Applications of the Standard Model of the Electroweak Interactions:

Let us return to a process that we discussed briefly at the end of Lecture 16 in the context of our phenomenological model for the weak interactions. Recall that we had reduced the apparent difficulties with the original Fermi model, which had point 4-fermion interactions, by introducing massive charged boson to mediate the charged current interactions. However, we noted at the end of Lecture 16 that the process $\nu \bar{\nu} \rightarrow W^+W^-$, where the $W$’s are longitudinally polarized, is still troublesome in that the cross section grows with energy without bound. The diagram we considered was electron exchange as indicated to the left, corresponding to electron exchange in the $t$ channel. In the Standard Model Electroweak theory this amplitude still has the same form. Let us define the 4-momenta as

$$q^\mu_{\nu} = (Q,0,0,Q)$$
$$q^\mu_{\bar{\nu}} = (Q,0,0,-Q)$$
$$k^\mu_{W^+} = (Q,K \sin \theta,0,K \cos \theta)$$
$$k^\mu_{W^-} = (Q,-K \sin \theta,0,-K \cos \theta)$$
$$P^\mu_e = q^\mu_{\nu} - k^\mu_{W^+} = (0,-K \sin \theta,0,Q - K \cos \theta),$$

with $Q^2 = s/4, K = \sqrt{Q^2 - M_W^2}, P^2_e = t$. Using the Feynman rules of the last lecture the matrix element is

$$M_e = (-i)^2 \frac{G_F M_W^2}{\sqrt{2}} \bar{\nu}_\nu (q_{\nu\bar{\nu}}) \gamma^\nu (1 - \gamma_5) \frac{P_e + m_e}{P_e^2 - m_e^2} \gamma^\mu (1 - \gamma_5) u_{\nu_e} (q_{\nu_e}).$$

The polarization vectors for longitudinally polarized vectors look like

$$\epsilon^{\mu}_{z,\text{long}} = \left( \frac{K}{M_W}, \frac{Q}{M_W} \hat{k}_{W^+} \right) \frac{k^\mu_{W^+}}{Q,K \gg M_W}. \quad (20.3)$$

Note that these expressions have the required properties,
\[ \epsilon^\mu_{\pm, \text{long}} k^\nu_{\pm, \mu} = 0, \epsilon^\mu_{\pm, \text{long}} \epsilon_{\pm, \text{long}, \mu} = -1. \]  

(20.4)

Substituting above we find

\[
M_e = -i \frac{G_F}{\sqrt{2}} \bar{v}_e (q_{\sigma}) \epsilon^\nu_w (1 - \gamma_5) \frac{P_e^\nu + m_e}{P_e^2 - m_e^2} \epsilon^\nu_{w*} (1 - \gamma_5) u_{\nu_e} (q_{\nu_e}).
\]

(20.5)

Using the Dirac equation for the massless \(i.e.,\) ignoring the tiny mass for now) neutrinos we can make the replacement

\[
\epsilon^\nu_w \rightarrow \gamma^\nu - \gamma_{\nu_e} = -P_e
\]

\[
\epsilon^\nu_{w*} \rightarrow \gamma^\nu - \gamma_{\nu_e} = P_e^\nu.
\]

(20.6)

and obtain (ignoring also the mass of the electron in the relativistic limit)

\[
M_e = i \frac{G_F}{\sqrt{2}} \bar{v}_e (q_{\sigma}) P_e^\nu (1 - \gamma_5) \frac{P_e^\nu}{P_e^2} (1 - \gamma_5) u_{\nu_e} (q_{\nu_e})
\]

\[
= i \sqrt{2} G_F \bar{v}_e (q_{\sigma}) P_e^\nu (1 - \gamma_5) u_{\nu_e} (q_{\nu_e}).
\]

(20.7)

Squaring and summing over spins we find

\[
|\bar{M}_e|^2 = 2G_F^2 \text{Tr} \left[ \frac{P_e^\nu (1 - \gamma_5)}{P_e^2} \gamma_{\nu_e} (1 + \gamma_5) P_e^\nu \gamma_{\nu_e} \right]
\]

\[
= 4G_F^2 \text{Tr} \left[ (1 - \gamma_5) \gamma_{\nu_e} P_e^\nu \gamma_{\nu_e} P_e^\nu \right]
\]

\[
= 16G_F^2 \left[ 2q_{\nu_e} \cdot P_{\nu_e} q_{\nu_e} \cdot P_e - q_{\nu_e} \times P_{\nu_e} \cdot P_e \right]
\]

\[
= 32G_F^2 Q^2 K^2 \sin^2 \theta = 2G_F^2 s^2 \sin^2 \theta.
\]

(20.8)

