Physics 558 – Lecture 27

The Neutral Kaon System: Technical Details

In this Lecture we want to make connection to some of the more formal aspects of the behavior of the neutral kaon system. We will study the time dependence of the system in Hamiltonian notation (i.e., this system can be usefully analyzed as a 2-state QM system),

\[ i \frac{\partial}{\partial t} \begin{pmatrix} K^0(t) \\ \bar{K}^0(t) \end{pmatrix} = \mathcal{H} \begin{pmatrix} K^0(t) \\ \bar{K}^0(t) \end{pmatrix}, \tag{27.1} \]

where, in principle, \( \mathcal{H} \) includes all relevant interactions, e.g., the strong, electroweak and Yukawa interactions,

\[ \mathcal{H} = \mathcal{H}_{\text{Strong}} + \mathcal{H}_{\text{Yukawa}} + \mathcal{H}_{\text{EM}} + \mathcal{H}_{\text{Weak}} + \cdots. \tag{27.2} \]

In this basis the Hamiltonian is a 2x2 matrix with the form,

\[ \mathcal{H} = M - i \frac{\Gamma}{2}, \tag{27.3} \]

where \( M \) and \( \Gamma \) are again 2x2 matrices. Due to the decays, the full Hamiltonian is no longer Hermitian but the individual components, \( M \) and \( \Gamma \), are Hermitian. Consider first the limit with \( \mathcal{H}_{\text{Weak}} = 0 \). Since \( K^0, \bar{K}^0 \) are eigenstates of strangeness, which is conserved by all interactions except the weak interactions, we have

\[ \left< \bar{K}^0 \left| \mathcal{H}_{\text{Strong}} + \mathcal{H}_{\text{Yukawa}} + \mathcal{H}_{\text{EM}} \right| K^0 \right> = 0. \tag{27.4} \]

Thus, in this limit, strangeness is conserved and there are no decays,

\[ \mathcal{H} \xrightarrow{\mathcal{H}_{\text{Weak}} \to 0} \begin{pmatrix} M_{K^0} & 0 \\ 0 & M_{\bar{K}^0} \end{pmatrix}, \tag{27.5} \]

where, due to CPT, the diagonal terms are equal, \( M_{K^0} = M_{\bar{K}^0} \equiv M_0 \). So what changes as we turn on the weak interactions? Now the neutral kaons can both decay and mix.
If we assume that $CPT$ is still a good symmetry, as it is in the Standard Model, the diagonal elements are still equal and we have

$$\mathbf{H} = \begin{pmatrix} M_0 - \frac{i}{2} i \Gamma_0 & M_{12} - \frac{i}{2} i \Gamma_{12} \\ M_{12}^* - \frac{i}{2} i \Gamma_{12}^* & M_0 - \frac{i}{2} i \Gamma_0 \end{pmatrix}. \quad (27.6)$$

If we also focus initially on the limit where $CP$ (which exchanges $K^0, \bar{K}^0$) is conserved, this matrix must be symmetric and we have

$$M_{12} = M_{21} = M_{12}^* \left( M_0 = M_0^* \right),$$

$$\Gamma_{12} = \Gamma_{21} = \Gamma_{12}^* \left( \Gamma_0 = \Gamma_0^* \right), \quad (27.7)$$

\text{i.e.}, both $M$ and $\Gamma$ are real. The resulting eigenvalues of the $CP$ conserving problem are

$$M_1 - i \frac{1}{2} \Gamma_1 = M_0 - M_{12} - i \frac{1}{2} \left( \Gamma_0 - \Gamma_{12} \right),$$

$$M_2 - i \frac{1}{2} \Gamma_2 = M_0 + M_{12} - i \frac{1}{2} \left( \Gamma_0 + \Gamma_{12} \right), \quad (27.8)$$

where we have chosen the signs to match the definitions of the previous lecture. The rotation matrix that performs the diagonalization is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (27.9)$$

so that we have, as noted above,

$$UMU^{-1} = \begin{pmatrix} M_0 - M_{12} & 0 \\ 0 & M_0 + M_{12} \end{pmatrix},$$

$$U\Gamma U^{-1} = \begin{pmatrix} \Gamma_0 - \Gamma_{12} & 0 \\ 0 & \Gamma_0 + \Gamma_{12} \end{pmatrix}. \quad (27.10)$$

