Physics 557 – Lecture 28

The Neutral Kaon System: Technical Details II (Decays and CP violation)

Recall that in the previous lecture we focused on the mixing that results from the second order weak processes corresponding to a net change in strangeness of ±2. In that analysis we assumed that $CPT$ and $CP$ (and thus $T$) were symmetries of the interactions. The primary parameter of interest was the mass splitting, $\Delta M$ of the resulting eigenstates, which we succeeded in relating (at least approximately) to the underlying Standard Model parameters. In this lecture we want to discuss the role of decays and consider what changes when we allow the possibility of $CP$ (and $CPT$) violation. Note that, since decays involve real (as opposed to virtual) intermediate states in the second order amplitudes, we have more control of the analysis.

We look again at the time evolution of the neutral kaon system in Hamiltonian language,

$$i \frac{\partial}{\partial t} \begin{pmatrix} K^0(t) \\ \bar{K}^0(t) \end{pmatrix} = \mathcal{H} \begin{pmatrix} K^0(t) \\ \bar{K}^0(t) \end{pmatrix},$$

(28.1)

where the Hamiltonian is represented by two Hermitian 2x2 matrices,

$$\mathcal{H} = M - i \frac{1}{2} \Gamma,$$

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}, \Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{22} \end{pmatrix}.$$

(28.2)

When $CPT$ is a symmetry, $[CPT, \mathcal{H}] = 0$, both $M$ and $\Gamma$ have equal terms on the diagonal,

$$\langle K^0 | \mathcal{H} | K^0 \rangle = \langle K^0 | \mathcal{H} (CPT)^{-1} CPT | K^0 \rangle$$

$$= \langle K^0 | (CPT)^{-1} \mathcal{H} CPT | K^0 \rangle$$

$$= \langle \bar{K}^0 | \mathcal{H} | \bar{K}^0 \rangle \Rightarrow M_{11} = M_{22} = M_0, \Gamma_{11} = \Gamma_{22} = \Gamma_0,$$

(28.3)

i.e., the mass and decay width of the $K^0$ equal that for the $\bar{K}^0$. If $CP$ is a symmetry, $[CP, \mathcal{H}] = 0$ (but $T$ might be violated), the diagonal terms are again equal and the off-diagonal terms are real,
\begin{linenomath*}
\begin{equation}
\begin{aligned}
\langle K^0 | \mathcal{H} | K^0 \rangle = & \langle K^0 | \mathcal{H} (CP)^{-1} CP | K^0 \rangle = \langle K^0 | (CP)^{-1} \mathcal{H} CP | K^0 \rangle \\
= & \langle \bar{K}^0 | \mathcal{H} | \bar{K}^0 \rangle \Rightarrow M_{11} = M_{22} = M_0, \Gamma_{11} = \Gamma_{22} = \Gamma_0,
\end{aligned}
\end{equation}
\end{linenomath*}

\begin{linenomath*}
\begin{equation}
\begin{aligned}
\langle K^0 | \mathcal{H} | \bar{K}^0 \rangle = & \langle K^0 | \mathcal{H} (CP)^{-1} CP | \bar{K}^0 \rangle = \langle K^0 | (CP)^{-1} \mathcal{H} CP | \bar{K}^0 \rangle \\
= & \langle \bar{K}^0 | \mathcal{H} | K^0 \rangle \Rightarrow M_{12} = M_{21} = M_{12}^*, \Gamma_{12} = \Gamma_{21} = \Gamma_{12}^*.
\end{aligned}
\end{equation}
\end{linenomath*}

Finally consider the implication of $T$ as a symmetry. This symmetry requires the reality of the off-diagonal terms,

\begin{linenomath*}
\begin{equation}
\begin{aligned}
\langle K^0 | \mathcal{H} | \bar{K}^0 \rangle = & \langle K^0 | \mathcal{H} T^{-1} T | \bar{K}^0 \rangle = \langle K^0 | T^{-1} \mathcal{H} T | \bar{K}^0 \rangle \\
= & \langle \bar{K}^0 | \mathcal{H} | K^0 \rangle \Rightarrow \Gamma_{12} = \Gamma_{21} = \Gamma_{12}^*.
\end{aligned}
\end{equation}
\end{linenomath*}

So the limiting options of interest are:

1) $CPT$, $CP$ and $T$ are all conserved, in which case the matrices $M$ and $\Gamma$ are both real (symmetric) and equal along the diagonal;

2) $CPT$ is a good symmetry but both $CP$ and $T$ are violated (likely to describe the real world), in which case the matrices $M$ and $\Gamma$ both have equal terms along the diagonal but the off-diagonal terms are not relatively real;

3) $T$ is a good symmetry but both $CP$ and $CPT$ are violated, in which case the terms along the diagonal are not equal but the off-diagonal terms are real.