The fact that the scattering vanishes in the forward and backward directions is easily understood by considering (as usual) angular momentum. The initial state is composed of a left-handed fermion colliding head-on with a right-handed antifermion creating a state with \(J_3 = -1\). The final state has two collinear, longitudinally polarized vector particles with \(J = 0\) along the direction of motion (any angular momentum due to orbital angular momentum is transverse to the scattering plane). Thus there can be no scattering when the final direction of motion is either aligned or
anti-aligned with the initial direction of motion. In the CM system the differential cross section becomes

\[
\frac{d\sigma}{d\Omega_{CM}} = \left| \frac{M}{64\pi^2 s} \frac{p'_{CM}}{p_{CM}} \right|^2 = \frac{G_F^2 s}{32\pi^2} \sin^2 \theta. \tag{20.9}
\]

Hence the integrated cross section,

\[
\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{G_F^2 s}{12\pi}, \tag{20.10}
\]

has the expected but inappropriate linear dependence on \(s\), growing forever. (Note that this behavior does not arise for transversely polarized W’s. Their components stay fixed in magnitude as \(s\) increases.) In the context of the SM theory of Electroweak interactions (unlike this phenomenological model with only W’s), the undesirable (non-unitary) behavior exhibited in Eq. (20.10) is cancelled by the second process that contributes at this order, which is not present in this phenomenological model. In particular, now we must also include the contribution of a Z particle appearing in the \(s\)-channel. This process involves the famous triple-boson coupling (and the phase will matter!). Using the Feynman rules from Lecture 19, we find the amplitude for this new process to be

\[
M_Z = -i \frac{g}{4\cos \theta_W} \bar{v}_e \left( q_{\tau_e} \right)^{\gamma^\mu} \left( 1 - \gamma_5 \right) u_e \left( q_{\nu_e} \right) \times \left( -i \right) \frac{g_{\mu\nu} - S_{\mu}S_{\nu}}{M_Z^2 - M_Z^2} \left( -i g \cos \theta_W \right) \times \left[ g^{\alpha\beta} \left( k_{\nu_e} - k_{\tau_e} \right)^\nu + g^{\alpha\nu} \left( S + k_{\nu_e} \right)^\beta + g^{\beta\nu} \left( -k_{\tau_e} - S \right)^\alpha \right] \times \epsilon^*_{\alpha,\alpha} \epsilon^*_{\beta,-\beta}, \tag{20.11}
\]

where \(S^\mu \equiv \left( q_{\nu_e}^\mu + q_{\tau_e}^\mu \right) = \left( k_{\nu_e}^\mu + k_{\tau_e}^\mu \right), S^2 = s\) and we have performed the appropriate transformation on the \(Z \rightarrow W^+W^-\) vertex in Lecture 19, Eq. (19.11) to correspond to the W’s now being outgoing (change the overall sign, because an incoming \(W^+\) becomes an outgoing \(W^-\) and vice versa, and change the signs of the \(W\) 4-momenta). By again using the Dirac equation for the neutrinos we note that the \(S^\nu S^\nu\) term in the propagator yields zero when multiplied into the neutrino vertex. We also need to
recall that the polarization vectors are orthogonal to the corresponding momenta, 
\( \varepsilon_+ \cdot k_{\nu^+} = \varepsilon_- \cdot k_{\nu^-} = 0 \), as noted above. Thus this amplitude takes the form

\[
M_Z = i \frac{g^2}{4(s-M_Z^2)} \nabla_{\nu_e} (q_{\tau_e}) \gamma_\nu (1-\gamma_5) u_{\nu_e} (q_{\nu_e}) 
\]

\[
\left[ \varepsilon_+^* \cdot \varepsilon_- \left( k_{\nu^+} - k_{\nu^-} \right) + 2 \varepsilon_+^* \varepsilon_-^* k_{\nu^+} - 2 \varepsilon_+^* \varepsilon_-^* \cdot k_{\nu^-} \right].
\]

(20.12)

Using our explicit expressions for the momenta and the longitudinal polarization vectors it is straightforward to obtain the following relations

\[
\varepsilon_+^* \cdot \varepsilon_-^* = \frac{Q^2 + K^2}{M_w^2},
\]

\[
\varepsilon_+^* \cdot k_{\nu^+} = \varepsilon_-^* \cdot k_{\nu^-} = \frac{2QK}{M_w},
\]

\[
\left( k_{\nu^+} - k_{\nu^-} \right)^\nu = \left( 0, K \hat{k}_{\nu^+} \right),
\]

\[
\varepsilon_+^* - \varepsilon_-^* = \left( 0, \frac{2Q}{M_w} \hat{k}_{\nu^+} \right) = \frac{Q}{K M_w} \left( k_{\nu^+} - k_{\nu^-} \right)^\nu.
\]

