In previous lectures we have outlined the quark/parton model as a description of hadrons. While this model supplies an intuitively appealing and approximately correct formalism for understanding the experimental data, it fails to provide a “fundamental” understanding of the physics (especially the symmetries) and, perhaps more importantly, fails to provide a systematic procedure for defining corrections to the zeroth order picture. We would prefer to have a true theory (instead of a model), which rigorously defines the structure of corrections. Prior to the mid-1970s many physicists believed that field theory was not the correct language for such a theory of particle physics. While quantum field theory was clearly the correct language for QED, it apparently failed to offer a description of either the weak or the strong interactions. However, once the richness of Yang-Mills theories with non-Abelian symmetries became apparent, this situation changed dramatically. So the next step in our discussion of symmetries and particle physics will be to introduce the very helpful language of field theory. In particular, with the language of field theory and Lagrangians we will be able to discuss the relationship between interactions (i.e., dynamics) and symmetries.

As a first and familiar example we will consider the electromagnetic interactions, QED, and the underlying local gauge symmetry. To proceed we must establish (recall) some formalism. First let us reach back to classical mechanics and define the Lagrangian describing an ensemble of particles labeled by coordinates \( q_i(t) \). The Lagrangian is formally given by the difference between the kinetic energy, a function of the velocities, and the potential energy, a function of the positions,

\[ L(q_i, \dot{q}_i) = T(\dot{q}_i) - V(q_i). \] (13.1)

The action, \( S \), is defined as the time integral of the Lagrangian

\[ S = \int_{t_i}^{t_f} dt \ L. \] (13.2)

By applying Hamilton’s Principle of Least Action, i.e., the action describing the “true” behavior of the system is an extremum, we are led to the Euler-Lagrange Equations or the usual Equations of Motion. We proceed by considering a variation in each of the coordinates of the form
\[ q_i(t) \rightarrow q_i(t) + \varepsilon \eta(t), \]  
(13.3)

where \( \varepsilon_i \) is a small parameter and \( \eta(t) \) is an arbitrary function of time subject only to the constraint that the variation of the coordinates must vanish at the boundaries, i.e.,

\[ \eta(t_1) = \eta(t_2) = 0. \]  
(13.4)

Now consider the dependence of the action on the parameter \( \varepsilon_i \)

\[ \frac{dS}{d\varepsilon_i} = \int_{t_i}^{t_f} dt \left[ \frac{\partial L}{\partial q_i} \frac{dq_i}{d\varepsilon_i} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{d\varepsilon_i} \right] = \int_{t_i}^{t_f} dt \left[ \frac{\partial L}{\partial q_i} \eta_i + \frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i \right]. \]  
(13.5)

For the action to be extremum we require that (i.e., the \( q_i \) in Eq. (13.3) define the extremum)

\[ \left. \frac{dS}{d\varepsilon_i} \right|_{\varepsilon_i=0} = 0. \]  
(13.6)

Thus, returning to the previous equation and integrating the last terms by parts, we require that

\[ 0 = \int_{t_i}^{t_f} dt \left[ \frac{\partial L}{\partial q_i} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \eta_i + \left. \frac{\partial L}{\partial \dot{q}_i} \eta_i \right|_{t_i}^{t_f}, \]  
(13.7)

where the last term vanishes due to the boundary conditions on \( \eta(t) \) in Eq. (13.4).

Finally the Principle of Least Action tells us that this expression should be true for any value of the index \( i \) and for any function \( \eta(t) \) that satisfies the boundary conditions. This latter constraint says that the content of the square brackets must vanish at any \( t \) (think about replacing \( \eta(t) \) with a delta function at any \( t \)). Thus for any \( i \) we have the Euler-Lagrange Equation(s)

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \]  
(13.8)

As a simple example, consider a single, nonrelativistic particle in a potential \( V \),

\[ q = \bar{x}, \ T(\dot{q}) = \frac{1}{2} m \dot{\bar{x}}^2, \ V = V(\bar{x}), \ L = \frac{1}{2} m \dot{x}^2 - V(\bar{x}). \]  
(13.9)
Noting that
\[
\frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \quad \frac{\partial L}{\partial x} = \nabla V,
\] (13.10)
the Euler-Lagrange Equation becomes Newton’s Equation
\[
m\frac{d^2x}{dt^2} = -\nabla V = \vec{F}.
\] (13.11)

For our purposes we want to generalize from particles to fields (still in the classical limit) that are defined at each point in space-time. Thus the Lagrangian becomes a spatial integral of a density
\[
L = \int d^3x \mathcal{L}
\] (13.12)
with the action defined in 4-D fashion
\[
S = \int d^4x \mathcal{L},
\] (13.13)
where both \(S\) and \(\mathcal{L}\) are Lorentz scalars. Since the “engineering” dimension of the action must be zero, \(\text{i.e.},\) so it can be the argument of an exponential, the dimension of the Lagrangian density, \(\mathcal{L}\), must be length\(^{-4}\) or energy\(^{+4}\) \([[\mathcal{L}] = E^4]).