The corresponding eigenstates are as defined in the previous lecture.
\begin{align}
\left| K_1 \right\rangle = U \left| K^0(t) \right\rangle = \frac{1}{\sqrt{2}} \left( \left| K^0(t) \right\rangle - \left| \bar{K}^0(t) \right\rangle \right),
\left| K_2 \right\rangle = \frac{1}{\sqrt{2}} \left( \left| K^0(t) \right\rangle + \left| \bar{K}^0(t) \right\rangle \right),
\end{align}

The relevant dimensionless parameters to describe the resulting mixing and oscillation, which we discussed in the last lecture, are typically chosen to be
\begin{align}
x = \frac{\Delta M}{\Gamma_0} = \frac{M_2 - M_1}{\Gamma_0} = \frac{2M_{12}}{\Gamma_0}, \quad y = \frac{\Delta \Gamma}{2\Gamma_0} = \frac{\Gamma_2 - \Gamma_1}{2\Gamma_0} = \frac{\Gamma_{12}}{\Gamma_0}.
\end{align}

For the case of kaons, we have
\begin{align}
\Gamma_1 \gg \Gamma_2 \Rightarrow \Gamma_{12} = -\Gamma_0 = -\frac{\Gamma_1}{2}, \quad y = -0.997,
\Delta M = 0.474 \Gamma_1 = (3.483 \pm 0.006) \times 10^{-12} \text{MeV}, \quad x = 0.945.
\end{align}

The first number tells us that the first eigenstate decays much more rapidly than the second, while the second results says that the oscillation time is comparable to the shorter of the two decay times. Thus, as we noted in the previous lecture, we essentially observe only a single oscillation. The dominant physics is the decay of the first eigenstate.

The corresponding numbers for the $D^0$ and $B^0$ systems are
\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
System & x & y \\
\hline
$D^0 - \bar{D}^0 \left( K^+ K^- \right)$ & 0.0100 ± 0.0025 & 0.0077 ± 0.0018 \\
$B^0 - \bar{B}^0$ & 0.771 ± 0.008 & <1% \\
\hline
\end{tabular}
\end{table}

From these numbers we conclude that we will not be able to observe mixing in the $D^0$ system but should see both mixing and real oscillations in the $B^0$ system.

So where do these numbers come from? Consider first just the $M$ component and keep terms through second order in the weak interaction,
\[ M_{11} = M_{K^0} + \left \langle K^0 \mid \mathcal{H}_\text{Weak} \mid K^0 \right \rangle + \sum_{n \neq K^0} \frac{\left \langle K^0 \mid \mathcal{H}_\text{Weak} \mid n \right \rangle^2}{M_{K^0} - E_n}, \]

\[ M_{22} = M_{\bar{K}^0} + \left \langle \bar{K}^0 \mid \mathcal{H}_\text{Weak} \mid \bar{K}^0 \right \rangle + \sum_{n \neq \bar{K}^0} \frac{\left \langle \bar{K}^0 \mid \mathcal{H}_\text{Weak} \mid n \right \rangle^2}{M_{\bar{K}^0} - E_n}. \]

As noted above, \( \mathcal{H}_\text{Weak} \) is invariant under CPT in the standard model and it follows that

\[ \left \langle K^0 \mid \mathcal{H}_\text{Weak} \mid K^0 \right \rangle = \left \langle \bar{K}^0 \mid \mathcal{H}_\text{Weak} \mid \bar{K}^0 \right \rangle, \]

\[ \left \langle \bar{K}^0 \mid \mathcal{H}_\text{Weak} \mid n \right \rangle = \left \langle \bar{K}^0 \mid (CPT)^{-1} \mathcal{H}_\text{Weak} CPT \mid n \right \rangle \]

\[ = -\left \langle \bar{n}' \mid \mathcal{H}_\text{Weak} \mid K^0 \right \rangle = -\left \langle K^0 \mid \mathcal{H}_\text{Weak} \mid \bar{n}' \right \rangle^*. \]

In the last expression the minus sign follows from our choice of the CP phase of the neutral kaons and the prime on \( n \) reminds us that the spins are all flipped due to \( T \). But the equation above includes a sum over all states. So, through second order, the diagonal terms are still identical. Thus \( M_{11} = M_{22} \) and both are real, since \( M \) is Hermitian, as noted above.