If we diagonalize the (general form of the) Hamiltonian using the standard techniques for 2x2 matrices (i.e., solve the characteristic equation), using the same sign conventions as in the previous lectures we find the following eigenvalues,

\begin{linenomath*}
\begin{equation}
\begin{aligned}
M_s - i \frac{1}{2} \Gamma_s & = \frac{1}{2} (M_{11} + M_{22}) - i \frac{1}{4} (\Gamma_{11} + \Gamma_{22}) \\
M_L - i \frac{1}{2} \Gamma_L & = \frac{1}{4} \left( (M_{11} - M_{22} - i \frac{1}{2} (\Gamma_{11} - \Gamma_{22}))^2 + (M_{12} - i \frac{1}{2} \Gamma_{12})(M_{12}^* - i \frac{1}{2} \Gamma_{12}^*) \right).
\end{aligned}
\end{equation}
\end{linenomath*}

To move from the strangeness basis used above to the $CP$ basis of the last lecture, it is helpful to introduce 2 small parameters, $\epsilon_S$ and $\epsilon_L$. In terms of these parameters we write the general (i.e., even when $CP$ and/or $CPT$ are violated) eigenstates of the original equation as
\[ |K_s\rangle = \frac{1}{\sqrt{2(1+|\epsilon_s|^2)}} \left[ (1+\epsilon_s) |K^0\rangle - (1-\epsilon_s) |\bar{K}^0\rangle \right], \]
\[ |K_L\rangle = \frac{1}{\sqrt{2(1+|\epsilon_L|^2)}} \left[ (1+\epsilon_L) |K^0\rangle + (1-\epsilon_L) |\bar{K}^0\rangle \right]. \] (28.7)

It will be helpful also to consider the 2 combinations
\[ \epsilon = \frac{1}{2}(\epsilon_s + \epsilon_L), \quad \bar{\epsilon} = \frac{1}{2}(\epsilon_s - \epsilon_L). \] (28.8)

In terms of the now familiar CP eigenstates,
\[ |K^0_1\rangle = \frac{1}{\sqrt{2}} \left[ |K^0\rangle - |\bar{K}^0\rangle \right], \]
\[ |K^0_2\rangle = \frac{1}{\sqrt{2}} \left[ |K^0\rangle + |\bar{K}^0\rangle \right], \] (28.9)
we have
\[ |K_s\rangle = \frac{1}{\sqrt{1+|\epsilon_s|^2}} \left[ |K^0_1\rangle + \epsilon_s |K^0_2\rangle \right], \]
\[ |K_L\rangle = \frac{1}{\sqrt{1+|\epsilon_L|^2}} \left[ |K^0_1\rangle + \epsilon_L |K^0_2\rangle \right]. \] (28.10)

With the privilege of foresight and hindsight we use the notation of Lecture 26, where the state with the larger width (shorter lifetime) but smaller mass is labeled as the “short” lived (S) state and the other is the “long” lived (L) state (see below). (As verified below in Eq. (28.15) CPT invariance guarantees that \( \epsilon_s = \epsilon_L = \epsilon \) in our previous notation.) The “physical” masses and widths are the appropriate real and imaginary parts of these expressions. As before, we have defined things so that the upper solution corresponds to a state that is (perhaps approximately) even under CP, decaying rapidly to \( \pi\pi \), while the lower solution is odd under CP and decays more slowly to \( \pi\pi\pi \). The forms of these parameters in the general case are not very illuminating and we will focus first on some special cases.

Consider the very interesting case of CPT conservation but with CP (and T) violations allowed (as in Lecture 26). The eigenvalues are
\[
\frac{M_S - i \frac{1}{2} \Gamma_S}{M_L - i \frac{1}{2} \Gamma_L} = M - i \frac{1}{2} \Gamma \mp \sqrt{(M_{12} - i \frac{1}{2} \Gamma_{12})(M_{12}^* - i \frac{1}{2} \Gamma_{12}^*)}, \tag{28.11}
\]

with eigenstates specified by the potentially complex ratio

\[
\frac{1 - \varepsilon_S}{1 + \varepsilon_S} = \frac{1 - \varepsilon_L}{1 + \varepsilon_L} = \chi = \sqrt{\frac{M_{12}^* - i \frac{1}{2} \Gamma_{12}^*}{M_{12} - i \frac{1}{2} \Gamma_{12}}}. \tag{28.12}
\]

In the strangeness basis we have (again using our previous conventions)

\[
\begin{bmatrix}
|K_S\rangle \\
|K_L\rangle
\end{bmatrix} = \frac{1}{\sqrt{1 + |\chi|^2}} \begin{bmatrix}
|K^0\rangle \mp \chi |\bar{K}^0\rangle
\end{bmatrix}. \tag{28.13}
\]

More explicitly this is the statement that

\[
\begin{pmatrix}
M_0 - i \frac{1}{2} \Gamma_0 & M_{12} - i \frac{1}{2} \Gamma_{12} \\
M_{12}^* - i \frac{1}{2} \Gamma_{12}^* & M_0^* - i \frac{1}{2} \Gamma_0^*
\end{pmatrix}
\begin{pmatrix}
1 \\
\mp \chi
\end{pmatrix} = \begin{pmatrix}
M_0 - i \frac{1}{2} \Gamma_0 \mp \chi (M_{12} - i \frac{1}{2} \Gamma_{12}) \\
M_{12}^* - i \frac{1}{2} \Gamma_{12}^* \mp \chi (M_0^* - i \frac{1}{2} \Gamma_0^*)
\end{pmatrix}
\tag{28.14}
\]

\[
= \begin{pmatrix}
M_0 - i \frac{1}{2} \Gamma_0 \mp \sqrt{(M_{12} - i \frac{1}{2} \Gamma_{12})(M_{12}^* - i \frac{1}{2} \Gamma_{12}^*)} \\
\mp \chi
\end{pmatrix}.
\]