(20.13)

Thus we can rewrite this matrix element as

\[
M_Z = i \frac{g^2}{4(s-M_Z^2)} \nabla_{\nu_e} (q_{\tau_e}) \gamma_\nu (1-\gamma_5) u_{\nu_e} (q_{\nu_e}) 
\]

\[
\left[ -\frac{Q^2 + K^2}{M_w^2} + 2 \frac{2QK}{M_w} \frac{Q}{K M_w} \left( k_{\nu^+} - k_{\nu^-} \right)^\nu \right]
\]

\[
= i \frac{g^2}{4(s-M_Z^2)} \frac{2Q^2 + M_w^2}{M_w^2} \nabla_{\nu_e} (q_{\tau_e}) \left( \hat{k}_{\nu^+} - \hat{k}_{\nu^-} \right) (1-\gamma_5) u_{\nu_e} (q_{\nu_e}) 
\]

\[
= -i \frac{g^2}{4(s-M_Z^2)} \frac{s + 2M_w^2}{M_w^2} \nabla_{\nu_e} (q_{\tau_e}) P_\nu (1-\gamma_5) u_{\nu_e} (q_{\nu_e}).
\]

(20.14)

In the last step we again used the Dirac equations for the neutrinos to make the replacement
\[ \mathcal{K}_w^- - \mathcal{K}_w^+ \rightarrow \mathcal{K}_w^+ - \mathcal{g}_{w^+} - (\mathcal{K}_w^- - \mathcal{g}_{w^-}) = -2P_e. \] (20.15)

We can rewrite \( g^2/4 \) as \( \sqrt{2G_F}M_W^2 \) in Eq. (20.14) and sum the amplitudes for \( t \)-channel electron exchange in Eq. (20.7) with \( s \)-channel \( Z \) exchange in Eq. (20.14) (since the initial and final states are the same, we sum the two contributions at the level of amplitudes). We find

\[
\mathcal{M}_e + \mathcal{M}_Z = i\sqrt{2G_F} \nabla_v \left(q_{\Gamma}\right) P_e \left(1 - \gamma_s\right) u_v \left(q_{\nu}\right) \left[1 - \frac{s + 2M_W^2}{s - M_Z^2}\right]
\]

\[\Rightarrow -i\sqrt{2G_F} \left(\frac{2M_W^2 + M_Z^2}{s - M_Z^2}\right) \nabla_v \left(q_{\Gamma}\right) P_e \left(1 - \gamma_s\right) u_v \left(q_{\nu}\right).\] (20.16)

The essential point is that the leading asymptotic behavior in the 2 amplitudes cancels and leaves us with a bounded result for the spin-averaged amplitude squared \( i.e.\), there is no longer a term linear in \( s \),

\[
\left|\mathcal{M}_e + \mathcal{M}_Z\right|^2 = 32G_F^2 Q^2 K^2 \sin^2 \theta \left(\frac{2M_W^2 + M_Z^2}{s - M_Z^2}\right)^2
\]

\[\xrightarrow{s \gg M_W^2, M_Z^2} 2G_F^2 \left(2M_W^2 + M_Z^2\right)^2 \sin^2 \theta \]

\[= \frac{g^4}{16} \left(\frac{2\cos^2 \theta_W + 1}{\cos^2 \theta_W}\right)^2 \sin^2 \theta.\] (20.17)

The corresponding cross section is no longer a threat to unitarity and, in fact, decreases with energy,

\[
\sigma_{e^+e^- \rightarrow W^+W^-} = \frac{g^4}{384\pi s} \left(\frac{2\cos^2 \theta_W + 1}{\cos^2 \theta_W}\right)^2 \left[1 + \mathcal{O}\left(M_Z^2/s\right)\right].\] (20.18)

This behavior, \( \sigma \propto 1/s \), is just what we would expect for the asymptotic behavior in a theory that is asymptotically free of dimensionfull parameters, \( i.e.\), the dimensions must come from the kinematic factors. The initial problem, which was in the “p-wave” \( i.e.\), \( \text{LEP II} \) process \( e^+e^- \rightarrow W^+W^- \), as we will discuss eventually...
in the HW. In this latter process, due to the mass of the electron (both left-handed and right-handed components can contribute – the latter suppressed by a factor of $m_e/\sqrt{s}$), there is also an s-wave issue, which is “fixed” by the direct Higgs particle contribution. In fact, at asymptotic energies, $E \gg v$ (the vev), the symmetry breaking is unimportant and the scattering structure we have been focusing on here is just that of the original Higgs theory with 4 scalar degrees of freedom, rather than a Higgs boson plus the longitudinal modes of the vector bosons. This issue is also addressed in future HW.