Physics 557 – Lecture 5 – Appendix

Why (and when) are the structure constants described by a completely antisymmetric tensor?

In this Appendix we will discuss a proof of this result (see, e.g., the book by Georgi) for the cases of primary interest to physicists, i.e., for Lie algebras and groups that exhibit nontrivial finite dimensional unitary representations. These algebras are labeled compact algebras (and groups). We will see that for these algebras we can choose a basis set for the generators (i.e., use our freedom as to the specific choice of the continuous parameters of the Lie group) so that the structure constants are completely antisymmetric. Note that this class of compact algebras does not exhaust all physically interesting groups. For example, it does not include the Lorentz group, SO(3,1) (recall that the parameter space of the boosts, unlike true rotations, is unbounded), but it does include the groups SO(n) and SU(n) that we will find most useful. The interested reader is encouraged to explore the symmetry properties of the structure constants for the case of SO(3,1), which can be obtained from the explicit commutation relations in the lecture for the matrices $J_k$ and $\tilde{K}_k$. These matrices provide an example of the generators of SO(3,1) in the fundamental representation (e.g., $Y_k = J_k, k = 1, 2, 3; Y_k = \tilde{K}_{k-3}, k = 4, 5, 6$).

Let us briefly review our starting point (and many of the concepts in the lecture). We consider Lie groups with elements depending smoothly (differentiably) on $N$ continuous parameters, $g(x_1, \ldots, x_N)$. In particular, derivatives (and a Taylor expansion) exist near the origin in parameter space (near the identity transformation in group space) and we can define the generators as

$$\left. \frac{1}{i} \frac{\partial g}{\partial x_k} \right|_{x_j=0, \text{all}} = Y_k. \quad (5.A.1)$$

Note that, since we are focusing are groups with unitary representations, we will use the “Hermitian” definitions, i.e., the choices of “$i$” factors, from the beginning so that the $Y_k$ are represented by imaginary matrices (these matrices are Hermitian in the case of compact algebras) and the $x_k$ are real. Near the identity transformation we can describe the group elements in terms of an infinitesimal transformation and the generators,

$$g \left( dx_k \right) = 1 + i dx_k Y_k, \quad (5.A.2)$$
where $dx_k$ is an *infinitesimal* parameter. We can iterate this expression to obtain a finite transformation. We can express this iteration as

$$g \left( x_k \right) = \lim_{m \to \infty} \left( 1 + i \frac{x_k}{m} Y_k \right)^m = e^{i x_k Y_k}. \quad (5.A.3)$$

Thus we obtain the exponential parameterization that we used in the lecture, which is valid in some finite region (where the above limit exists) around the identity transformation, $1$.

Next we can map the group property of being closed under the defined multiplication unto a property of the generators by using this exponential expression. Even though

$$e^{i x_k Y_k} e^{i \tilde{x}_k Y_k} \neq e^{i (x_k + \tilde{x}_k) Y_k}, \quad (5.A.4)$$

for $\tilde{x}_k \neq \alpha x_k$ with $\alpha$ a constant (because the generators do not, in general, commute), we must still be able to write

$$e^{i x_k Y_k} e^{i \tilde{x}_k Y_k} = e^{i z_k Y_k} \quad (5.A.5)$$

for some set of parameters $z_k$ in order for the group to close under multiplication. We can determine the constraint that must obtain for this to be possible, *i.e.*, the condition on the generators for this exponential expression to faithfully represent the group, by expanding the exponentials and performing some (old fashioned) algebra. In particular, expanding we find that

$$iz_k Y_k = \ln \left( e^{i x_k Y_k} e^{i \tilde{x}_k Y_k} \right) = \ln \left( 1 + \left( e^{i x_k Y_k} e^{i \tilde{x}_k Y_k} - 1 \right) \right)$$

$$= i x_k Y_k + i \tilde{x}_k Y_k - x_k Y_k \tilde{x}_k Y_k$$

$$- \frac{1}{2} (x_k Y_k)^2 - \frac{1}{2} (\tilde{x}_k Y_k)^2 + \frac{1}{4} (x_k Y_k + \tilde{x}_k Y_k)^2 + \cdots$$

