Physics 558 – Lecture 29

The Neutral B System:

Let us close this discussion of mixing phenomena and CP violation with a brief summary of the neutral B system. (Note that there are both the \( B_d^0 \left( B_d^0 \right) \) and the \( B_s^0 \left( B_s^0 \right) \) systems. We will focus primarily on the former, but data are being collected on both.) Some of these results will repeat what has been mentioned in the previous 3 lectures and are intended to provide a contrast with the kaon system. In the language of the 2x2 Hamiltonian matrix in the flavor basis (assuming CPT symmetry) we have

\[
\mathcal{H} = \begin{pmatrix}
M_0 - i \frac{\Gamma_0}{2} & M_{12} - i \frac{\Gamma_{12}}{2} \\
M_{12}^* - i \frac{\Gamma_{12}^*}{2} & M_0 - i \frac{\Gamma_0}{2}
\end{pmatrix},
\]

(29.1)

where the kaon system is characterized by the fact that \( \Gamma_0, M_{12} \) and \( \Gamma_{12} \) are all of the approximately the same magnitude. This situation obtains because the same 2-pion channels dominate the total decay rate (\( \Gamma_0 \)), the mixing (\( M_{12} \)) and the off-diagonal width terms \( \Gamma_{12} \). On the other hand the CP violating phase in the CKM matrix appears only in contributions involving a virtual top quark that are a small part (~ \( 10^{-3} \)) of the mixing terms. Indirect CP violation is seen at this \( 10^{-3} \) level, while direct CP violation is down by another factor of \( 10^{-3} \).

In the \( B \) system, our observations and expectations are somewhat different. The first difference is that the overall mass scale is an order of magnitude larger,

\[
M_{B^0} = 5279.58 \pm 0.17 \text{ MeV}.
\]

(29.2)

As a result the lifetime is two orders of magnitude shorter than that of the \( K^0 \),

\[
\tau_{B^0} = (1.519 \pm 0.007) \times 10^{-12} \text{ s},
\]

(29.3)

(Instead of the \( 10^{-10} \) s number for the \( K^0 \)) and thus the full width is about a factor of 100 larger than for the \( K^0 \). Furthermore the bulk of this larger width does not arise from channels that can lead to mixing. For example, nearly 80% of the final states in \( B^0 \) decay contain a \( K^+ \), while \( B^0 \) decay s contain a \( K^- \) (+ anything). Flavor neutral
states like $\psi/J K_S$ contribute only at the $10^{-3}$ level. The observed mass splitting between the mass eigenstates (of the full Hamiltonian) is ($H =$ heavy, $L =$ light)

$$\Delta M_{B^0} \equiv M_{B^{0}_{H}} - M_{B^{0}_{L}} = (0.507 \pm 0.004) \times 10^{12} \, s^{-1} \quad \Rightarrow \quad (3.337 \pm 0.033) \times 10^{-10} \, \text{MeV.}$$

Note that it is again the case that $\Delta M$ is of order $1/\tau$ just as it was for the $K^0$ (although the magnitudes of the 2 quantities are different from the $K^0$ case). The PDG number for the ratio is

$$x_d = \frac{\Delta M_{B^0}}{\Gamma_{B^0}} = 0.770 \pm 0.008.$$  \hspace{1cm} (29.5)

So, as in the $K^0$ case, the decay and the oscillations occur on the same time scale. These numbers are to be contrasted with the $B^0_s \left( \bar{B}^0_s \right)$ system where

$$M_{B^0_s} = 5366.77 \pm 0.24 \, \text{MeV},$$

$$\tau_{B^0_s} = 1.497 \pm 0.015 \times 10^{-12} \, s,$$

$$\Delta M_{B^0_s} \equiv M_{B^0_{s,H}} - M_{B^0_{s,L}} = (17.69 \pm 0.08) \times 10^{12} \, s^{-1},$$

$$x_s = \frac{\Delta M_{B^0_s}}{\Gamma_{B^0_s}} = 26.49 \pm 0.29.$$ \hspace{1cm} (29.6)

Hence in the $B^0_s \left( \bar{B}^0_s \right)$ system (unlike the $B^0 \left( \bar{B}^0 \right)$ or $K^0 \left( \bar{K}^0 \right)$) multiple oscillations can be (and have been) seen before the particles decay.

