theories with a gauge symmetry, we simply take the $m \to 0$ limit above, yielding $V \sim \alpha/r$, which is essentially guaranteed by dimensional considerations ($V$ has units of energy and the only dimensionfull quantity is $r$ with units of 1/energy).