Physics 557 – Lecture 9 Appendix: Spin, helicity, handedness, etc.

Let us try first to make connection to what we learned about these topics in the context of E&M (see Jackson) with propagating plane waves. In the absence of sources we have \( \nabla \cdot \vec{E} = 0 \) (and \( \nabla \cdot \vec{B} = 0 \)) so that, if the wave is propagating in the \( z \) direction, the vector \( \vec{E} \)-field (and pseudovector \( \vec{B} \)-field) must lie in the \( x-y \) plane. Thus the polarization of the plane wave (the 3-vector direction of the electric field) is transverse to the direction of motion (this is directly related to the fact that the electric field couples to a conserved current). If the direction of the \( \vec{E} \)-field is constant, independent of \( z \) and \( t \), we say the wave has linear (transverse) polarization. The basis states for this linear polarization are the unit vectors in the \( x \) and \( y \) directions, \( \hat{x} \) and \( \hat{y} \), i.e., there are 2 independent possible linear polarizations. This is directly analogous to the \( J_1 \) and \( J_2 \) or \( I_1 \) and \( I_2 \) states that we know in the context of angular momentum and isospin. Just as we saw with those situations, there is also a basis set containing the states \( \hat{x} \pm i\hat{y} \), which lead to an electric field with the behavior

\[
\vec{E}_\pm(t, \vec{r}) = E_0 \text{Re} \left[ \left( \hat{x} \pm i\hat{y} \right) e^{ikz - \omega t} \right] \\
= E_0 \left[ \hat{x} \cos \left( kz - \omega t \right) \mp \hat{y} \sin \left( kz - \omega t \right) \right].
\]

(9.A.1)

As suggested by the expression on the second line, this behavior is referred to as circular polarization (the direction of the electric field is rotating in the \( x-y \) plane). Now comes one point of confusion in the language! Historically in the study of optics, people focused on the behavior of the polarization (the direction of the \( \vec{E} \)-field) as a function of time at fixed \( z \), say \( z = 0 \), \( E_\pm(t) = E_0 \left[ \hat{x} \cos \omega t \pm \hat{y} \sin \omega t \right] \). An observer facing the oncoming wave sees the polarization with the + sign rotate in a counterclockwise direction \( \hat{x} \rightarrow \hat{y} \rightarrow -\hat{x} \rightarrow -\hat{y} \rightarrow \hat{x} \) and this polarization state was called left circularly polarized in optics. The minus signed case was naturally called right circularly polarized. However, in more modern terms, we wish to focus on the angular momentum carried by the fields, \( \hat{J}_z \), by the photons. If we ask about the component of angular momentum along the direction of motion, \( \hat{J}_z \), we find (see Jackson) that the + sign above corresponds to a positive \( \hat{J}_z \) component while the – sign correspond to a negative \( \hat{J}_z \) component. For a single photon the angular momentum is just \( \pm 1 \) in our units (\( \hbar \)). The word helicity is historically associated with the sign of the \( \hat{J} \) component that is along the direction of motion (\( \hat{J}_z \) here). Thus these two states are called the plus helicity and minus helicity states, respectively. The change in basis from linear polarization to circular polarization is just like the change from the \( x,y,z \) basis to \(+,,-,0 \) basis that we discussed with respect to the pions.
in isospace. However, as we will see in more detail later, the fact that E&M corresponds to an unbroken symmetry (a conserved current) leads to the fact that “real” photons are massless and never appear in the \( z \) or 0 or “longitudinal” polarization state.

In the context of modern particle physics we also use the language of handedness. Again the connection is to spin. Classically a particle’s angular momentum along the direction of motion is determined by applying the “right-hand rule”. Thus a particle with positive \( J_z \) and positive \( p_z \), and thus positive helicity, should be “spinning” in a clockwise direction when viewed along the \( z \)-axis in the plus direction or counterclockwise when looking back at it in the minus \( z \) direction. Such a particle, including a photon with the rotation being that of the E-vector, is called right-handed (in direct contrast to the left-hand circular polarization label above). The right-hand rule can also be thought of as relating the direction the spin points to the direction that a right-handed screw moves when you turn it in the right-hand, clockwise direction – forward in \( z \), or left-hand, counterclockwise direction – backward in \( z \). For the massless photon we have the following connections for a photon moving in the + \( z \) direction,

\[
J_z > 0 \iff \text{Helicity} = + \iff \text{Right-handed}
\]

\[
J_z < 0 \iff \text{Helicity} = - \iff \text{Left-handed}.
\]