In terms of the parameters directly relevant to the \(CP\) basis we have

\[
\varepsilon_S = \varepsilon_L = \varepsilon = \frac{1 - \chi}{1 + \chi}, \quad \bar{\varepsilon} = 0, \tag{28.15}
\]

which is the form in Eq. (26.20). As we noted in our previous discussion (and make explicit below), if we impose \(CP\) conservation, we expect the off diagonal terms to be real so that \(\chi = 1, \varepsilon = 0\) and we recover the previously discussed \(CP\) conserving eigenstates. As the above expression suggests, however, we really only need \(M_{12}\) and \(\Gamma_{12}\) to be \textit{relatively} real in order to avoid indirect \(CP\) violation (\textit{i.e.}, \(CP\) violation in the mixing). In this case \(\chi\) is just a phase, \textit{i.e.}, \(\chi = e^{i\phi}\), and can be rotated away by redefining the phase of the \(\bar{K}^0\) state. Indirect \(CP\) violation will arise only when \(M_{12}\) and \(\Gamma_{12}\) acquire \textit{different} phases.
To understand where such phases could come from let us return to our discussion of the second order weak interaction. Recall that the diagonal terms in $M$ look like

$$M = M_K + \langle K^0 | H_{\text{Weak}} | K^0 \rangle + \sum_{n=K^0} \frac{|\langle K^0 | H_{\text{Weak}} | n \rangle|^2}{M_{K^0} - E_n}.$$  

$$= M_{\bar{K}^0} + \langle \bar{K}^0 | H_{\text{Weak}} | \bar{K}^0 \rangle + \sum_{n=\bar{K}^0} \frac{|\langle \bar{K}^0 | H_{\text{Weak}} | n \rangle|^2}{M_{\bar{K}^0} - E_n}.$$  

Implicitly we take the principal value (i.e., we do not keep the $n = K^0$ term in the sum) of the third term in order to obtain the real part of the expression. This is explicitly what we did, in the language of field theory, for the off-diagonal term $M_{12}$ in the last lecture. Here we want to focus on the “imaginary” part of the same expression to obtain the matrix $\Gamma$, which we can relate to the decay data. Recall from your studies of analytic functions that when integrating “through a pole” the imaginary part is essentially a Dirac delta function at the pole. Thinking of the summation above as an integral we have

$$\Gamma \propto \int dn \delta (E_n - M_K) \langle K^0 | H_{\text{Weak}} | n \rangle^2.$$  

The delta function picks out only those states $n$ that are physically allowed in kaon decay. This is just the form we would expect from our general discussion of decay widths in Lecture 6. Recall that we expressed the partial decay width of particle $a$ into final state $n$ as (see Eq. (6.20))

$$d\Gamma_{a \rightarrow n} = \frac{(2\pi)^4 |M_{a \rightarrow n}|^2}{2m_a} d\Phi_n,$$  

where the phase space factor, $d\Phi_n$, includes an energy-momentum conserving delta function. The full width arises from a sum over all allowed states. (The relationship between the imaginary part of an amplitude and the sum over “physically allowed” intermediate states is also at the heart of the optical theorem in Lecture 6.)

For the real decays of $K^0$ and $\bar{K}^0$, which we discussed in Lecture 26, we have 4 amplitudes for each initial state (assuming CP conservation $[H_{\text{Weak}}, CP] = 0$, for now, and using our choice of the $CP$ phase for the neutral kaon states)
\[ \left\langle K^0 \left| \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \right\rangle = -\left\langle K^0 \right| \mathcal{CP} \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \right\rangle \right. \]
\[ = -\left\langle \bar{K}^0 \right| \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \right\rangle = -\left\langle \bar{K}^0 \right| \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \right\rangle, \]
\[ \left\langle K^0 \left| \mathcal{H}_{\text{Weak}} \left| \pi^+ \pi^- \right\rangle = -\left\langle \bar{K}^0 \right| \mathcal{H}_{\text{Weak}} \left| \pi^+ \pi^- \right\rangle, \right. \]
\[ \left(28.19\right) \]

Thus, in the CP-conserving limit, the properly normalized diagonal decay terms are

\[ \Gamma_0 = \Gamma_{11} = \Gamma_{22} = \frac{1}{2} \left( \Gamma_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0} + \Gamma_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0} \right), \]
\[ \Gamma_{\pi^+ \pi^-} = \Gamma_{\pi^+ \pi^-} + \Gamma_{\pi^+ \pi^-}, \]
\[ \left(28.20\right) \]

where \( \Gamma_{K^0, \bar{K}^0 \rightarrow \pi^0 \pi^0} \gg \Gamma_{K^0, \bar{K}^0 \rightarrow \pi^0 \pi^0} \) due to the differences in the available phase space. The off-diagonal decay terms involve the following amplitudes