$$= i x_k Y_k + i \tilde{x}_k Y_k - \frac{1}{2} \left[ x_k Y_k, \tilde{x}_k Y_k \right] + \cdots, \quad (5.A.6)$$

where the terms not displayed explicitly (the “…”s) are higher order in $x_k, \tilde{x}_k$. The higher order terms also all involve commutators, confirming what we noted earlier. The inequality above is an inequality only because the $Y_k$ represent linear operators, not numbers, which do not, in general, commute. Reorganizing and explicitly using the fact that the generators provide a basis set we can write
\[ [x_k Y_k, \bar{x}_k Y_k] = -2i \left( z_k - x_k - \bar{x}_k \right) Y_k + \cdots \equiv i \bar{z}_k Y_k, \quad (5.A.7) \]

for some new set of constants, \( \bar{z}_k \). In order for this expression to be valid for all values of \( x_k, \bar{x}_k \) we must have

\[ \bar{z}_m = x_k \bar{x}_l C_{klm}, \quad (5.A.8) \]

for some tensor of (real) constants, \( C_{klm} \). Thus we obtain the defining equation of the algebra

\[ [Y_k, Y_l] = iC_{klm} Y_m. \quad (5.A.9) \]

For the moment we are assured only that the structure constants, \( C_{klm} \), are represented by a tensor antisymmetric in the first 2 indices,

\[ C_{klm} = -C_{lkm}. \quad (5.A.10) \]

Note that we can now write

\[ z_k = x_k + \bar{x}_k - \frac{1}{2} \bar{z}_k + \cdots, \quad (5.A.11) \]

again connecting the earlier inequality to the nonzero commutators, i.e., the nonzero structure constants.

The next step is to note, as in the lecture, that the structure constants themselves provide a representation of the generators and thus of the group, the adjoint representation. To see this, recall the Jacobi Identity, which is a general property of commutators (and also of more general, and more abstract, Lie products),

\[ \left[ Y_k, \left[ Y_l, Y_m \right] \right] + \text{cyclic permutations} = 0, \quad (5.A.12) \]

which can be verified by simply writing out the terms. Thus, since the algebra implies that

\[ \left[ Y_k, \left[ Y_l, Y_m \right] \right] = \left[ Y_k, iC_{lmn} Y_n \right] = -C_{lma} C_{knp} Y_p, \quad (5.A.13) \]

the Jacobi Identity yields
\[ C_{lmn}C_{knp} + C_{klm}C_{mnp} + C_{mkn}C_{lmp} = 0. \] \tag{5.A.14}

So, (almost) as in the lecture, if we define a set of matrices by (note the order of the indices)

\[ [T_k]_{lm} \equiv -iC_{klm}, \] \tag{5.A.15}

the previous result, after (appropriate) substitution, now reads

\[-(T_i \cdot T_k)_{mp} - (T_k \cdot T_m)_{lp} + iC_{mkn}(T_l)_{np} = 0,\]
\[(T_m \cdot T_k)_{lp} - (T_k \cdot T_m)_{lp} = [T_m, T_k]_{lp} = iC_{mkn}(T_n)_{lp},\] \tag{5.A.16}

where the second line (twice) makes use of the antisymmetry in the first 2 indices of the structure constants. Thus, in standard notation, we have the relation defining the algebra of the group,

\[ [T_k, T_l] = iC_{klm}T_m, \] \tag{5.A.17}

confirming that the adjoint representation defined by the \( T_k \) is a faithful representation.