(9.A.2)

Now what about spinors? First let us say a few more words about the Dirac equation. Recall that the goal was to have a Lorentz invariant, first-order equation of motion. This goal can be met because the Dirac matrices, satisfying

\[
\{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu \nu} \times 1,
\]

exist (the 1 refers to the matrix structure with respect to the spinor indices). These matrices supply us with (another) 4-D representation of the Lorentz group. Recall that in Lecture 5 we discussed the general Lorentz transformation of 4-D vectors in the form

\[
\Lambda(\omega^{\mu \nu}) = e^{(i/2)\omega_{\mu \nu} J^{\mu \nu}}.
\]

(9.A.4)

In the new representation provided by 4 component Dirac spinors (i.e., the solutions of the Dirac equation) the transformation operator looks like
\[ \Lambda_{\frac{1}{2}}(\sigma^{\mu \nu}) = e^{(j\frac{1}{2})_{\mu \nu} S^{\mu \nu}}, \]
\[ S^{\mu \nu} = \frac{i}{4} [\gamma^{\mu}, \gamma^{\nu}], \tag{9.A.5} \]

where the subscript is to remind us that (we think) we are describing spin $\frac{1}{2}$ particles. The final property of the Dirac matrices that we need is the commutator
\[ \left[ \gamma^{\mu}, S^{\rho \sigma} \right] = (J^{\rho \sigma})^{\mu}_{\nu} \gamma^{\nu}, \tag{9.A.6} \]

which follows from all of the above definitions (if you can keep all of the indices straight). From this expression it follows that
\[ \Lambda_{\frac{1}{2}}^{-1} \gamma^{\mu} \Lambda_{\frac{1}{2}} = \Lambda_{\nu}^{\mu} \gamma^{\nu}, \tag{9.A.7} \]

or, in words, boosting the spinor indices of the Dirac matrix is equivalent to an “opposite” boost applied to the Lorentz index. This means that the differential operator in the Dirac equation
\[ (i \gamma^{\mu} \partial_{\mu} - m)\psi(x) = (i \vec{\partial} - m)\psi(x) = 0, \tag{9.A.8} \]

is a Lorentz invariant. To see this consider the Lorentz transformed equation (we inverse boost the coordinate $x$ and boost the spinor field)
\[
\begin{align*}
(i \gamma^{\mu} \partial_{\mu} - m)\psi(x) &\rightarrow \left[ i \gamma^{\mu} \left( \Lambda^{-1} \right)^{\nu}_{\mu} \partial_{\nu} - m \right] \Lambda_{\frac{1}{2}} \psi \left( \Lambda^{-1} x \right) \\
&= \Lambda_{\frac{1}{2}} \Lambda_{\frac{1}{2}}^{-1} \left[ i \gamma^{\mu} \left( \Lambda^{-1} \right)^{\nu}_{\mu} \partial_{\nu} - m \right] \Lambda_{\frac{1}{2}} \psi \left( \Lambda^{-1} x \right) \\
&= \Lambda_{\frac{1}{2}} \left[ i \Lambda_{\frac{1}{2}}^{-1} \gamma^{\mu} \Lambda_{\frac{1}{2}} \left( \Lambda^{-1} \right)^{\nu}_{\mu} \partial_{\nu} - m \right] \psi \left( \Lambda^{-1} x \right) \\
&= \Lambda_{\frac{1}{2}} \left[ i \Lambda_{\frac{1}{2}} \gamma^{\sigma} \left( \Lambda^{-1} \right)^{\nu}_{\mu} \partial_{\nu} - m \right] \psi \left( \Lambda^{-1} x \right) \\
&= \Lambda_{\frac{1}{2}} \left[ i \gamma^{\nu} \partial_{\nu} - m \right] \psi \left( \Lambda^{-1} x \right) = 0. \tag{9.A.9}
\end{align*}
\]