Moving back to the $B^0 \left( \bar{B}^0 \right)$ system, we note that in moving from the kaon system to the $B$ system, the CKM and quark mass factors in our estimate of the mixing matrix element (i.e., the off-diagonal mass term $M_{12}$) have been changed by the replacement

$$V_{cs} V_{cd}^* \to V_{cb} V_{ub}^*, \quad m^2_c \to \lambda^2 m^2_c \Rightarrow \lambda^2 m^2_t \to \lambda^6 m^2_t.$$ \hspace{1cm} (29.7)
This corresponds to an enhancement factor of about 30, essentially matching the increase in the total width.

**ASIDE:** The same (naïve) analysis of the change in $\Delta M$ in going from the $B_d^0\left(\bar{B}_d^0\right)$ system to the $B_s^0\left(\bar{B}_s^0\right)$ system, *i.e.*, Eq. (29.7), yields

$$|V_{tb}V_{td}|^2 m_t^2 \rightarrow |V_{tb}V_{ts}|^2 m_s^2 \Rightarrow \lambda^6 m_t^2 \rightarrow \lambda^4 m_s^2. \quad (29.8)$$

Thus there is an enhancement by a factor of $1/\lambda^2 \approx 20$, in quite good agree with the numbers in Eq. (29.6).

Returning to the comparison of the neutral kaon system to the $B_d^0\left(\bar{B}_d^0\right)$ system, we see that the role of the off-diagonal width has *decreased* dramatically,

$$\frac{\left|\Gamma_{12}\right|}{M_{12}} \approx \frac{\left|\Gamma_{12}\right|}{\Gamma_0} \approx 10^{-3}, \frac{\Delta\Gamma}{\Gamma_0} \approx 10^{-3}. \quad (29.9)$$

Thus both eigenstates decay at essentially the same (high) rate and, *unlike* the $K^0$ case, an initial $B_d^0$ or $\bar{B}_d^0$ state does not automatically decay into a single $CP$ eigenstate. Since it was exactly this situation that allowed $CP$ violation to first be observed in the kaon system, we expect that we will have to proceed differently to study $CP$ violation in the $B_d^0$ system. If we simply ignore $\Delta\Gamma$, Eq. (29.1) becomes

$$\mathcal{H}_{B_d^0} = \begin{pmatrix} M_{b^0} - i\frac{\Gamma_0}{2} & M_{12} \\ M_{12}^* & M_{b^0} - i\frac{\Gamma_0}{2} \end{pmatrix}, \quad (29.10)$$

where

$$M_{12} \propto (V_{tb}V_{td}^*)^2 \propto e^{2i\phi_{\text{mixing}}}. \quad (29.11)$$

With the definitions of the phases in the CKM matrix provided in the last lecture (and by the PDG), we have $\phi_{\text{mixing}} = \beta$.

We can use the results of Lecture 27 to write the eigenvalues and eigenstates as
\[
\frac{M_1}{M_2} = \frac{M_{B^0}}{M_{B^0}} \mp |M_{12}|, \quad \Gamma_1 = \Gamma_2 = \Gamma_0, \quad (29.12)
\]
and
\[
\begin{align*}
|B_1\rangle & = \frac{1}{\sqrt{1 + |\chi|^2}} \left[ B^0 \pm \chi |\bar{B}^0\rangle \right], \\
|B_2\rangle & = \frac{1}{\sqrt{M_{12}}} B^0, \\
\chi & = \frac{M_{12}}{M_{12}}. \quad (29.13)
\end{align*}
\]

With the choice
\[
CP |B^0\rangle = -|\bar{B}^0\rangle, \quad (29.14)
\]
we identify \(B_1\) with a state that is essentially \(CP\) even and \(B_2\) is nearly \(CP\) odd. In this limit of ignoring the splitting of the widths, we see that \(\chi\) is just a phase (the negative of the phase of \(M_{12}\)),
\[
\chi = e^{-i\phi_{\text{mixing}}}. \quad (29.15)
\]
Experimentally we find \(|\chi| = 1.0002 \pm 0.0028\), consistent with being just a phase. This means that \(CP\) violation is a tiny effect in the mixing of the \(B\) system and we will need to focus more on direct \(CP\) violation in the decays.