Again since \( M \) is Hermitian, we also have real off-diagonal contributions

\[ M_{12} = M_{12}^* = M_{21} \]

\[ = \left \langle K^0 \mid \mathcal{H}_\text{Weak} \mid \bar{K}^0 \right \rangle + \sum_{n \neq K^0, \bar{K}^0} \frac{\left \langle K^0 \mid \mathcal{H}_\text{Weak} \mid n \right \rangle \left \langle n \mid \mathcal{H}_\text{Weak} \mid \bar{K}^0 \right \rangle}{M_{K^0} - E_n}. \]

Since \( \mathcal{H}_\text{Weak} \) has \( |\Delta S| \leq 1 \), the first order off-diagonal term vanishes and we have only the second order term. Thus to this order we can express the mass splitting as

\[ \Delta M = 2M_{12} = 2 \sum_{n \neq K^0, \bar{K}^0} \frac{\left \langle K^0 \mid \mathcal{H}_\text{Weak} \mid n \right \rangle \left \langle n \mid \mathcal{H}_\text{Weak} \mid \bar{K}^0 \right \rangle}{M_{K^0} - E_n}. \]

In a 2-generation world with no CP violation, the relevant intermediate states in the language of the elementary degrees of freedom are either 2 \( W \)'s or the various
combinations of \((u, \bar{u}), (u, \bar{c}), (c, \bar{u}), (c, \bar{c})\), as suggested in the figures (where, with the choice above, time runs from left to right, but this cannot matter with \(T\) conserved).

For now let us forget the issues of color and the kaon wave function (i.e., confinement issues) and consider the perturbative form of the box diagrams exhibited above. Appropriately interpreted (i.e., ignoring “extra” quark-antiquark pairs) these are the squares of the quark diagrams for the decay processes discussed in the previous lecture. The sum over \(n\) now becomes a continuous integral over the momentum \(k\) flowing around the box. Formally the contribution to the real part \(M\) arises from the principal value of the integral, while the contribution to the imaginary part \(\Gamma\) comes from the discontinuity (imaginary part) arising from “real” physical intermediate states (i.e., the states that the \(K^0\) can actually decay into). First consider the \(u, \bar{u}\) intermediate state. The matrix element corresponding to the second figure has the form (recall that there are 4 flavor changing vertices and an \(i\) factor for the loop)

\[
M_{u\bar{u}} = \left(-i \frac{g}{2\sqrt{2}} \sin \theta_C \right)^2 \left(-i \frac{g}{2\sqrt{2}} \cos \theta_C \right)^2 \times i \int \frac{d^4k}{(2\pi)^4} \times \ldots
\]

where we have made the further approximation of setting all the external momenta to zero. This is a reasonable approximation since they will be kept small compared to \(M_w\) by an appropriate wave function for the \(K^0\). Notice that, due to the usual properties of the helicity projection operators, we can ignore the terms with the quark mass in the numerator. Only the \(k\) terms will contribute. We next notice that the integral appears to be quadratically divergent in the UV. This troubling feature is fixed when we include the GIM mechanism, i.e., sum over the various possible
quarks in the intermediate state. In our model world with 2-generations this yields the following factor arising from the mixing angles and fermion propagators

$$M_{u\pi} + M_{c\pi} + M_{u\pi} + M_{c\pi}$$

$$\propto \frac{\sin^2 \theta_c \cos^2 \theta_c}{(k^2 - m_u^2)^2} - 2 \frac{\sin^2 \theta_c \cos^2 \theta_c}{(k^2 - m_u^2)(k^2 - m_c^2)} + \frac{\sin^2 \theta_c \cos^2 \theta_c}{(k^2 - m_c^2)^2}$$

$$= \sin^2 \theta_c \cos^2 \theta_c \frac{(m_c^2 - m_u^2)^2}{(k^2 - m_u^2)^2 (k^2 - m_c^2)^2}.$$ (27.19)