Again in words, the properties of the Dirac matrices are such that the operator $\gamma^{\mu} \partial_{\mu}$ is indeed a Lorentz scalar as desired, i.e., it has the same form in both frames.
After this (re)introduction to the Dirac equation, consider first the massless version of the equation, \( i \partial \psi = 0 \). Instead of using the explicit representations of the Dirac or gamma matrices presented in Eq. (9.43), which were useful for considering the nonrelativistic case and talking about particles versus antiparticles, consider the following forms (often called the Weyl or chiral representation with various choices of the signs seen in the literature, e.g., compared to the definitions used here, Peskin and Schroeder have \( \gamma^k \rightarrow -\gamma^k, \gamma^5 \rightarrow -\gamma^5 \))

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\gamma^1 &= \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\gamma^2 &= \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & -i & 0 \\
0 & -i & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}, \\
\gamma^3 &= \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\gamma^5 &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\end{align*}
\] (9.A.10)

The two matrices \( \gamma^0 \) and \( \gamma^5 \) have switched forms and \( \gamma^1, \gamma^2 \) and \( \gamma^3 \) have changed sign from our previous choices, but as a group they still satisfy the required anticommutation relations,

\[
\{ \gamma^\mu, \gamma^\nu \} = 2 g^{\mu\nu} = 2 \text{Diag} [+1, -1, -1, -1], \{ \gamma^\mu, \gamma^5 \} = 0.
\] (9.A.11)

With this choice of matrices the boost and rotation generators take the simple forms

\[
S^{0k} = \frac{i}{4} [\gamma^0, \gamma^k] = \frac{i}{2} \begin{pmatrix}
\sigma^k & 0 \\
0 & -\sigma^k
\end{pmatrix},
\]

\[
S^{kl} = \frac{i}{4} [\gamma^k, \gamma^l] = \frac{i}{2} \varepsilon^{klm} \begin{pmatrix}
\sigma^m & 0 \\
0 & \sigma^m
\end{pmatrix}.
\] (9.A.12)

These block diagonal forms clearly suggest that the Dirac representation of the Lorentz group is reducible in the massless limit. This is generally expressed in terms of two 2-dimensional representations called Weyl or chiral spinors,
\[ \psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}, \quad (9.1) \]

with the components referred to as right-handed and left-handed respectively (hopefully, this notation will become clear shortly). To see the reduction of the Dirac representation we multiply the \textit{massless} Dirac equation on the left by \(-i\gamma^1\gamma^2\gamma^3\). We also note that (in 2x2 matrix notation)

\[ i\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\gamma} = \begin{pmatrix} 0 & -\bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix}, \]

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (9.13) \]

and recall that

\[ x^\mu = (x^0, \vec{x}), \quad \partial_\mu = \partial/\partial x^\mu = (\partial/\partial x^0, \vec{\nabla}). \quad (9.14) \]

Thus we can reduce the resulting expression to

\[ \left(-i\gamma^1\gamma^2\gamma^3\right)i\left(\gamma^0\partial_0 + \bar{\gamma} \cdot \vec{\nabla}\right)\psi = i\left(\gamma^5\partial_0 + \left(\bar{\sigma} \cdot \vec{\nabla}\right)\right)\psi = \left\{ \begin{array}{ccc}
0 & 0 \\
0 & -i\left(\partial_0 - \bar{\sigma} \cdot \vec{\nabla}\right) \end{array} \right\} \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = 0 \quad (9.15) \]

\[ \Rightarrow \quad i\left(\partial_0 + \bar{\sigma} \cdot \vec{\nabla}\right)\Psi_R = 0, \quad \Rightarrow \quad i\left(\partial_0 - \bar{\sigma} \cdot \vec{\nabla}\right)\Psi_L = 0. \quad (9.16) \]

As expected, in this massless limit, the Dirac equation breaks into two equations that are sometimes called the Weyl equations. The two \((R,L)\) fields evolve separately and form two distinct representations of the Lorentz group. This is just the standard statement that you cannot boost to the rest frame of a massless particle and so cannot mix the left-handed state with the right-handed state (although we still have not defined just what this language means). We can start to give some meaning to this
language by noting that these Weyl or chiral spinors are eigenstates of $\gamma^5$ in the sense that

$$\gamma^5 \psi_R \equiv \gamma^5 \left( \begin{array}{c} \Psi_R \\ 0 \end{array} \right) = + \psi_R; \quad \gamma^5 \psi_L = \gamma^5 \left( \begin{array}{c} 0 \\ \Psi_L \end{array} \right) = - \psi_L.$$  \hspace{1cm} (9.A.17)

Finally note that this decomposition of the Dirac representation is not complete when there is a nonzero mass. In this case the Dirac equation times $-i \gamma^1 \gamma^2 \gamma^3$ has the form

$$\left( -i \gamma^1 \gamma^2 \gamma^3 \right) \left( i \gamma^\mu \partial_\mu - m \right) \psi = 0,$$

so that the mass term clearly mixes the left- and right-handed representations.