To analyze such decays in the \(B\) system we start by noting that the time evolution of the energy eigenstates defined above is just
\[
\begin{align*}
|B_1(t)\rangle & = e^{-iM_{B^0}t} e^{-i\Gamma_{B^0}t} |B_1(0)\rangle, \\
|B_2(t)\rangle & = e^{-iM_{B^0}t} e^{-i\Gamma_{B^0}t} e^{-i\Delta M_{B^0}t} |B_2(0)\rangle, \quad (29.16)
\end{align*}
\]
\[
\Delta M_{B^0} = M_2 - M_1 = 2|M_{12}|. 
\]
Thus the time evolution of the amplitude for a state that is pure \(B^0 (\bar{B}^0)\) at \(t = 0\) is given by
\[ |B^0(t)\rangle = \frac{1}{2} \left\{ e^{-iM_B t} e^{-i\Delta M t} \right\} \left[ (1 + e^{-i\Delta M t}) |B^0(0)\rangle - \chi (1 - e^{-i\Delta M t}) |\bar{B}^0(0)\rangle \right], \]
\[ |\bar{B}^0(t)\rangle = \frac{1}{2} \left\{ e^{-iM_B t} e^{-i\Delta M t} \right\} \left[ (1 + e^{-i\Delta M t}) |\bar{B}^0(0)\rangle - \frac{1}{\chi} (1 - e^{-i\Delta M t}) |B^0(0)\rangle \right]. \]  

Thus the probability that an initial $B^0$ has not decayed and is still a $B^0$ at time $t$ is given by
\[ P\left[B^0 \rightarrow B^0(t)\right] = |\langle B^0(0) | B^0(t) \rangle|^2 = \frac{e^{-\Gamma_G t}}{2} (1 + \cos \Delta M_{B^0} t). \]  

The corresponding probability to have oscillated into a $\bar{B}^0$ is
\[ P\left[B^0 \rightarrow \bar{B}^0(t)\right] = |\langle \bar{B}^0(0) | B^0(t) \rangle|^2 = \frac{e^{-\Gamma_G t}}{2} |\chi|^2 (1 - \cos \Delta M_{\bar{B}^0} t) \]
\[ = \frac{e^{-\Gamma_G t}}{2} (1 - \cos \Delta M_{\bar{B}^0} t). \]

This is just what we would expect from our study of the $K^0$ system in the limit of equal widths (in which case $\chi$ is just a phase).

Now imagine that we observe the specific final state $f$ (i.e., we are able to detect it) and we define the decay amplitudes (allowing $|\bar{f}\rangle = CP|f\rangle \neq |f\rangle$)
\[ A(f) = \langle f | \mathcal{H} | B^0 \rangle, \quad \bar{A}(f) = \langle f | \mathcal{H} | \bar{B}^0 \rangle, \]
\[ A(\bar{f}) = \langle \bar{f} | \mathcal{H} | B^0 \rangle, \quad \bar{A}(\bar{f}) = \langle \bar{f} | \mathcal{H} | \bar{B}^0 \rangle. \]  

If $CP$ is conserved in the decay, $[CP, \mathcal{H}] = 0$, and focusing for now on final states that are $CP$ eigenstates, $CP|f\rangle = \eta_f |f\rangle \quad (\eta_f = \pm 1)$, we have
\[ A(f) = \langle f \mid \mathcal{H} \mid B^0 \rangle = -(\eta_f)^{-1} \langle f \mid (CP)^{-1} \mathcal{H} \mid CP B^0 \rangle \]
\[ = -\eta_f \langle f \mid \mathcal{H} \mid B \rangle = -\eta_f \bar{A}(f), \]  
(29.21)

\[ \rho(f) = \bar{\rho}(f) = -\eta_f. \]

If \( CP \) is violated in the decay but there is only a single decay amplitude involved (or at least just a single phase), then \( \rho \) is just a phase and we can write

\[ \bar{\rho}(f) = -\eta_f e^{-2i\phi_{\text{decay}}}. \]  
(29.22)