The integral is now well behaved in the UV (and the IR). In fact, if the quarks were degenerate in mass ($m_u = m_c$), the contribution of these box diagrams would vanish. (A similar GIM cancellation also occurs when we include the full 3-generation structure of the CKM mixing matrix, as we will discuss below.) To leading order in the small expansion parameter $m_c^2 / M_w^2$ we can focus on the just the $g^{\lambda \sigma} g^{\mu \rho}$ terms from the $W$ propagators. We now have

$$M_{u\pi} + M_{c\pi} + M_{u\pi} + M_{c\pi} \equiv M_{q\bar{q}} \equiv ig^4 \frac{\sin^2 \theta_c \cos^2 \theta_c (m_c^2 - m_u^2)^2}{16}$$

$$\times \left[ \bar{u}_d \gamma_\lambda \gamma_\mu \gamma_\rho (1 - \gamma_5) u_s \right] \times \left[ \nabla_\delta \gamma^\rho \gamma^\nu (1 - \gamma_5) v_s \right] \times I_{\mu \nu}.$$ (27.20)

So finally the integral has the form

$$I_{\mu \nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 - m_u^2)^2 (k^2 - m_c^2)^2 (k^2 - M_w^2)^2}.$$ (27.21)

In the limit $M_w^2 \gg m_c^2 \gg m_u^2$ one can show (see the Appendix) that this integral has the value (note that the integral must be proportional to $g^{\mu \nu}$ since that is the only possible tensor)

$$I_{\mu \nu} \approx \frac{g^{\mu \nu}}{64\pi^2 iM_w^4 m_c^2}.$$ (27.22)

Thus we have
\[ M_{q\bar{q}} = g^4 \frac{\sin^2 \theta_C \cos^2 \theta_C}{16} \frac{m_c^2}{64\pi^2 M_W^4} \times \left[ \bar{u}_d \gamma_\lambda \gamma_\nu \gamma_\rho \left(1 - \gamma_5\right) u_s \right] \times \left[ \nabla_d \gamma^\mu \gamma^\nu \gamma^\lambda \left(1 - \gamma_5\right) v_s \right]. \]  

(27.23)

We can further simplify this expression with the following useful spinor identity (where we include a related identity for completeness – note the order of indices - you are encouraged to check these),

\[ \{ \gamma_\lambda \gamma_\nu \gamma_\rho \left(1 - \gamma_5\right) \} \{ \gamma^\rho \gamma^\nu \gamma^\lambda \left(1 - \gamma_5\right) \} = 4 \{ \gamma_\nu \left(1 - \gamma_5\right) \} \{ \gamma^\nu \left(1 - \gamma_5\right) \}, \]

\[ \{ \gamma_\rho \gamma_\nu \gamma_\lambda \left(1 - \gamma_5\right) \} \{ \gamma^\rho \gamma^\nu \gamma^\lambda \left(1 - \gamma_5\right) \} = 16 \{ \gamma_\nu \left(1 - \gamma_5\right) \} \{ \gamma^\nu \left(1 - \gamma_5\right) \}. \]  

(27.24)

Thus we obtain a factor of 4 and remove 4 of the 6 gamma matrices. With an appropriate Fierz transformation we can put the contribution of the second box diagram, the WW intermediate state, in an identical form.

ASIDE: Fierz transformations are discussed, for example, in Chapter 3 of Peskin and Schroeder, especially exercise 3.6. In general, they allow us to rearrange the order of the spinors in matrix elements of interest. The characteristic structure is

\[ \left( \bar{u}_1 \Gamma^K u_2 \right) \left( \bar{u}_3 \Gamma^B u_4 \right) = \sum_{CD} C_{CD}^{AB} \left( \bar{u}_1 \Gamma^C u_4 \right) \left( \bar{u}_3 \Gamma^D u_2 \right), \]

(27.25)

\[ \Gamma^K = \{ 1, \gamma^5, \gamma^\mu, \ldots \}. \]

Naively, adding the two (identical) contributions leads to another factor of two. However, if we now include the color quantum number for the quarks and the fact that the incoming and outgoing \( q\bar{q} \) states are color singlets (the quark and antiquark have the same color and the states are properly normalized), we find that the WW box has weight 1 but the \( q\bar{q} \) box has weight of only 1/3 for a total factor of 4/3 rather than 2.