To proceed with our discussion for the massive case it is helpful to transform from configuration space fields to momentum space fields. The useful basis is that of plane waves (i.e., perform a Fourier transform), where we focus on the solutions with $p^0 > 0$,

$$\psi (x) = u(p) e^{-ip^\mu x_\mu}.$$  \hspace{1cm} (9.A.19)

In momentum space the (transformed) Dirac equation becomes

$$\left( \gamma^\mu p^\mu - m \right) u(p) \equiv (p^\mu - m) u(p) = 0,$$

where $p^2 = m^2$ (with our choice of metric). First consider the rest frame, $p^\mu = (m, \vec{0})$, where we have (in 2x2 notation)

$$\left( m \gamma^0 - m \right) u(p) = m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p).$$  \hspace{1cm} (9.A.21)

The general solution of this equation can be written in terms of an arbitrary 2-component spinor, $\xi$,

$$u(p) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix},$$  \hspace{1cm} (9.A.22)
where the prefactor is chosen for later convenience and $\xi$ is normalized as $\xi^+ \xi = 1$. The 2-component spinor $\xi$ transforms in the expected way under rotations and we can interpret the two independent basis states, $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, as spin up and spin down along the $z$-axis (or any other useful axis). Note that the Dirac equation allows just 2 independent (positive energy) states as we expect for a field describing a spin $\frac{1}{2}$ particle.

Now consider a moving particle, which we obtain by boosting to another frame. If we want the particle to have a rapidity $y$ (moving along the $z$-axis), we boost the observer (i.e., us) by $-y$.

$$\Lambda_y (-y) u(m) = e^{-is_{00}} u(m) = \exp \left[ \frac{1}{2} y \left( \begin{array}{cc} 0 & \sigma^3 \\ \sigma^3 & 0 \end{array} \right) \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \cosh \left( \frac{y}{2} \right) 1 & 0 \\ 0 & \cosh \left( \frac{y}{2} \right) \end{pmatrix} + \sinh \left( \frac{y}{2} \right) \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} e^{y/2} \left( \frac{1+\sigma^3}{2} \right) + e^{-y/2} \left( \frac{1-\sigma^3}{2} \right) & 0 \\ 0 & e^{y/2} \left( \frac{1-\sigma^3}{2} \right) + e^{-y/2} \left( \frac{1+\sigma^3}{2} \right) \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

where we have used the usual definition of the 4-vector components of the momentum of a particle with rapidity $y$,

$$E = m \cosh y, \quad p^3 = m \sinh y,$$

$$E \pm p^3 = m e^{\pm y}.$$ 

Note that for these spinors we have
where we see, as stated earlier, that \( u^\dagger u \) is not a scalar but rather the zeroth component of a vector (the scalar density component of a current), while the true scalar, \( \bar{u}u \), is not so useful for \( m = 0 \).

Now we make the specific choices
\[
\xi \Rightarrow \xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi \Rightarrow \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
to define
\[
u_+ (p) = \begin{pmatrix} \sqrt{E + p^3} \\ \sqrt{E - p^3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \nu_- (p) = \begin{pmatrix} \sqrt{E - p^3} \\ \sqrt{E + p^3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

We see that in the relativistic (\( E \gg m \)) limit these states are identical to the Weyl states we discussed earlier. To identify the states at finite rapidity we consider the helicity operator that measures the component of spin along the direction of motion,
\[
h \equiv \hat{p} \cdot \hat{S} = \frac{1}{2} \hat{p}_k \begin{pmatrix} \sigma^k \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \sigma^k \end{pmatrix}
\]