This is the situation in the Standard Model where a single CKM phase dominates the \( B^0 \) system. The time evolution of the decay rate for \( B^0 \to f \) is given by

\[ P[B^0 \to f] \propto e^{-\Gamma f t} \left| A(f) \right|^2 \times \left\{ 1 + \cos \Delta M_{B^0} t \right\} + \left( 1 - \cos \Delta M_{B^0} t \right) \left| \bar{\rho}(f) \right|^2 + 2 \sin \left( \Delta M_{B^0} t \right) \text{Im} \left( \chi \bar{\rho}(f) \right), \]  
(29.23)

\[ P[B^0 \to f] \propto e^{-\Gamma f t} \left| \bar{A}(f) \right|^2 \times \left\{ 1 + \cos \Delta M_{B^0} t \right\} + \left( 1 - \cos \Delta M_{B^0} t \right) \left| \rho(f) \right|^2 + 2 \sin \left( \Delta M_{B^0} t \right) \text{Im} \left( \chi^{-1} \rho(f) \right). \]

Using this \( (CP \) eigenstate) decay channel we can detect (direct) \( CP \) violation by measuring the time-dependent asymmetry in the rates defined by (also sometimes labeled \( \mathcal{A}_{\text{FCP}} \)),

\[ a_f(t) = \frac{P[B^0 \to f] - P[B^0 \to f]}{P[B^0 \to f] + P[B^0 \to f]}. \]  
(29.24)

For a situation where the decay is dominated by a single amplitude (or at least there is just a single phase in the decay amplitude) and we ignore the splitting of the decay widths, both \( \chi \) and \( \rho \) are just phases as noted earlier. In this case (we leave the details for the homework) the decay asymmetry is given by a remarkably simple form, free of any issues about the evaluation of the details of the mixing or decay,

\[ a_f(t) = -\eta_f \sin \left( 2\phi_{\text{mixing}} + 2\phi_{\text{decay}} \right) \sin \Delta M_{B^0} t. \]  
(29.25)
Note that, while the specific values of the individual angles, $\phi_{\text{mixing}}$ and $\phi_{\text{decay}}$ may be dependent on choices of conventions, the sum is independent of these choices and is simply the angle $\beta$ (recall the figure on page 13 of Lecture 28). The $CP$ violation described in this way corresponds to interference between decays both with and without mixing (i.e., before and after mixing). Note also that, as expected in a situation where $|\chi|=|\rho|=1$, there is no $\cos \Delta M_{B^0} t$ term in the asymmetry.

A particularly attractive choice for $f$ (often referred to as the Golden Mode) is the decay $B^0 \left( \bar{B}^0 \right) \rightarrow J/\psi K_s^0$. This is a $CP$ odd state (the reader is encouraged to convince herself of this point) and we have

\begin{equation}
\eta_{J/\psi K_s} = -1 \tag{29.26}
\end{equation}

and

\begin{equation}
a_{J/\psi K_s} = \sin 2\beta \sin \Delta M_{B^0} t, \tag{29.27}
\end{equation}

where we have previously identified $\beta$ as the phase of the CKM element $V_{ud}^*$. In the terminology used by the PDG reviews the coefficient of $\sin \Delta M_{B^0} t$ in $a_f(t)$ is labeled $S_f$ and the PDG value is $S_{J/\psi K_s} = 0.679 \pm 0.020$. This is in good agreement with other measurements of $\sin 2\beta$ suggesting the consistency of the CKM matrix explanation of $CP$ violation.

We can also look for (direct) $CP$ violation by considering exclusive decays into a specific final state, $f$, where $|f\rangle = CP |f\rangle \neq |f\rangle$. In particular, we can directly look at the charge asymmetry

\begin{equation}
A_{CP}(f) = \frac{P[\bar{B}^0 \rightarrow \bar{f}] - P[B^0 \rightarrow f]}{P[\bar{B}^0 \rightarrow \bar{f}] + P[B^0 \rightarrow f]}. \tag{29.28}
\end{equation}

An informative example is the case $f = K^+ \pi^-$, for which the quoted PDG (average) asymmetry is $A_{CP}(K^+ \pi^-) = -0.097 \pm 0.012$, i.e., a clear observation of $CP$ violation (although the clarity is only in the recent data).

For the moment, the behavior of the $B^0 \left( \bar{B}^0 \right)$ system is fully consistent with the expectations of the Standard Model. However, with a large array of decay channels being precisely studied, at both the LHC and at dedicated $e^+e^-$ colliders, we can
anticipate rapidly improving tests of the Standard Model and, perhaps, evidence of Beyond the Standard Model physics to arise from the $B^0 (\bar{B}^0)$ system (see the presentations at the Terascale Workshop held in this Department last year with webpage here).

