In this lecture we will move from the “big picture” of Lecture 2 to a smaller picture where we can develop the tools we need to pursue particle physics in more detail. In particular, we want to see how we can both organize our knowledge and solve interesting problems (i.e., the homework) by utilizing symmetries. This utilization of the ideas of symmetries will be a common theme throughout the rest of the course. We will want to enumerate all the sorts of transformations under which the universe (i.e., the physics) is invariant (or at least approximately invariant).

Perhaps one of the most important symmetries in particle physics is that of the invariance of the basics physics under translations in time and space. This tells us (through Noether’s theorem – see Capter 4.1 in Griffiths or Appendix C.2 in Rolnick or Chapter 9.6 in Peskin and Schroeder or later in this course) that both the (total) energy and the (total) momentum are conserved. When particles (even a large number as in the collision of two automobiles) interact, resulting, perhaps, in even more particles after the interaction, both the total energy and the total momentum will remain the same (i.e., be identical before and after the interaction). These conservation laws are useful even in the absence of detailed knowledge of the dynamics of the interaction. In particular, if 2 particles collide and make 2 other particles, the sum of the energies and momenta before the collision are the same as the sum after the collision. The easiest language in which to implement these conservation laws is that of 4-vectors, which brings us to Lorentz transformations. (See Chapter 3 in Griffiths)

Not only is physics invariant under translations in time and space but also under transformations between reference frames differing by space-time rotations. This symmetry is illustrated by the Lorentz invariance of the basic physics, as encoded, for example, in the action (i.e., the action is a Lorentz scalar – we take the word scalar to denote an invariant just as it is used in the study of spatial rotations). More generally, the quantities of interest will (must!) be expressed as quantities with definite properties under Lorentz transformations – scalars, vectors, etc. (i.e., representations of the Lorentz group). We can often solve problems (as in the homework) by comparing the results of calculating a Lorentz scalar quantity in two different inertial or Lorentz frames (frames that differ by a relative velocity) and requiring that the results be identical.

Of special interest is the 4-vector representing the energy and momentum of a particle - $p^\mu \equiv \left( E, p_x, p_y, p_z \right)$. The choice of using the Greek superscript, $\mu =$
(0,1,2,3) helps to remind us that this is in Lorentz notation. Strictly the superscript form is the “contravariant” form. There is also a subscript or covariant form -

\[ p_\nu \equiv g_{\mu\nu} p^\mu = \left( E, -p_x, -p_y, -p_z \right), \]

where we have introduced the metric

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

This choice for the metric (as opposed to the form \text{diag}(-1,1,1,1)) is often called the “West Coast” metric, while the other choice is called the “East Coast” metric. You will see both choices in the literature (and on chalkboards in the physics building). Both Griffiths and Rolnick use the West Coast metric as here. The physics is independent of this choice but the sign of specific quantities will vary with the choice.

A Lorentz scalar is constructed by multiplying a contravariant vector by a covariant vector (i.e., multiplying two contravariant or covariant vectors with the metric “in between”). For example, the square of the energy-momentum vector of a particle is

\[ p^\mu p_\mu = \left( \gamma m, \gamma m \vec{v} \right)^2 = E^2 - p_x^2 - p_y^2 - p_z^2 = m^2 \]

(note that this would yield \(-m^2\) in the other metric). It is useful (and standard) to introduce two quantities in terms of the particle’s velocity \( v \) –

\[
\begin{align*}
\beta &= \frac{v}{c} \xrightarrow{c \to 1} v, \\
\gamma &= \frac{1}{\sqrt{1 - \beta^2}}.
\end{align*}
\]

In this notation we have \( E = \gamma m, \vec{p} = \gamma m \vec{v} \). A related concept, especially useful when our interest is focused primarily on one direction, say the \( z \) direction (typically the direction of the beam particles at an accelerator), is the “rapidity” notation. We think of the momentum as having a longitudinal component in the \( z \) direction and a transverse component, as in \( p^\mu = (E, \vec{p}_T, p_z) \), where \( \vec{p}_T \) is a two component vector \((x, y)\). We define the rapidity \( y \) as

\[
y = \frac{1}{2} \ln \left( \frac{E + p_z}{E - p_z} \right)
\]

and define in the rapidity notation.
\[ E = \sqrt{m^2 + p_T^2} \cosh y \equiv m_T \cosh y, \]
\[ p_z = \sqrt{m^2 + p_T^2} \sinh y \equiv m_T \sinh y. \quad (4.3) \]