ASIDE: To see this color factor think of the color singlet meson wave function as having two color indices, one for the quark and one for the antiquark. The expected form is

\[ \psi_{ab}^M = \frac{1}{3} \delta_{ab}, \]

(27.26)
where the Kronecker delta function sets the colors equal (except, of course, if the quark is red, the anti-quark is redbar) and the $1/3$ gives the correct normalization for each possible color. The box graph with the WW intermediate states corresponds to the quark annihilating with the antiquark from the same meson (initial or final state) and yields a color factor

$$F_{WW} = \sum_{a,b,c,d} \psi^*_d \psi_a \delta_{cd} \delta_{ab} = \frac{1}{9} \sum_{a,b,c,d} \delta_{cd}^2 \delta_{ab}^2 = \frac{1}{9} \sum_{a,b} = 1.$$  

(27.27)

In contrast, for the $q\bar{q}$ intermediate state the quark and anti-quark color flows across the diagram and connects the incoming state with the outgoing one. The corresponding factor looks like

$$F_{q\bar{q}} = \sum_{a,b,c,d} \psi^*_d \psi_a \delta_{ac} \delta_{bd} = \frac{1}{9} \sum_{a,b,c,d} \delta_{ac} \delta_{bd} \delta_{cd} \delta_{ab} = \frac{1}{9} \sum_{a,b} = \frac{1}{3},$$

(27.28)

i.e., the color connection runs all the way around “through the wave functions” and allows only one free sum over colors.

If we include these color factors and substitute for $g$ in terms of $G_F$, i.e., $g^2 = 4\sqrt{2}G_F M_w^2$, we obtain

$$M_{q\bar{q}} + M_{WW} \approx G_F \frac{\sin^2 \theta_c \cos^2 \theta_c}{6\pi^2} m_c^2$$

\[
\times [\bar{u}_d \gamma^\nu (1 - \gamma_5) u_s] \times [\bar{v}_d \gamma^\nu (1 - \gamma_5) v_s].
\]

(27.29)

Finally we replace the spinors with the usual creation /annihilation operators ($s, d$) and express the effective Hamiltonian as a 4-fermion term

$$\mathcal{H}_{\text{eff}} = M_{q\bar{q}} + M_{WW} \approx G_F \frac{\sin^2 \theta_c \cos^2 \theta_c}{6\pi^2} m_c^2$$

\[
\times d \gamma^\rho (1 - \gamma_5) s \cdot d \gamma^\rho (1 - \gamma_5) s.
\]

(27.30)

Thus we can write

$$\Delta M = 2M_{12} \approx 2 \left\langle K^0 \left| \mathcal{H}_{\text{eff}} \right| \bar{K}^0 \right\rangle / 2M_K^0,$$

(27.31)
where the factor in the denominator is to account for the relativistic normalization of the bra and ket.

Now comes the “hard” part – how do we evaluate this expectation value? One of the goals of lattice QCD is to be able to accurately calculate just such quantities. That numerical task is essentially done, but here we proceed analytically using the following (initially surprising) approximation. In the 4-fermion operator above we imagine inserting the unit operator as represented by a sum over states, $1 = \sum_n |n\rangle \langle n|$, and then approximate the sum by keeping just the vacuum sum, $\sum_n |n\rangle \langle n| \sim |0\rangle \langle 0|$. What an approximation!? It’s called the “vacuum insertion approximation”. Actually it is not so crazy. Due to the virtual $W$’s in the box graphs we have been studying the 4 weak interaction vertices are all within a volume characterized by a radius $r \sim 1/M_W$, which is quite small on the scale over which hadronic wave functions vary, i.e., $r \sim 1/m$. So it is not so crazy to evaluate the matrix element in terms of the probability to find the quark and anti-quark at the same point, i.e., the wave function at the origin (squared) just as we did for the leptonic decays, which is just what the projection onto the vacuum state does. So we use the identification