\[ \left\langle K^0 \left| \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \right\rangle \left\langle \pi^0 \pi^0 \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle \right. \]
\[ = -\left\langle K^0 \right| \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \right\rangle \left\langle \pi^0 \pi^0 \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle, \]
\[ \left\langle K^0 \left| \mathcal{H}_{\text{Weak}} \left| \pi^+ \pi^- \right\rangle \left\langle \pi^+ \pi^- \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle \right. \]
\[ = -\left\langle K^0 \right| \mathcal{H}_{\text{Weak}} \left| \pi^+ \pi^- \right\rangle \left\langle \pi^+ \pi^- \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle, \]
\[ \left\langle K^0 \left| \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \pi^0 \right\rangle \left\langle \pi^0 \pi^0 \pi^0 \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle \right. \]
\[ = \left\langle K^0 \right| \mathcal{H}_{\text{Weak}} \left| \pi^0 \pi^0 \pi^0 \right\rangle \left\langle \pi^0 \pi^0 \pi^0 \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle, \]
\[ \left\langle K^0 \left| \mathcal{H}_{\text{Weak}} \left| \pi^+ \pi^- \pi^0 \right\rangle \left\langle \pi^+ \pi^- \pi^0 \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle \right. \]
\[ = \left\langle K^0 \right| \mathcal{H}_{\text{Weak}} \left| \pi^+ \pi^- \pi^0 \right\rangle \left\langle \pi^+ \pi^- \pi^0 \right| \mathcal{H}_{\text{Weak}} \left| K^0 \right\rangle, \]
\[ \left(28.21\right) \]

and thus can be expressed in terms of the physical decay widths in the following form

\[ \Gamma_{12} = \Gamma_{21} = \frac{1}{2} \left( -\Gamma_{K^0, \bar{K}^0 \rightarrow \pi^0 \pi^0} + \Gamma_{K^0, \bar{K}^0 \rightarrow \pi^0 \pi^0} \right) \approx -\frac{1}{2} \Gamma_{K^0, \bar{K}^0 \rightarrow \pi^0 \pi^0}. \]
\[ \left(28.22\right) \]
When we use these forms for the eigenvalues, we find the expected results for the $CP$ eigenstates (effectively we have $\chi = 1$)

$$
M_S - i \frac{1}{2} \Gamma_S = M_0 - M_{12} - i \frac{1}{2} \left( \Gamma_0 - \Gamma_{12} \right)
$$

$$
M_L - i \frac{1}{2} \Gamma_L = M_0 + M_{12} - i \frac{1}{2} \left( \Gamma_0 + \Gamma_{12} \right)
$$

$$
\Gamma_S = \frac{1}{2} \left( \Gamma_{\pi\pi} + \Gamma_{\pi\pi} \right) \mp \frac{1}{2} \left( -\Gamma_{\pi\pi} + \Gamma_{\pi\pi} \right)
$$

Thus, due to the numerical dominance of the 2-pion decay channel, we have

$$
\Delta \Gamma \equiv \Gamma_L - \Gamma_S = 2 \Gamma_{12} \approx -\Gamma_S = -2 \Gamma_0 = -\Gamma_{\pi\pi}
$$

for the kaon system, as we discussed in the last lecture. The observed fact that

$$
\Delta M \equiv M_L - M_S = 2 M_{12} = -\frac{\Delta \Gamma}{2} = -\Gamma_{12} \approx \Gamma_{\pi\pi}
$$

suggests that the principle value and the imaginary (on energy-shell) part of the second order weak terms discussed above are of comparable magnitudes ($M_{12} \approx -\Gamma_{12}/2$).

Finally we want to turn to the issue of $CP$ violation in the neutral kaon system. As noted in the last lecture, the $M_{12}$ component of the Hamiltonian, as represented by the box diagram, can pick up a (small) phase from the contribution of the (virtual) $t$ quark, which will involve CKM matrix elements with a phase. Specifically, we expect that

$$
\text{Im} M_{12} = \frac{G_F m_t^2}{6 \pi^2} f_K^2 m_K \text{Im} \left[ V_{ts}^2 V_{td}^* \right] \mathcal{F} \left( \frac{m^2}{M_W^2} \right)
$$

where the last factor, $\mathcal{F} \left( \frac{m^2}{M_W^2} \right)$, describes the detailed dependence of the box diagram on the mass ratio (the details we ignored in the Appendix to Lecture 27). This factor is generically of order 1. We can represent the CKM matrix in the Wolfenstein parameterization (see the PDG review) as
\[ V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \]

\[ \approx \begin{pmatrix} 1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\rho - i \eta) \\ -\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\ A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1 \end{pmatrix} + O(\lambda^4), \quad (28.27) \]

where data suggest that $\lambda = 0.22$ and $A$, $\rho$ and $\eta$ are real parameters with magnitudes expected to be of order 1. We see that $M_{12}$ is sensitive to the phase in the CKM matrix through the imaginary part of $V_{td}$. For future reference we note that fairly standard notation (see the PDG) for the (related) phases in the CKM matrix is given by

\[ V_{td} = |V_{td}| e^{-i \beta}, \tan \beta = \frac{\eta}{1 - \rho}, \quad (28.28) \]
\[ V_{ub} = |V_{ub}| e^{-i \gamma}, \tan \gamma = \frac{\eta}{\rho}. \]

In contrast, the decay widths, in particular $\Gamma_{12}$, are not sensitive to the CKM phases in leading order. Since the widths only receive contributions from physically accessible states, there is no leading order $t$ (or $b$) quark contribution to kaon decay. Thus we expect that $\text{Im} \Gamma_{12} \ll \text{Im} M_{12}$ (although the real parts of the two terms are comparable). This means that we expect the magnitude of the parameter $\chi$ introduced above to deviate from unity (or more generally that it is not a pure phase) and $\varepsilon$ to be nonzero.