As suggested in the last lecture, the observation of $CP$ violation implies that a corresponding $T$ violation must also be present to preserve the expectation (in standard field theory descriptions) of $CPT$ conservation. The trick is to observe these effects separately, i.e., in terms of some observation that must be $T$ violating and not $CP$ violating. The so-called B-factory experiments offer exactly this possibility and here we want to discuss the results from the $BaBar$ experiment reported last autumn, *Phys. Rev. Lett.* **109**, 211801 (2012). The central elements of the experiment are colliding electron-positrons beams tuned to produce the $\Upsilon(4s)$ (Upsilon-4s) state. Recall this is the (excited) $bb$ vector meson with mass 10.58 GeV, i.e., just above the $BB$ threshold (recall the figure on page 10 in Lecture 12). This state decays approximately equally into $B^+B^-$ and $B^0\bar{B}^0$. Also the beams are asymmetric with 9.0 GeV electrons colliding head-on with 3.1 GeV positrons. Thus the $\Upsilon(4s)$ and the center-of-mass frame of the $BB$ system are moving in the lab with $\gamma v = 0.56$. This serves to spatially separate and allow “easy” analysis of the distinct decays of the $B$ and the $\bar{B}$. More importantly, since the $B$ and the $\bar{B}$ come from the decay of the single $\Upsilon$ (4s) state, the wave functions of the two $B$’s are quantum mechanically entangled (think “quantum computers”). This feature is extremely powerful for the $B^0\bar{B}^0$ system, upon which we now focus. (Actually we could pursue a similar path for the neutral kaon system using the $\phi$ meson, but this experiment has not been performed.) If the first decay identifies the decaying particle to be a $B^0$ say, then we know that, at that time, the other (not yet decayed) particle is a $\bar{B}^0$! And vice versa, if the first decay is a $\bar{B}^0$, the remaining particle is a $B^0$.

Even better, there are distinct decays that yield “tagging” of the particles. Recall that, as for the neutral kaon case, we have two different 2D basis sets – the flavor basis, $B^0$ and $\bar{B}^0$, and the overall eigenstates of the Hamiltonian $B_1$ and $B_2$, of Eq.(29.13), which are nearly $CP$ eigenstates. These latter states have simple time evolution as in Eq. (29.16). Still, like the neutral kaon system, we must allow for CP violation in both the mixing (indirect violation) and in the decays (direct violation). We can proceed by defining states in terms of the observed states into which the $B$ states decay, a natural approach from the experimental statement. So we define the various tagging decays as
Flavor: $B^0 \to \ell^+ X$, $\bar{B}^0 \to \ell^- X$, \hfill (29.29)

$CP$: $B_+ \to J/\psi K^0_L$, $B_- \to J/\psi K^0_S$.

Note that for the $CP$ case the $J/\psi$ is $CP$ even as is the $K^0_L$ while $K^0_S$ is $CP$ odd (i.e., we ignore $CP$ violation in the neutral kaon system in this discussion). However, since the $J/\psi$ is a vector particle while the other particles are pseudoscalars, the final state is orbital angular momentum 1, which adds an extra (-1) to the $CP$ analysis. We repeat that, while the first line defines the usual flavor eigenstates, the second line defines state that, in general, are not exactly equal to either $CP$ eigenstates or the Hamiltonian eigenstates $B_1$ and $B_2$. On the other, in the SM case of interest here, where there is a single $CP$ violating phase, the states $B_1$ and $B_2$ are orthogonal, $\langle B_+ | B_- \rangle = 0$, which is essential for the following analysis (can you prove this?). In particular, recall that, if the $CP$ tag is used for the first decay, we know that the remaining state is the orthogonal state.