In the purely collinear case, \( p_T = 0 \), we have simply
\[ \gamma = \cosh y, \]
\[ \beta \gamma = \sinh y. \quad (4.4) \]

Another related quantity is the “pseudorapidity” \( \eta \) defined by
\[ \eta = \frac{1}{2} \ln \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right) = \ln \cot \frac{\theta}{2}, \]
\[ \tan \theta = \frac{p_T}{p_z}. \quad (4.5) \]

The pseudorapidity becomes a good approximation to the true rapidity in the relativistic limit, \( y \xrightarrow{p \gg m} \eta \). In particular, a little arithmetic shows that
\[ y = \eta - \frac{m^2}{2|\vec{p}|^2} \frac{\cos \theta}{\sin^2 \theta} \quad (4.6) \]

in the limit where the last term is small (\(|\vec{p}| \gg m, \theta \neq 0, \pi \)). Thus, if we are interested only in highly energetic particles, a geometric measurement of \( \theta \) is (nearly) as good as a measurement of the kinematic variable \( y \) (and generally a lot easier to perform). This connection has proved essential in the study of cosmic rays with emulsions as were performed by the cosmic ray group in this Department for years and also plays a role at particle colliders. Note the connections between the various regimes
\[ \theta \to 0 : \eta \to \infty, \]
\[ \theta \to \pi : \eta \to -\infty, \]
\[ \theta = \frac{\pi}{2} : \eta = 0. \quad (4.7) \]

For completeness, we should mention the “light cone” notation, which is also useful for highly relativistic particles moving along the \( z \) axis (and is in common
usage). We define $+$ and $-$ light cone components as

$$p^\pm = \frac{E \pm p_z}{\sqrt{2}}$$  \hspace{1cm} (4.8)

so that

$$p^\mu = \left(p^+, p^-, \vec{p}_r\right), \quad p^\mu p_\mu = 2p^+p^- - p_r^2,$$
$$p_1^\mu p_2^\mu = p_1^+ p_2^- + p_1^- p_2^+ - \vec{p}_{r1} \cdot \vec{p}_{r2}.$$  \hspace{1cm} (4.9)

In the rapidity notation we have the simple (and useful) form

$$p^\pm = \frac{m_{p_L} e^{\pm y}}{\sqrt{2}}.$$  \hspace{1cm} (4.10)

Now return to the subject of Lorentz transformations. Consider the Lorentz transformation (boost) to a frame $S'$ moving with respect to a frame $S$ with velocity $u$ in the $z$ direction. Define analogous quantities to those above for the transformation itself –

$$\beta_u = u,$$
$$\gamma_u = \frac{1}{\sqrt{1 - \beta_u^2}}.$$  \hspace{1cm} (4.11)

If the expression for $p^\mu$ in the frame $S$ is as above, the corresponding form in the frame $S'$, \textit{i.e.}, the particle’s momentum as seen in the frame $S'$ is

$$E' = \gamma_u \left(E - \beta_u p_z\right),$$
$$p'_z = \gamma_u \left(p_z - \beta_u E\right),$$
$$p'_x = p_x,$$
$$p'_y = p_y.$$  \hspace{1cm} (4.12)

Since the observer is now moving (faster) in the $z$ direction, the particle’s momentum in that direction (and its $E$) is reduced (as seen in the boosted frame).
From the form of the transformation it follows that

\[ p'_\mu p'^\mu = E'^2 - p'^2_x - p'^2_y - p'^2_z \]

\[ = \gamma^2_u \left( E^2 - 2Ep_z \beta_u + \beta^2_u p^2_z \right) - \gamma^2_u \left( p^2_z - 2Ep_z \beta_u - \beta^2_u E^2 \right) 
- p^2_x - p^2_y \]

\[ = \gamma^2_u \left( E^2 - p^2_z \right) \left( 1 - \beta^2_u \right) - p^2_x - p^2_y \]

\[ = E^2 - p^2_x - p^2_y - p^2_z = p_\mu p^\mu = m^2. \]  

(4.13)

As expected, the square of the momentum (i.e., the rest mass\(^2\)) is a Lorentz invariant or Lorentz scalar. To remind us of this fact the product of any two Lorentz vectors, \( v_1^\mu v^\nu_2 \), is called the scalar product. One of the especially attractive features of the rapidity variable introduced earlier is its simple behavior under boosts. The boost transformation is like the collinear vector above and we can define the corresponding change of rapidity –

\[ \gamma_u = \cosh^{-1} \gamma_a. \]

(4.14)

Under the action of the Lorentz transformation the rapidity of a particle changes as a simple translation

\[ y \rightarrow y - y_u, \]  

(4.15)

i.e., the particle has a smaller rapidity in the new frame.