$$\langle 0 | \bar{d} \gamma^\rho (1-\gamma_5) s | K^0 \rangle \equiv i\sqrt{2} f_K q_\rho,$$  \hspace{1cm} (27.32)

with the decay constant defined as we did above (and for the pion). Without FCNC we cannot directly measure this quantity in leptonic decays of the $K^0$, but we can measure the corresponding quantity for the charged kaons. Then we invoke the fact that the strong interactions conserve strong isospin to justify using the same number, i.e., the same wave function, for the neutral $K^0$ (note that we have already included the correct color factors for the two forms of the box diagram). Thus we now have

$$\langle K^0 | \bar{d} \gamma^\rho (1-\gamma_5) s | 0 \rangle \langle 0 | \bar{d} \gamma^\rho (1-\gamma_5) s | K^0 \rangle = 2 f_k^2 m_K^2,$$  \hspace{1cm} (27.33)

and, finally,

$$\Delta M = \frac{G_F^2 m_e^2}{3\pi^2} f_k^2 m_K \sin^2 \theta_c \cos^2 \theta_c.$$  \hspace{1cm} (27.34)

Plugging in numbers, $G_F = 1.166 \times 10^{-11}$ MeV$^{-2}$, $f_K = 1.2 m_\pi = 170$ MeV, $m_K = 498$ MeV, $\sin^2 \theta_C = 0.049$, $\cos^2 \theta_C = 0.951$, $m_c = 1500$ MeV we find...
\[ \Delta M \sim 6.8 \times 10^{-12} \, \text{MeV}, \] (27.35)

which works \((i.e., \) agrees with data) at the factor of 2 level! This formalism was, in fact, used by Gaillard and Lee in 1974 to estimate that the mass of the charmed quark was \( \sim 1.5 \, \text{GeV} \), before it was seen.

In the (real) world of \(3\) generations we should also include the contribution of the top quark. For this contribution we make the following replacement in terms of the CKM mixing matrix elements

\[
m_c^2 \sin^2 \theta_c \cos^2 \theta_c = m_c^2 |V_{cs} V_{cd}^*|^2 \rightarrow m_t^2 |V_{ts} V_{td}^*|^2.
\] (27.36)

The charm quark contribution \((i.e., \) the quantity above\) is of order \( \sim 0.10 \, \text{GeV}^2 \), while, due to the strong suppression within the mixing matrix, the top contribution is of order \( \sim 0.003 \, \text{GeV}^2 \). (Strictly speaking, we should also keep the corrections of order \( m_t^2 / M_W^2 \) and higher that we ignored for the charm quark. These are now of order 1 or 2, but will not make up for the mixing suppression.)

To see the suppression more explicitly, note that we can characterize the magnitudes and signs of the CKM matrix elements (ignoring the phase) as

\[
"|V|" = \begin{pmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ \lambda^3 & -\lambda^2 & 1 \end{pmatrix},
\] (27.37)

where the rows are labeled \((d,s,b)\) and the columns \((u,c,t)\) and \( \lambda \sim 0.22 \). Thus the charm mixing factor is \( |V_{cs} V_{cd}^*|^2 \sim \lambda^2 \sim 0.05 \), while the top quark factor is

\[ |V_{ts} V_{td}^*|^2 \sim \lambda^{10} \sim 3 \times 10^{-7}. \] This large suppression more than compensates for factor of \(1.4 \times 10^4\) in the mass ratio squared. It also qualitatively explains why the \(CP\) violating contribution to \(M\), which arises from the phase in the top term, is a small effect. If we keep the phases, the mixing matrix looks crudely like
\[
V \approx \begin{pmatrix}
1 & \lambda & \lambda^3 e^{-i\delta} \\
-\lambda & 1 & \lambda^2 \\
\lambda^3 (1 - e^{-i\delta}) & -\lambda^2 & 1
\end{pmatrix}
\]

(27.38)

We will pursue the subject of CP violation in the next lecture.