**ASIDE:** Let us take a short detour to consider the states $B_+$ and $B_-$, and the orthogonality issue in more detail. We define the state $\tilde{B}_+$ by the fact that $\langle J/\psi, K^0_L | H | B_+ \rangle \neq 0$. Then consider the state $\tilde{B}_-$ that is both orthogonal and does not decay into this final state, $\langle B_+ | \tilde{B}_- \rangle = 0$ and $\langle J/\psi, K^0_L | H | \tilde{B}_- \rangle = 0$. Using the (well-defined) flavor basis we have

$$\tilde{B}_- = \frac{1}{\sqrt{1 + \alpha^2}} \left[ | B^0 \rangle - \alpha | \bar{B}^0 \rangle \right], \hfill (29.30)$$

where, recalling Eq. (29.20), we have

$$\langle J/\psi, K^0_L | H | \tilde{B}_- \rangle = 0$$

$$\Rightarrow \alpha = \frac{\langle J/\psi, K^0_L | H | B^0 \rangle}{\langle J/\psi, K^0_L | H | \bar{B}^0 \rangle} = \frac{A \langle J/\psi, K^0_L \rangle}{\bar{A} \langle J/\psi, K^0_L \rangle} = \rho \langle J/\psi, K^0_L \rangle. \hfill (29.31)$$

The assumed orthogonality then yields the form of $B_+$,
\[ \langle B_+ | \overline{B}_- \rangle = 0 \Rightarrow |B_+\rangle = \frac{1}{\sqrt{1 + |\alpha|^2}} \left[ |B^0\rangle + \alpha^* |B^0\rangle \right]. \] (29.32)

Repeating a similar analysis for \( B_- \) we have

\[ |\overline{B}_+\rangle = \frac{1}{\sqrt{1 + |\beta|^2}} \left[ |B^0\rangle - \beta |B^0\rangle \right], \]

\[ \langle J/\psi , K_S^0 | H | \overline{B}_+ \rangle = 0 \]

\[ \Rightarrow \beta = \frac{\langle J/\psi , K_S^0 | H | B^0 \rangle}{\langle J/\psi , K_S^0 | H | B^0 \rangle} = \frac{A(J/\psi , K_S^0)}{A(J/\psi , K_S^0)} = \rho(J/\psi , K_S^0), \] (29.33)

\[ \langle B_- | \overline{B}_+ \rangle = 0 \Rightarrow |B_-\rangle = \frac{1}{\sqrt{1 + |\beta|^2}} \left[ |B^0\rangle + \beta^* |B^0\rangle \right]. \]

So finally we consider the desired scalar product

\[ \langle B_- | B_+ \rangle = \frac{1 + \alpha^* \beta}{\sqrt{1 + |\alpha|^2} \sqrt{1 + |\beta|^2}}. \] (29.34)

For a system with a single \( CP \) violating phase, recall Eq. (29.22), we have the desired results

\[ \eta_{J/\psi , K_S^0} = -\eta_{J/\psi , K_L^0}, \quad \left| \rho(J/\psi , K_L^0) \right| = \left| \rho(J/\psi , K_S^0) \right| = 1 \]

\[ \Rightarrow \alpha = \rho(J/\psi , K_L^0) = -\rho(J/\psi , K_S^0) = -\beta \]

\[ \Rightarrow \alpha^* \beta = -\alpha^* \alpha = -1 \Rightarrow 1 + \alpha^* \beta = 0 \] (29.35)

\[ \Rightarrow \langle B_- | B_+ \rangle = 0. \]

So now we consider events where the two decays are into 2 of these 4 final states and then look at how the distributions of the decays vary as a function of the time between the two decays,

\[ \Delta \tau = t_2 - t_1, t_2 > t_1. \] (29.36)
A given event is labeled by $\Delta \tau$ and the two observed decays. The observed decays tag the transition undergone by the second (longer lived) particle between the first and second decays. For example, an event corresponding to a transition $B^0(t_1) \rightarrow B_-(t_2)$ between the two decays is labeled by $(\ell^\ast X, J/\psi K_0^0)$, where $\ell^\ast X$ means that the first decay at $t_1$ is a $B^0$ decay leaving a $\bar{B}^0$. The $J/\psi K_0^0$ means that it is a $B$. that decays at $t_2$. The corresponding $T$ reversed transition, $B_-(t_1) \rightarrow \bar{B}^0(t_2)$, is labeled by $(J/\psi K_0^0, \ell^\ast X)$. Any observed difference in the rates (as functions of $\Delta \tau$) of these two transitions is an indication of $T$ invariance violation. In summary the interesting transitions (between the first and second decays) for testing $T$ and their corresponding labels are indicated in the following table.