Lorentz transformations can also be represented in matrix notation as follows

\[ p'^\mu \equiv \Lambda^\mu_\nu p^\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta \gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}. \]  

(4.16)

The invariance of scalar products implies the following property for the matrix representation of the boost
\[ p_1^\mu p_2^\mu = \Lambda^\mu_{\nu} p_1^\nu g_{\mu\sigma} \Lambda^\sigma_\delta p_2^\delta = p_1^\nu g_{\nu\delta} p_2^\delta = p_1 p_2 \quad \text{for any } p_1, p_2 \]  
\[ (4.17) \]

If we take the transpose of the first \( \Lambda \), we can write this last relationship in pure matrix notation (where the indices are understood and are in the standard order).

\[ \Lambda^T g \Lambda = g. \quad (4.18) \]

Now we can use the properties of the metric, \( g^2 = 1 \), \( g = g^{-1} = g^T \) (where the symbol \( 1 \) stands for the unit matrix \( \text{diag}(1,1,1,1) \)), and multiply the equation above by \( g \) to derive (note that these results are independent of the choice of metric) that

\[ 1 = g \Lambda^T g \Lambda \equiv \Lambda^{-1} \Lambda \text{ or } \Lambda^{-1} = g \Lambda^T g. \quad (4.19) \]

So we have verified that both \( \Lambda \) and \( \Lambda^{-1} \) exist and it is easy to show that the result of sequential boosts is just another boost, \( \Lambda_1 \Lambda_2 = \Lambda_3 \). As you may recall from past studies of group theory, these are just the properties that we require in order for the boosts to form a group. It is also straightforward to verify that, like rotations, boosts do not, in general, commute (i.e., \( \Lambda_1 \Lambda_2 \) is not equal to \( \Lambda_2 \Lambda_1 \)). Thus we expect this matrix representation of the boosts to be the representation of a non-Abelian group (i.e., a group with elements that do not commute, while a group with commuting elements would be an Abelian group). The group in question is the Lorentz group – \( \text{SO}(3,1) \) (or \( \text{SL}(2,\mathbb{C}) \)) where the O is for “orthogonal” and the notation \( (3,1) \) reminds we have 3 space dimensions and 1 time dimension with different signs in the metric.

We can study the group \( \text{SO}(3,1) \) in more detail by a careful consideration of these matrices. In particular, we can determine how many parameters are needed to characterize the Lorentz group. Since general 4x4 real matrices have 16 (real) parameters, we start at 16, but there are constraints due to the properties of the group. The matrix equation(s) above constitute 10 constraints (1 each for the terms on the diagonal and 6 for the off-diagonal terms, the ones above the diagonal are identical to those below since the transpose of the equation returns the same equation \([ (\Lambda^T g \Lambda)^T = \Lambda^T g \Lambda ] \). Thus there are 6 = 16 – 10 parameters describing the group. We also do not allow reflections in the “proper” Lorentz group (the “S” in \( \text{SO}(3,1) \)) and so require \( \text{det}[\Lambda] = +1 \) (and not \(-1\)). We also do not allow the form \( \text{diag}(-1,-1,-1,-1) \), which is a reflection in all 4 dimensions.
These 6 parameters can be easily understood as the 3 Euler angles describing the usual rotations in 3-space, i.e., orthogonal transformations (transformations which leave the lengths of 3-vectors invariant) plus 3 parameters that describe the boosts themselves. These latter transformations can be thought of as “hyperbolic” (i.e., imaginary) rotations (recall how the explicit form involved hyperbolic functions) that mix the time dimension with one of the space dimensions. They are orthogonal in the sense that they preserve the length (i.e., square) of 4-vectors defined in terms of a proper scalar product. The fact that the metric does not treat the 4 dimensions in the same way explains the (3,1) notation. (In Euclidean 4-D space, where the metric is diag(1,1,1,1), one finds instead SO(4).) The connection is particularly clear if we compare the form of the usual rotation by $\phi$ around the $z$-axis, which mixes the $x$ and $y$ components, to the boost in the $z$ direction written in terms of the rapidity of the boost, which mixes the $t$ and $z$ components.

\[
\Lambda(\{x, y\}, \phi) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi & \sin \phi & 0 \\
0 & -\sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
\Lambda(\{t, z\}, y_u) = \begin{bmatrix}
\cosh y_u & 0 & 0 & -\sinh y_u \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh y_u & 0 & 0 & \cosh y_u
\end{bmatrix}.
\]

The basic form is clearly similar. The boosts simply have hyperbolic functions (and “funny” signs). Note that both transformations are “passive” in the sense that we are thinking of rotating or boosting the reference frame. One can also consider “active” transformations where the 4-momentum of a particle is itself rotated or boosted while the reference frame remains fixed. The difference between these two cases is seen in the signs of the off-diagonal terms.