We see that the states we have defined above are eigenstates of this operator,
\[
h u_+ (p) = \frac{1}{2} \begin{pmatrix} \sigma^3 \\ 0 \end{pmatrix} u_+ (p) = \begin{pmatrix} + \frac{1}{2} \end{pmatrix} u_+ (p)
\]
\[
h u_- (p) = \frac{1}{2} \begin{pmatrix} \sigma^3 \\ 0 \end{pmatrix} u_- (p) = \begin{pmatrix} - \frac{1}{2} \end{pmatrix} u_- (p)
\]
imdependent of the value of \( y \) as long as we do not change the sign of \( y \) (or \( \hat{p} \)), i.e., the fact that these states are eigenstates of helicity is (nearly) boost invariant. By analogy with our earlier discussion of optics and photons, the states with \( h = +1/2 \) are often referred to as right-handed, while the \( h = -1/2 \) states are called left-handed. The confusing part of this discussion is that, except for massless particles or in the limit \( m/E \to 0 \), these states are not identical to the Weyl or chiral states defined earlier that are also often referred to as right- or left-handed. To see this in more detail we note that we can always define eigenstates of \( \gamma^5 \) by using the following projection operators, for which we introduce chiral projection operators,

\[
P_{5^+} \equiv \frac{1+\gamma^5}{2}, \quad P_{5^-} \equiv \frac{1-\gamma^5}{2}
\]

so that

\[
\psi = P_{5^+}\psi + P_{5^-}\psi = \psi_{5^+} + \psi_{5^-} = \psi_R + \psi_L.
\] (9.A.30)

Note that \( P_{5^+} + P_{5^-} = 1 \), \( P_{5^+}P_{5^-} = P_{5^-}P_{5^+} = 0 \), \( P_{5^+}P_{5^-} = P_{5^-}P_{5^+} = 0 \) as required for projection operators. Independent of the specific basis used, these projection operators always yield \( \gamma^5 \) eigenstates,

\[
\gamma^5\psi_{5^+} = \frac{\gamma^5 + 1}{2}\psi = \frac{1 + \gamma^5}{2}\psi = (+1)\psi_{5^+},
\]

\[
\gamma^5\psi_{5^-} = \frac{\gamma^5 - 1}{2}\psi = \frac{1 - \gamma^5}{2}\psi = (-1)\psi_{5^-}.
\] (9.A.31)

If we apply these projection operators to the helicity eigenstates above (using our specific representation of the Dirac matrices), we find structure similar to the Weyl spinors but with nonzero masses and implied mixing,

\[
P_{5^+} u_+ (p) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
\sqrt{E + p^3} & 1 \\
0 & 0 \\
\sqrt{E - p^3} & 1 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
\sqrt{E + p^3} & 1 \\
0 & 0 \\
\sqrt{E - p^3} & 1 \\
0 & 0
\end{pmatrix},
\]
\[ P_{5-} u_+(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{E + p^3} \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{E - p^3} \\ 1 \\ 0 \end{pmatrix}, \]

\[ P_{5-} u_-(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{E - p^3} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{E - p^3} \\ 0 \\ 0 \end{pmatrix}, \]

\[ P_{5+} u_+(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{E + p^3} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{E + p^3} \\ 0 \\ 1 \end{pmatrix}. \quad (9.A.32) \]

To simplify these expressions we can rewrite them in the limit \( E \gg m \) and keep just the leading terms in \( m/E \),

\[ P_{5+} u_+(p) \xrightarrow{E \gg m} \left[ \sqrt{2E} - \frac{m^2}{\sqrt{2E (4E)}} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{m \to 0} \left( \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \]

\[ P_{5-} u_+(p) \xrightarrow{E \gg m} \frac{m}{\sqrt{2E}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{m \to 0} 0, \]

\[ P_{5+} u_-(p) \xrightarrow{E \gg m} \frac{m}{\sqrt{2E}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{m \to 0} 0, \]

\[ P_{5-} u_-(p) \xrightarrow{E \gg m} \left[ \sqrt{2E} - \frac{m^2}{\sqrt{2E (4E)}} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{m \to 0} \left( \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \quad (9.A.33) \]