With a phase convention where the dominant 2-pion decay amplitude is real and assuming that all of the imaginary parts of amplitudes are small, we can write
\[
\varepsilon = \frac{1 - \chi}{1 + \chi} = \frac{1 - \sqrt{\left(M_{12}^* - i \frac{1}{2} \Gamma_{12}^*\right) / \left(M_{12} - i \frac{1}{2} \Gamma_{12}\right)}}{1 + \sqrt{\left(M_{12}^* - i \frac{1}{2} \Gamma_{12}^*\right) / \left(M_{12} - i \frac{1}{2} \Gamma_{12}\right)}}
\]

\[
\approx \frac{1}{2} \left[ 1 - \sqrt{\frac{\text{Re} M_{12}^* - i \frac{1}{2} \text{Re} \Gamma_{12} - i \text{Im} M_{12} - \frac{1}{2} \text{Im} \Gamma_{12}}{\text{Re} M_{12}^* - i \frac{1}{2} \text{Re} \Gamma_{12} + i \text{Im} M_{12} + \frac{1}{2} \text{Im} \Gamma_{12}}} \right]
\]

\[
\approx \frac{i \text{Im} M_{12} + \frac{1}{2} \text{Im} \Gamma_{12}}{\Delta M - \frac{1}{2} i \Delta \Gamma} = \frac{i \text{Im} M_{12}}{\Delta M - \frac{1}{2} i \Delta \Gamma} = \frac{-\frac{1}{2} \Delta \Gamma - i \Delta M}{\frac{1}{4} \Delta \Gamma^2 + \Delta M^2} \text{Im} M_{12}.
\]

(28.29)

The next to last line assumes that \(\text{Im} \Gamma_{12} \ll \text{Im} M_{12}\), which we motivated above. Thus (assuming that \(\text{Im} M_{12} > 0\)) we find the phase of \(\varepsilon\) to be given by

\[
\phi_\varepsilon = \arctan \left( \frac{2 \left( M_L - M_S \right)}{\Gamma_S - \Gamma_L} \right) \equiv \phi_0 \approx 43.5^\circ,
\]

(28.30)

where the numerical result comes from substituting the experimental values for the masses and widths. This result explains the phases near \(\pi/4\) that we first introduced in Lecture 26. This analysis of indirect CP violation is apparently self-consistent and in good agreement with the data.

With our (admittedly somewhat crude) results for \(\text{Im} M_{12}\) and \(\Delta M\) (using the good approximation \(\Delta M = -\Delta \Gamma/2\)) and using the representation of the CKM matrix above, we expect

\[
|\varepsilon| \approx \frac{\text{Im} M_{12}}{\sqrt{2 \Delta M}} \approx \frac{1}{2 \sqrt{2}} \frac{m_t^2}{m_c^2} \left( \frac{\lambda^{10}}{\lambda^2} \right) |2\eta(1 - \rho)| \approx 0.02 \times |\sin 2\beta|.
\]

(28.31)

Thus the measurement of CP violation in the neutral kaon system provides us with a measurement of the phase in the CKM matrix. From our crude numbers (recall that we have only approximately evaluated the required hadronic matrix elements) we expect that \(\sin 2\beta \sim 0.1\). The experimental results are consistent with reasonable values for the CKM parameters but it is clearly important to test and clarify the
picture with further measurements, \textit{e.g.}, in the neutral B system as we will discuss in
the next lecture. In terms of the parameter \( \varepsilon \), as already noted in Lecture 26, we also have
\[
\delta_L = \frac{\Gamma(K_L \to \pi^- l^+ \nu_l) - \Gamma(K_L \to \pi^+ l^- \bar{\nu}_l)}{\Gamma(K_L \to \pi^- l^+ \nu_l) + \Gamma(K_L \to \pi^+ l^- \bar{\nu}_l)} \approx 2 \Re \varepsilon 
\]
\[
= (3.32 \pm 0.06) \times 10^{-3}. 
\] (28.32)

Before proceeding to discuss direct CP violation, it is informative to consider how the
discussion above changes if we allow \textit{CPT} and \textit{CP} violation (but \textit{T} conservation).
From the discussion at the beginning of the lecture, we know that we can introduce
\textit{CPT} (and \textit{CP}) violation by explicitly splitting the masses of the \( K^0 \) and \( \bar{K}^0 \) mesons. As
a simple expression of this consider the form
\[
\mathcal{H}_{\text{CPT}} = \begin{pmatrix}
M_0 - \delta m - i \frac{1}{2} \Gamma_0 & M_{12} - i \frac{1}{2} \Gamma_{12} \\
M_{12}^* - i \frac{1}{2} \Gamma_{12}^* & M_0 + \delta m - i \frac{1}{2} \Gamma_0 
\end{pmatrix}.
\] (28.33)

Employing the same techniques as earlier we find eigenvalues
\[
M_S - i \frac{1}{2} \delta \Gamma_S = M_0 - i \frac{1}{2} \Gamma_0 \mp \sqrt{\delta m^2 + (M_{12} - i \frac{1}{2} \Gamma_{12}^*)^2 (M_{12} - i \frac{1}{2} \Gamma_{12}^*)}. 
\] (28.34)

To keep things simple we will assume for now that \( M_{12} - i \frac{1}{2} \Gamma_{12} \) and \( M_{12}^* - i \frac{1}{2} \Gamma_{12}^* \) are
relatively real so that all of the \textit{CP} violation arises entirely from the explicit \textit{CPT}
violation and further that the amount of explicit \textit{CPT} violation is small compared to
the mixing, \textit{i.e.}, \( \delta m \ll M_{12} \).