We can express the full transformation in exponential form as

\[
\Lambda = e^{\mathcal{L}} = 1 + \mathcal{L} + \frac{\mathcal{L}^2}{2} + \ldots
\]
where \( L \) is itself a real 4x4 matrix. We can think of \( L \) as generating a “small” transformation, which is iterated to produce the full transformation \( \Lambda \). Since we want only the “proper” transformation, we have

\[
\det[\Lambda] = e^{\text{Tr}[L]} = 1 \Rightarrow \text{Tr}[L] = 0. \tag{4.22}
\]

Thus \( L \) is must be traceless and real. Similarly we have

\[
\Lambda^T = e^{L^T} = 1 + L^T + \frac{L^T L^T}{2} + \ldots \tag{4.23}
\]
and

\[
\Lambda^{-1} = g \Lambda^T g = e^{-L} = g \left( 1 + L^T + \frac{L^T L^T}{2} + \ldots \right) g = g g + g L^T g + \frac{g L^T g g L^T g}{2} + \ldots = e^{gL^T g}. \tag{4.24}
\]
Taking the logarithm of this expression and using the fact that \( g = g^{-1} = g^T \), we find

\[
g L^T g = -L \Rightarrow L^T g = (gL)^T = -gL. \tag{4.25}
\]
Thus \( gL \) is both traceless and anti-symmetric, while \( L \) is traceless and of mixed symmetry. It has exactly the 6 free components that we desire to represent \( \text{SO}(3,1) \) (i.e., the 6 off-diagonal components of a traceless, anti-symmetric 4x4 tensor). We can write the general form of \( L \) as

\[
L = \begin{bmatrix}
0 & L_{01} & L_{02} & L_{03} \\
L_{01} & 0 & L_{12} & L_{13} \\
L_{02} & -L_{12} & 0 & L_{23} \\
L_{03} & -L_{13} & -L_{23} & 0
\end{bmatrix} \quad (L = \text{boosts, } L = \text{rotations}). \tag{4.26}
\]
In the 6-D vector space of these matrices we can take the following set of matrices as the basis set (we will see shortly that these matrices form a representation of the generators of \( \text{SO}(3,1) \), \textit{i.e.}, of the members of its algebra).

\[
S_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix};
S_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix};
S_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix};
\]

\[
K_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix};
K_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix};
K_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

(4.27)

We recognize the first row as the usual basis set for rotations: \( S_1 \) generates a rotation about the 1 or \( x \)-axis (mixing \( y \) and \( z \)), \( S_2 \) about the 2 or \( y \)-axis and \( S_3 \) about the 3 or \( z \)-axis. The matrices in the second row are the corresponding generators of boosts in the 1 (\( x \)), 2 (\( y \)) and 3 (\( z \)) directions. It is easy to verify that both \( S_i^2 \) and \( K_i^2 \) (all \( i \)) are diagonal matrices (each \( S_i^2 \) has 2 minus 1’s on the diagonal while each \( K_i^2 \) has 2 plus 1’s) while \( S_i^3 = - S_i \) and \( K_i^3 = + K_i \). For example,

\[
S_3^2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
K_3^2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

(4.28)

Now define two real 3-vectors

\[
\vec{\theta} = (\theta_1, \theta_2, \theta_3),
\]

\[
\vec{y} = (y_1, y_2, y_3),
\]

(4.29)

which contain the six parameters that we are looking for. Taking the obvious products

\[
\vec{\theta} \cdot \vec{S} \equiv \theta_1 S_1 + \theta_2 S_2 + \theta_3 S_3,
\]

\[
\vec{y} \cdot \vec{K} = y_1 K_1 + y_2 K_2 + y_3 K_3,
\]

(4.30)
it is straightforward to show (from the results above) that

\[
\begin{align*}
(\vec{\theta} \cdot \vec{S})^3 &= -\vec{\theta} \cdot \vec{S} |\vec{\theta}|^2, \\
(\vec{y} \cdot \vec{K})^3 &= \vec{y} \cdot \vec{K} |\vec{y}|^2.
\end{align*}
\] (4.31)

These relations make it easy to expand the following expressions

\[
\begin{align*}
L &= -\vec{\theta} \cdot \vec{S} - \vec{y} \cdot \vec{K}; \\
\Lambda(\vec{\theta}, \vec{y}) &= e^{-\vec{\theta} \cdot \vec{S} - \vec{y} \cdot \vec{K}},
\end{align*}
\] (4.32)

where the choice of the sign (-1) again corresponds to these being “passive” transformations, i.e., transforming the reference frame and not the state vectors.