The last step takes us back to where we started, the massless case. Recall that the form of the Weyl equations describing the massless case clearly indicates that the
solutions are eigenstates of helicity, explaining the \( R, L \) notation. We also notice that (in words) the amplitude to be both + helicity and – under \( \gamma^5 \) (or - helicity and + under \( \gamma^5 \)), \textit{i.e.}, the amplitude for the \( \gamma^5 \) eigenstates to have the “opposite” helicity, is proportional to the mass for \( m \ll E \). It is exactly this mixing that leads to the factor of the lepton mass in the decay amplitude for the pion. This same factor will arise for any spin zero (scalar or pseudoscalar) particle that decays to fermion-antifermion via a vector or pseudovector current interaction (or any combination as in the V-A charged Weak current and pion decay). In more detail, the vector and pseudovector interactions are non-chiral or chirality conserving interactions. Hence a chiral fermion while retain its chirality if scattered by such an interaction. The fermion and antifermion in a pair annihilated or produced by such an interaction will have \textit{opposite} chiralities. To the extent that these fermions are massless or highly relativistic, they will also exhibit \textit{opposite} helicities. Thus if a scalar or pseudoscalar particle decays through such an interaction, angular momentum conservation requires that the fermion and antifermion have the \textit{same} helicity (in the rest frame of the decaying particle) and the decay can only occur through the mixing of opposite helicity and chirality states. This again explains the lepton mass dependence of the observed weak decays of the charged pions.

We can also define a helicity projection operator by analogy with the \( \gamma^5 \) projection operator,

\[
P_{h^\pm} = \frac{1 \pm 2h}{2} = \frac{1}{2} \begin{pmatrix} 1 \pm \sigma^3 & 0 \\ 0 & 1 \pm \sigma^3 \end{pmatrix}.
\]

(9.A.34)

We note that the two kinds of projections operators commute, \([P_{h^\pm}, P_{h^\pm}] = 0\).

\textbf{SUMMARY:} The general (positive energy) solution to the Dirac equation has 2 degrees of freedom that can be characterized in terms of two potentially different basis sets, the eigenstates of helicity (spin component along the direction of motion) or eigenstates of \( \gamma^5 \) (similar considerations apply also to the negative energy solutions – the antiparticles). While analogy with the language of E&M suggests that the labels right-handed and left-handed should be associated with the helicity eigenstates \( \left( P_{h^+} = P_R, P_{h^-} = P_L \right) \), it is also common practice to use the handedness labels with the \( \gamma^5 \) eigenstates \( \left( P_{5^+} = P_R, P_{5^-} = P_L \right) \), especially in the context of the weak interactions where the charged current coupling is proportional to \((1-\gamma^5)\) for fermions and \((1+\gamma^5)\) for antifermions. For massless fermions the definitions are identical and there is no ambiguity (and the 2 eigenstates form separate representations of the Lorentz group).
For nonzero mass fermions, the eigenstates of the 2 operators are not identical, with a difference (i.e., mixing) that scales with the mass $m$. We will try to be careful with the language used in this class.

Recap of spinor solutions of the Dirac Equation: positive energy solution (particle) moving in $+z$ direction

$$
\psi(x) = u(p) e^{-ip \cdot x},
\quad
u_r(p) = \begin{pmatrix}
\sqrt{E + p^3 \left(\frac{1+\sigma^3}{2}\right)} + \sqrt{E - p^3 \left(\frac{1-\sigma^3}{2}\right)} \\
\sqrt{E + p^3 \left(\frac{1-\sigma^3}{2}\right)} + \sqrt{E - p^3 \left(\frac{1+\sigma^3}{2}\right)}
\end{pmatrix} \xi_r, \quad r = 1, 2, \quad (9.4.35)
$$

$$
\bar{u}_r(p) u_s(p) = 2m\delta_{rs}.
$$

The corresponding “negative energy” (antiparticle) solutions look very similar

$$
\psi(x) = v(p) e^{+ip \cdot x},
\quad
v_r(p) = \begin{pmatrix}
\sqrt{E + p^3 \left(\frac{1+\sigma^3}{2}\right)} + \sqrt{E - p^3 \left(\frac{1-\sigma^3}{2}\right)} \\
-\sqrt{E + p^3 \left(\frac{1-\sigma^3}{2}\right)} - \sqrt{E - p^3 \left(\frac{1+\sigma^3}{2}\right)}
\end{pmatrix} \eta_r, \quad r = 1, 2, \quad (9.4.36)
$$

$$
\bar{v}_r(p) v_s(p) = -2m\delta_{rs}
\quad
\bar{v}_r(p) u_s(p) = \bar{u}_r(p) v_s(p) = 0.
$$