Hence the structure of the eigenvalues has changed very little from the previous, \textit{CPT}
conserving, case. The corresponding eigenstates in the strangeness basis look like
\[
\left[ |K^0\rangle \mp \sqrt{\chi^2 + \delta m^2 / (M_{12} - i \frac{1}{2} \Gamma_{12})^2} \mp \delta m / (M_{12} - i \frac{1}{2} \Gamma_{12}) \right] |\bar{K}^0\rangle 
\]
\[
\mp \left[ |K^0\rangle \mp \delta m / (M_{12} - i \frac{1}{2} \Gamma_{12}) \right] |\bar{K}^0\rangle. 
\] (28.35)
where we have set $\chi$ explicitly to unity in the last step. Once more we can project onto the $CP$ basis and identify the parameters $\varepsilon_S$ and $\varepsilon_L$ to leading order in $\delta m/M_{12}$. We find

$$
\varepsilon_S \equiv \pm \frac{\delta m}{2\left(M_{12} - i\frac{1}{2}\Gamma_{12}\right)} = \pm \frac{\delta m}{\Delta M - i\frac{1}{2}\Delta \Gamma},
$$

$$
\varepsilon_L = \frac{\delta m}{\Delta M - i\frac{1}{2}\Delta \Gamma}, \varepsilon = 0.
$$

Note the distinctly different structure from the $CPT$ conserving scenario discussed earlier, i.e., in Eqs. (28.15) and (28.29). The $CP$ violating parameters $\varepsilon_S$ and $\varepsilon_L$ are now of opposite sign and differ in phase from the previous result by $90^\circ$ (i.e., the factor of $i$ is missing), while the parameters $\varepsilon$ and $\overline{\varepsilon}$ have, in some sense, switched roles. This (extreme) scenario is clearly ruled out by the data. In fact, the neutral kaon system offers some of the clearest confirmations of $CPT$ as a valid symmetry.

In a more comprehensive analysis (see the PDG web page on Tests of Conservation Laws and the brief discussion in the Appendix), the following limit can be deduced from the neutral kaon data,

$$
\left| m_{K^0} - m_{\bar{K}^0} \right| < 8 \times 10^{-19}. \quad (28.37)
$$

The above discussion focuses on how “indirect” $CP$ violation can occur in the kaon system, i.e., through mixing in the wave function, but we still need to address “direct” violation in the decay process. As outlined in our earlier discussions and in the HW, this possibility can be observed only if we can compare the phases of two distinct decay amplitudes (i.e., the phase of a single amplitude is not physically observable). In the neutral kaon system we can use the two (strong) isospin amplitudes for the 2-pion decays, $I = 0$ and 2. To the extent that they have different phases, there is a second small parameter, which characterizes direct $CP$ violation,

$$
\varepsilon' \equiv \frac{ie^{i(\delta_2 - \delta_0)}}{\sqrt{2}} \text{Im} \left( \frac{A_2}{A_0} \right).
$$

As previously defined, the $A_I$ are the weak amplitudes while the $\delta_I$ are the phase shifts generated by the strong interactions between the pions in the final state.

Unfortunately, as we have already observed (the $\Delta I = \frac{1}{2}$ rule), $|A_2| \sim |A_0|/20$, and direct
CP violation in the kaon system is truly tiny. In fact, it was initially felt that, in the Standard Model, this parameter should vanish as the decays involve only the low mass quarks for which the CKM matrix elements are essentially real. However, it was eventually recognized that there is a class of diagrams, the so-called “penguin” diagrams (there are a series of implausible explanations of the name, although there is general agreement to blame it on my “brother” John Ellis), one of which is illustrated here, that contribute to $A_0$ and contain virtual loops with the required CKM elements.

![Penguin diagram](image1)

(A) Penguin diagram

![LO diagram](image2)

(B) LO diagram

The interference term between the penguin diagram and the lowest order diagram is again dependent on the CKM element $\text{Im} \left[ V_{ts} V^*_{td} \right]$ and introduces (a small) direct CP violation term. We will not attempt to evaluate these diagrams in this class. However, since this higher order effect is small, $\varepsilon' \ll \varepsilon$, and with the results noted in Lecture 26 (and the HW)

$$
\eta^{+} = \frac{\text{ampl}(K_L \rightarrow \pi^+ \pi^-)}{\text{ampl}(K_S \rightarrow \pi^+ \pi^-)} \approx \varepsilon + \varepsilon',
$$

$$
\eta^{00} = \frac{\text{ampl}(K_L \rightarrow \pi^0 \pi^0)}{\text{ampl}(K_S \rightarrow \pi^0 \pi^0)} \approx \varepsilon - 2\varepsilon',
$$

it is easy to understand (to leading order in small numbers) the magnitude and phase of $\eta^{+}$ and $\eta^{00}$,

$$
|\eta^{+}| = |\eta^{00}| \approx |\varepsilon| \approx 2.2 \times 10^{-3}, \phi_{-} = \phi_{00} = \phi_{\varepsilon} \approx 43.5^\circ.
$$