<table>
<thead>
<tr>
<th>Reference</th>
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<tbody>
<tr>
<td>Transition</td>
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<td>$\bar{B}^0 \rightarrow B_-$</td>
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</tr>
<tr>
<td>$B_+ \rightarrow B^0$</td>
<td>$(J/\psi K_0^0, \ell^\ast X)$</td>
</tr>
<tr>
<td>$\bar{B}^0 \rightarrow B_+$</td>
<td>$(\ell^\ast X, J/\psi K_0^0)$</td>
</tr>
<tr>
<td>$B_- \rightarrow B^0$</td>
<td>$(J/\psi K_0^0, \ell^\ast X)$</td>
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With the same set of transitions and observed final states we can also consider both $CP$ and $CPT$ conjugates. These are listed in the next long table.

<table>
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</tr>
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</tbody>
</table>
Thus measurements of these transitions will provide tests of all three possible invariances. We will present the numerical results of the BaBar study shortly, but first we consider the general form of the $\Delta\tau$ dependence. We work with the decay rates $R_{\alpha,\beta}^\pm (\Delta\tau)$ where the lower indices are defined by $\alpha \in \{\ell^+, \ell^-\}$ and $\beta \in \{K_S, K_L\}$ and indicate the definite flavor states $\{\ell^+X, \ell^-X\}$ and $CP$ eigenstates $\{J/\psi K_S, J/\psi K_L\}$. The upper index + or – indicates whether the decay to the flavor state $\alpha$ occurs before (at $t_1$) or after (at $t_2$) the decay to the $CP$-eigenstate $\beta$. To analyze this system we can use the previous results in Eqs. (29.20) through (29.23) and Eq. (29.26). It follows, as we review in the HW, that, in the limit that $\Delta\Gamma = 0$ (see Eq. (29.9) and which is a good approximation for the neutral $B$ system), the various decay rates can be expressed in the general form

$$R_{\alpha,\beta}^\pm (\Delta\tau) \propto e^{-\frac{\Gamma\Delta\tau}{2}} \left[ 1 + C_{\alpha,\beta}^\pm \cos\left(\Delta M_{\beta\beta}^\pm \Delta\tau\right) + S_{\alpha,\beta}^\pm \sin\left(\Delta M_{\beta\beta}^\pm \Delta\tau\right) \right]. \quad (29.37)$$

Here $C_{\alpha,\beta}^\pm$ and $S_{\alpha,\beta}^\pm$ are (initially) generic coefficients that we can fit to the data and the other parameters were defined earlier (see Eqs. (29.16) and (29.17)). The sine term arises from the interference between amplitudes with and without mixing, while the cosine term arises from the interference between decay amplitudes with different weak and strong phases. As suggested above, differences between the data fitted values of say $S_{\ell^+K_S}^+$ and $S_{\ell^-K_L}^-$ correspond to an observation of $T$ invariance violation.

As we verify in the HW, in a scenario with $CPT$ invariance and $\Delta\Gamma = 0$ (e.g., the SM) the various coefficients are simply related to each other. The coefficient of the sine term changes sign when we switch flavor states $\left( B^0 \leftrightarrow \bar{B}^0 \right)$, when we switch from a $CP$ even $f$ to a $CP$ odd $f$ or vice versa, or when we switch the order of the decays (flavor $\leftrightarrow$ $CP$). We have the following forms for the coefficients (recall Eq. (29.23))

$$S = \frac{2 \text{Im}(\chi\bar{\rho})}{1 + |\chi\bar{\rho}|^2} = S_{\ell^+K_L}^+ = S_{\ell^-K_S}^+ = -S_{\ell^+K_S}^+ = -S_{\ell^-K_L}^+, \quad S_{\ell^+K_L}^- = S_{\ell^-K_S}^- = -S_{\ell^+K_S}^- = -S_{\ell^-K_L}^-, \quad (29.38)$$

$$C = \frac{1 - |\chi\bar{\rho}|^2}{1 + |\chi\bar{\rho}|^2} = C_{\ell^+K_L}^+ = C_{\ell^-K_S}^+ = -C_{\ell^+K_S}^+ = -C_{\ell^-K_L}^+, \quad C_{\ell^+K_L}^- = C_{\ell^-K_S}^- = -C_{\ell^+K_S}^- = -C_{\ell^-K_L}^-.$$
(A detailed discussion can be found in arXiv:1203.0171. But note that the signs for the $C$ terms are different here – check this in the HW!) Here the parameter $\chi$ is as defined in Eq. (29.13). In particular, in the SM, as we have already noted, $\chi$ and $\bar{\rho}$ are phases, $\chi \bar{\rho} = e^{2i\beta}$ (see the discussion following Eq. (29.27)), and we have

$$S = \sin (2\beta),$$
$$C = 0.$$  
(29.39)