Finally we should consider a couple of specific examples to illustrate what we have been discussing. Choosing the forms \( \vec{\theta} = (0,0,\phi), \vec{y} = (0,0,0) \), it is straightforward to verify that

\[
\begin{align*}
-\vec{\theta} \cdot \vec{S} &= -\phi S_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & 0 \\ 0 & -\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
(-\vec{\theta} \cdot \vec{S})^2 &= \phi^2 S_3^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\phi^2 & 0 & 0 \\ 0 & 0 & -\phi^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
(-\vec{\theta} \cdot \vec{S})^3 &= -\phi^2 (\phi S_3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\phi^3 & 0 & 0 \\ 0 & 0 & -\phi^3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\] (4.33)

Remembering the series expansions for the trigonometric functions, it is easy to see that
\[ e^{-\phi S_3} = -S_3 \left( \phi - \frac{\phi^3}{3!} + \ldots \right) - S_3^2 \left( 1 - \frac{\phi^2}{2!} + \ldots \right) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ = -S_3 \sin \phi - S_3^2 \cos \phi + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & \sin \phi & 0 \\ 0 & -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.34} \]

which we recognize as a “passive” rotation of the axes about the \( z \)-axis.

Correspondingly with the choice \( \bar{\theta} = (0, 0, 0), \bar{y} = (0, 0, y_u) \), we have

\[-\bar{y} \cdot \vec{K} = -y_u K_3 = \begin{bmatrix} 0 & 0 & 0 & -y_u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -y_u & 0 & 0 & 0 \end{bmatrix}, \]

\[ ( -\bar{y} \cdot \vec{K} )^2 = y_u^2 K_3^2 = \begin{bmatrix} y_u^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_u^2 \end{bmatrix}, \tag{4.35} \]

\[ ( -\bar{y} \cdot \vec{K} )^3 = -y_u^2 ( y_u K_3 ) = \begin{bmatrix} 0 & 0 & 0 & -y_u^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -y_u^3 & 0 & 0 & 0 \end{bmatrix}. \]
Thus the full boost has the expected form

\[ e^{-\gamma_{\kappa_i}} = -K_3 \left( y_u + \frac{y_u^3}{3!} + \ldots \right) + K_3^2 \left( 1 + \frac{y_u^2}{2!} + \ldots \right) \]

\[ = -K_3 \sinh y_u + K_3^2 \cosh y_u \]

\[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ (4.36) \]

Hence we can write

\[ e^{-\gamma_{\kappa_i}} = \begin{pmatrix} \cosh y_u & 0 & 0 & -\sinh y_u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh y_u & 0 & 0 & \cosh y_u \end{pmatrix} \]

\[ (4.37) \]

To prepare us for our subsequent discussion of group theory, note that a group whose elements are parameterized in terms of continuous variables (e.g., the \( \theta_i \) and \( y_i \)) is called a Lie Group. The partial derivatives of the group elements with respect to these parameters near the identity element, i.e., near the origin in the parameter space,

\[ \left. \frac{\partial \Lambda (\bar{\theta}, \bar{y})}{\partial \theta_k} \right|_{\bar{\theta}=0, \bar{y}=0} = -S_k, \]

\[ \left. \frac{\partial \Lambda (\bar{\theta}, \bar{y})}{\partial y_l} \right|_{\bar{\theta}=0, \bar{y}=0} = -K_l, \]

\[ (4.38) \]

are called the *generators* of the group (modulo factors of \( i \)). As note earlier, this name is appropriate as these matrices (operators) are associated with generating infinitesimal transformations. The commutators of the generators define the (Lie) Algebra of the (Lie) Group. We have, for example,
\begin{align*}
\left[ S_1, S_2 \right] &= S_3, \quad \left[ S_2, S_3 \right] = S_1, \\
\left[ K_1, K_2 \right] &= -S_3, \quad \left[ S_1, K_2 \right] = K_3,
\end{align*}
\tag{4.39}

and so on. Can you guess the general case? We will discuss these points further in the next lecture.