We close this discussion by mentioning a subject related to how the results of measurements of the CKM elements are displayed and checked. Since CKM is a unitary matrix, there are various unitarity conditions on the elements. For example, we have

$$
V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0.
$$
Substituting for the “diagonal” elements \(\sim 1\), we have
\[
s_{12}V_{cb}^* = \lambda V_{cb}^* = V_{ub}^* + V_{td},
\] (28.41)
which describes the relationship between these 3 complex quantities. This relationship is typically represented in terms of a triangle in the complex plane, as indicated in part (A) of the figure to the left. Note that the phases defined earlier are explicitly represented in the triangle. The second triangle, (B), shows the result after scaling out the common factor \(A\lambda^3\). Such triangles are a feature of the study of the CKM elements.

A summary of the various forms of experimental data that constrain our knowledge of this triangle is presented in the final figure, taken from a recent PDG review.

In the next lecture we will consider the neutral \(B\) system.

Figure 12.2: Constraints on the \(\rho, \eta\) plane. The shaded areas have 95\% CI.
Appendix – More about \textit{CPT}

Here we will discuss the behavior of the neutral kaon system if \textit{CP} violation is present both due to the usual CKM phase in the off-diagonal mixing term (presumably the dominant cause) and due a \textit{CPT} violating mass difference for the kaons. The relevant parameters are

\begin{equation}
\chi = \sqrt{\frac{(M_{12}^* - i \frac{1}{2} \Gamma_{12}^*)}{(M_{12} - i \frac{1}{2} \Gamma_{12})}}, \quad 2\delta m = m_{K^0} - m_{K^+},
\end{equation}

as already introduced in the lecture. We will proceed by keeping only the leading terms in the small parameters. The corresponding eigenvalues are

\begin{equation}
M_S - i \frac{1}{2} \Gamma_S \quad \text{and} \quad M_L - i \frac{1}{2} \Gamma_L = M - i \frac{1}{2} \Gamma \mp \sqrt{\delta m^2 + (M_{12} - i \frac{1}{2} \Gamma_{12})(M_{12}^* - i \frac{1}{2} \Gamma_{12}^*)},
\end{equation}

with eigenstates defined by

\begin{align}
\frac{1 - \varepsilon_S}{1 + \varepsilon_S} &= \sqrt{\chi^2 + \delta m^2/(M_{12} - i \frac{1}{2} \Gamma_{12})^2} - \delta m/(M_{12} - i \frac{1}{2} \Gamma_{12}), \\
\frac{1 - \varepsilon_L}{1 + \varepsilon_L} &= \sqrt{\chi^2 + \delta m^2/(M_{12} - i \frac{1}{2} \Gamma_{12})^2} + \delta m/(M_{12} - i \frac{1}{2} \Gamma_{12}).
\end{align}

To leading order in the (presumed) small quantities we have

\begin{align}
\varepsilon &= \frac{\varepsilon_S + \varepsilon_L}{2} \approx \frac{1 - \chi}{2}, \\
\bar{\varepsilon} &= \frac{\varepsilon_S - \varepsilon_L}{2} \approx \frac{\delta m}{2 M_{12} - i \frac{1}{2} \Gamma_{12}} \approx \frac{\delta m}{\Delta M - i \frac{1}{2} \Delta \Gamma}.
\end{align}

Now consider the (nonzero) overlap between the eigenstates,
\[ \langle K_S | K_L \rangle = \frac{(1 + \varepsilon_S^*) (1 + \varepsilon_L) - (1 - \varepsilon_S^*) (1 - \varepsilon_L)}{2 \sqrt{1 + |\varepsilon_S|^2} \sqrt{1 + |\varepsilon_L|^2}} \approx \varepsilon_S^* + \varepsilon_L \]

\[ \approx 2 \left( \text{Re}(\varepsilon) - i \text{Im}(\bar{\varepsilon}) \right). \]

This expression introduces the desired dependence on the kaon mass splitting. We proceed to relate these quantities to other experimental quantities by considering the time dependence of this matrix element both in terms of the eigenvalues and in terms of the actual decay amplitudes. We have

\[ i \frac{d}{dt} \langle K_S | K_L \rangle = \{ M_L - i \frac{1}{2} \Gamma_S - M_S - i \frac{1}{2} \Gamma_L \} \langle K_S | K_L \rangle = \{ 2\Delta M - i \left( \Gamma_S + \Gamma_L \right) \} \left( \text{Re}(\varepsilon) - i \text{Im}(\bar{\varepsilon}) \right) \]

\[ = \sum_n \langle K_S | \mathcal{H}_{\text{Weak}} | n \rangle \langle n | \mathcal{H}_{\text{Weak}} | K_L \rangle. \]