The results from the $BaBar$ analysis of autumn 2012 yield the following numbers

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Result</th>
<th>SM expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta S^+ = S_{e^-,K^0}^+ - S_{e^+,K^0}^+$</td>
<td>$-1.37 \pm 0.14 \pm 0.06$</td>
<td>$-2\sin (2\beta)$</td>
</tr>
<tr>
<td>$\Delta S^- = S_{e^+,K^0}^- - S_{e^-,K^0}^+$</td>
<td>$1.17 \pm 0.18 \pm 0.11$</td>
<td>$2\sin (2\beta)$</td>
</tr>
<tr>
<td>$\Delta C^+ = C_{e^-,K^0}^+ - C_{e^+,K^0}^+$</td>
<td>$0.10 \pm 0.14 \pm 0.08$</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta C^- = C_{e^+,K^0}^- - C_{e^-,K^0}^+$</td>
<td>$0.04 \pm 0.14 \pm 0.08$</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta S^+<em>{CP} = S</em>{e^-,K^0}^+ - S_{e^+,K^0}^+$</td>
<td>$-1.30 \pm 0.11 \pm 0.07$</td>
<td>$-2\sin (2\beta)$</td>
</tr>
<tr>
<td>$\Delta S^-<em>{CP} = S</em>{e^+,K^0}^- - S_{e^-,K^0}^+$</td>
<td>$1.33 \pm 0.12 \pm 0.06$</td>
<td>$2\sin (2\beta)$</td>
</tr>
<tr>
<td>$\Delta C^+<em>{CP} = C</em>{e^-,K^0}^+ - C_{e^+,K^0}^+$</td>
<td>$0.07 \pm 0.09 \pm 0.03$</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta C^-<em>{CP} = C</em>{e^+,K^0}^- - C_{e^-,K^0}^+$</td>
<td>$0.08 \pm 0.10 \pm 0.04$</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta S^+<em>{CPT} = S</em>{e^-,K^0}^+ - S_{e^+,K^0}^+$</td>
<td>$0.16 \pm 0.21 \pm 0.09$</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta S^-<em>{CPT} = S</em>{e^+,K^0}^- - S_{e^-,K^0}^+$</td>
<td>$-0.03 \pm 0.13 \pm 0.06$</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta C^+<em>{CPT} = C</em>{e^-,K^0}^+ - C_{e^+,K^0}^+$</td>
<td>$0.14 \pm 0.15 \pm 0.07$</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta C^-<em>{CPT} = C</em>{e^+,K^0}^- - C_{e^-,K^0}^+$</td>
<td>$0.03 \pm 0.12 \pm 0.08$</td>
<td>0</td>
</tr>
<tr>
<td>$S_{e^-,K^0}^+$</td>
<td>$0.55 \pm 0.09 \pm 0.06$</td>
<td>$\sin (2\beta)$</td>
</tr>
<tr>
<td>$S_{e^+,K^0}^+$</td>
<td>$-0.66 \pm 0.06 \pm 0.04$</td>
<td>$-\sin (2\beta)$</td>
</tr>
<tr>
<td>$C_{e^-,K^0}^+$</td>
<td>$0.01 \pm 0.07 \pm 0.05$</td>
<td>0</td>
</tr>
<tr>
<td>$C_{e^+,K^0}^+$</td>
<td>$-0.05 \pm 0.06 \pm 0.03$</td>
<td>0</td>
</tr>
</tbody>
</table>

Clearly the $BaBar$ results are consistent (within uncertainties) with the expectations of the Standard Model with a world average value (PDG) $\sin(2\beta) = 0.679 \pm 0.020$, CPT invariance and correlated $CP$ and $T$ violations.
Note that the definition of the quantity $\Delta S_{CP}^+ = S^+_{\ell^- K^0_\pi} - S^+_{\ell^- K^0_\pi}$ is the negative of the expression in Eq. (29.27) and so the two results agree.