The final step in this discussion is to define a scalar product so that the space defined by the \( Y_k \) (and the \( T_k \)) is not just a linear space (with addition and multiplication), but also a vector space. An appropriate candidate for the scalar product is the trace in the adjoint representation,

\[ (T_i \cdot T_k)_{mp} = \text{Tr}(T_i T_k), \] \tag{5.A.18}

which defines a real, symmetric (by the cyclical properties of the trace) matrix. We want to put this matrix in canonical form by using the allowed linear transformations in the space of the \( Y_k \),

\[ Y_k \rightarrow Y'_k = M_{kl}Y_l. \] \tag{5.A.19}

The fundamental commutator in the new basis is now

\[ [Y'_k, Y'_l] = iM_{km}M_{lp}C_{mpr}Y_r \\
= iM_{km}M_{lp}C_{mpr}M^{-1}_{rs}M_{st}Y_r \\
= iM_{km}M_{lp}C_{mpr}M^{-1}_{rs}Y'_s, \] \tag{5.A.20}
This result tells us that, under this general transformation, the new structure constants are

$$C'_{klm} = M_{kp} M_{lr} C_{prs} M_{sm}^{-1}. \quad (5.A.21)$$

The corresponding transformed adjoint representation is given by

$$[T_k]_{lm} \rightarrow [T'_k]_{lm} = -iC'_{klm} = M_{kp} M_{lr} [T_p]_{rs} M_{sm}^{-1}, \quad (5.A.22)$$

or, in more general notation,

$$[T_k] \rightarrow [T'_k] = M_{kp} M^{-1} [T_p] M. \quad (5.A.23)$$

In other words, the linear transformation of the $Y_k$ induces both a corresponding linear transformation of the $T_k$ (on the index $k$) and a similarity transformation (on the usual matrix indices – the $MTM^{-1}$ form). However, our scalar product, expressed in terms of a trace, is insensitive to the similarity transformation and takes the form

$$\text{Tr}[T_k T_l] \rightarrow \text{Tr}[T'_k T'_l] = M_{km} M_{lp} \text{Tr}[T_m T_p]. \quad (5.A.24)$$

We can now choose (and it is always possible) the transformation $M$ (it is just an orthogonal transformation) to diagonalize the trace. In the new basis (but dropping the primes) we have

$$\text{Tr}[T_k T_l] = \lambda_k \delta_{kl} \quad (5.A.25)$$

where no sum is implied by the repeated index $k$. We still have the freedom to use diagonal linear transformations $M$ to rescale the $\lambda_k$ to have absolute value 1. Note, however, that, since the transformation is bilinear in $M$, we cannot change the signs of the $\lambda_k$. It is at this point that we reduce the scope of our discussion to only the case of compact algebras. For such algebras the $\lambda_k$ are all positive and this constraint serves as a definition of this class of algebras (the scalar product is always positive as in Euclidean space). You may want to verify that for the one noncompact case we have mentioned, SO(3,1), there are necessarily negative $\lambda_k$ values (recall the metric). For the compact case we can write (for some convenient positive value of $\lambda$)

$$\text{Tr}[T_k T_l] = \lambda \delta_{kl} \quad (5.A.26)$$
With this definition of the “metric” in the vector space \((i.e., this diagonal scalar product)\) of the \(T_k\), we can (finally!) express the structure constants as

\[
C_{klm} = -i\lambda^{-1}\text{Tr}([T_k, T_l]T_m).
\]

Due to the cyclical nature of the trace, the obvious antisymmetry in the first 2 indices is propagated to the other pairs as well,

\[
\text{Tr}([T_k, T_l]T_m) = \text{Tr}(T_k TT_m - T_l TT_m) = \text{Tr}(T_l T_m T_k - T_m T_l T_k)
\]

\[
= \text{Tr}([T_l, T_m]T_k),
\]

\[
C_{klm} = C_{lmk} = -C_{mlk} = -C_{klm}.
\]

Thus for the class of groups with compact algebras, and nontrivial finite dimensional unitary representations, we have demonstrated that the structure constants are described by a \textit{completely} antisymmetric tensor, \(i.e.,\) antisymmetric with respect to the interchange of \textit{any} pair of indices. Note also that, as expected, the adjoint representation yields matrices for the generators that are imaginary (the structure constants are real) and antisymmetric. Thus the generators are Hermitian and the representation of the group is unitary.

For the non-compact case, where some of the \(\lambda_k\) values above are negative, we cannot factor out a common \(\lambda\) factor \(i.e.,\) the sign of \(\lambda_k\) depends on \(k\) and the cyclical nature of the trace is not enough to guarantee complete antisymmetry. Again you are encouraged to look at the case of SO(3,1) in detail.