Now we assume that the sum over intermediate states is saturated by the two-pion states and use our previous definitions to write (where there is an implicit sum/integral of the 2-pion phase space and we are using the same normalization as in the Lecture)

\[ i \frac{d}{dt} \langle K_S | K_L \rangle \]

\[ \approx \langle K_S | \mathcal{H}_{\text{Weak}} | \pi^+ \pi^- \rangle \langle \pi^+ \pi^- | \mathcal{H}_{\text{Weak}} | K_L \rangle + \langle K_S | \mathcal{H}_{\text{Weak}} | \pi^0 \pi^0 \rangle \langle \pi^0 \pi^0 | \mathcal{H}_{\text{Weak}} | K_L \rangle \]

\[ \approx \left| \frac{\langle \pi^+ \pi^- | \mathcal{H}_{\text{Weak}} | K_L \rangle}{\langle \pi^+ \pi^- | \mathcal{H}_{\text{Weak}} | K_S \rangle} \right|^2 \left| \frac{\langle K_S | \mathcal{H}_{\text{Weak}} | \pi^+ \pi^- \rangle}{\langle K_S | \mathcal{H}_{\text{Weak}} | K_L \rangle} \right|^2 \]

\[ + \left| \frac{\langle \pi^0 \pi^0 | \mathcal{H}_{\text{Weak}} | K_L \rangle}{\langle \pi^0 \pi^0 | \mathcal{H}_{\text{Weak}} | K_S \rangle} \right|^2 \left| \frac{\langle K_S | \mathcal{H}_{\text{Weak}} | \pi^0 \pi^0 \rangle}{\langle K_S | \mathcal{H}_{\text{Weak}} | K_L \rangle} \right|^2 \]

\[ \approx \frac{1}{2} \left( \eta^{++} \Gamma_S^{++} + \eta^{00} \Gamma_S^{00} \right). \]

So the expression we want to simplify (reorganize) is
\[ \text{Re}(\varepsilon) - i \text{Im}(\varepsilon) \approx \frac{1}{2} \left( \eta^{+} \Gamma_{s}^{+} + \eta^{00} \Gamma_{s}^{00} \right) - i \left( \frac{1}{2} \eta \Gamma_{s}^{+} \Gamma_{L} \right) \]

\[ \approx \frac{1}{2} \left( \eta^{+} \Gamma_{s}^{+} + \eta^{00} \Gamma_{s}^{00} \right) \approx \frac{1}{2} \left( \frac{1}{4} \Gamma^{2} + i \Delta \right) \]

\[ \approx -\frac{1}{2} \left| \frac{\eta}{\Gamma} \right| \cos \phi_0 \left( 1 + i \frac{2}{3} \phi_+ + \frac{1}{3} \phi_0 - i \phi_0 \right). \]

In the second line we used the dominance of the \( I = 0 \) channel (the \( \Delta I = \frac{1}{2} \) rule) and the corresponding Clebsch-Gordan coefficients. The form of \( \varepsilon \) obtained above yields the following expression for the imaginary part

\[ \text{Im}(\varepsilon) \approx \frac{2 \delta m \Delta \Gamma}{4 \Delta M^2 + \Delta^2} = \frac{\Delta M^\kappa \cos \phi_0}{\sqrt{4 \Delta M^2 + \Delta^2}} = \frac{\Delta M^\kappa \cos \phi_0}{2 \Delta M} \sin \phi_0. \]  

Finally we can take the imaginary part of the previous expression and equate the two results (normalizing to the kaon mass itself)

\[ \frac{\Delta M^\kappa}{m^\kappa} \approx \frac{\Delta M}{m^\kappa} \left| \frac{\frac{2}{3} \phi_+ + \frac{1}{3} \phi_0 - \phi_0}{\sin \phi_0} \right|. \]  

Substituting the various experimental values from Lecture 27 we find the expected small limit (\( i.e., \) each factor in this expression is a small number)

\[ \frac{\Delta M^\kappa}{m^\kappa} \leq \left( 7 \times 10^{-15} \right) \times \left( 2.2 \times 10^{-3} \right) \times \frac{\frac{2}{3} \times 43.51^\circ + \frac{1}{3} \times 43.52 - 43.51^\circ}{\sin(43.5^\circ)} \]  

\[ \leq 10^{-18}, \]  

and the quoted (PDG) number includes the uncertainties, especially in the angles, which yield zero within the errors. Clearly there is little experimental room for \( CPT \)
violation in the neutral kaon system and this system provides one of the most stringent experimental tests.

This brings us to the related question of observing the $T$ violation implied by the conservation of $CPT$ in the presence of the violation of $CP$. It turns out that the neutral kaon system, which, as we have noted several times, is dominated by the large differences in the decay physics for the $2\ CP$ eigenstates, is not very well suited to this study. While $T$ violating observations in the neutral kaon system have been claimed, the situation is rather murky as explained, for example, by Lincoln Wolfenstein, *Phys. Rev. Lett.* 83 (1999) 911; *Int. J. Mod. Phys.* E 8 (1999) 501. To see $T$ violation explicitly you would like to run the same process in both directions (in time), which is difficult for decay processes (where the multi-particle final states correspond to very challenging initial states). Further, the analysis of the kaon decay system is complicated by the appearance of the phases due to the strong interactions in the hadronic final states. The neutral B system (as observed at B factories) allows a rather different analysis and the separate observation of $CP$ and $T$ violation. We discuss the neutral B system next.