The Lagrangian density is a function of fields and their derivatives, \(\text{i.e.},\) we substitute above with
\[
q \rightarrow \varphi(x^\nu), \quad \dot{q} \rightarrow \partial_\mu \varphi(x^\nu) = \frac{\partial \varphi(x^\nu)}{\partial x^\mu},
\] (13.14)
and find the new form of the Euler-Lagrange equation to be
\[
\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \varphi(x) \right)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi(x)} = 0
\] (13.15)
for all values of \(x\). [The constraint on the allowed variations of the fields is that they must vanish at the 4-surface defining the boundary of the integral for the action, which may be at infinity.] If the Lagrangian depends on more than one field, there will one such equation for each field.
A simple example is a free scalar field of mass $m$. The corresponding Lagrangian density is
\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \varphi \right) \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2, \tag{13.16}
\]
where we interpret the first terms as the relativistic analog of the kinematic energy and the second term as the potential for a free, massive field. From the discussion of dimensions above we note that the engineering dimension of such a scalar field must be $E^1$ or $L^{-1}$ ($[\varphi] = E^1$). Applying the Euler-Lagrange expression in Eq. (13.15) we obtain the expected Klein-Gordon Equation as the equation of motion for such a scalar field,
\[
\partial_\mu \partial^{\mu} \varphi - m^2 \varphi = 0, \tag{13.17}
\]
with $\partial_\mu \partial^{\mu} = \partial_t^2 - \nabla^2$.

To move to the quantum version of this theory the first step is to make the usual quantum mechanical identification of operators (and take $\hbar = 1$ instead of $\hbar = 0$). We have
\[
p^\mu = (E, \vec{p}) = i \partial^{\mu} = i \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right) \tag{13.18}
\]
(note the sign) and $\partial_\mu \partial^{\mu} = -p^\mu p_\mu$, while the Euler-Lagrange Equation becomes the “mass shell” constraint
\[
(m^2 - p^2) \varphi = 0. \tag{13.19}
\]

Note that this Lagrangian has a discrete symmetry, i.e., the action and $\mathcal{L}$ are invariant under the transformation $\varphi \to -\varphi$. It is generally more interesting to consider a continuous symmetry. Imagine that the field $\varphi$ depends on some continuous parameter $\alpha$ (with no spatial dependence) and that the Lagrangian is invariant under changes in $\alpha$. Then we can consider variations parameterized by changes in $\alpha$,
\[
d \varphi = \frac{\partial \varphi}{\partial \alpha} d \alpha, \quad d \left( \partial_\mu \varphi \right) = \frac{\partial}{\partial \alpha} \left( \partial_\mu \varphi \right) d \alpha = \partial_\mu \left( \frac{\partial \varphi}{\partial \alpha} \right) d \alpha. \tag{13.20}
\]
Since the Lagrangian is invariant, we can use these expressions to write

$$d L = 0 = \frac{\partial L}{\partial \phi} \frac{\partial L}{\partial \alpha} d \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\mu \left( \frac{\partial \phi}{\partial \alpha} \right) d \alpha. \quad (13.21)$$

This result must be true for any $d \alpha$ so that the coefficient of $d \alpha$ must vanish. Finally, we can use the Euler-Lagrange equation to rewrite the first term and find

$$\partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \frac{\partial \phi}{\partial \alpha} + \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\mu \left( \frac{\partial \phi}{\partial \alpha} \right) = \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \frac{\partial \phi}{\partial \alpha} \right) = 0. \quad (13.22)$$

Thus the quantity in the final bracket can be interpreted as a conserved current,

$$J^\mu \equiv \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \frac{\partial \phi}{\partial \alpha}, \right), \partial_\mu J^\mu = 0. \quad (13.23)$$

If the Lagrangian depends on more than one field, each of which depends on $\alpha$, the current defined in this way will be a sum of terms, one for each such field. The German mathematician Emmy Noether (1882-1935) first derived this connection between a continuous symmetry of the Lagrangian and the existence of a conserved current in 1918 and such currents are often called Noether currents. She was the daughter of a professor of mathematics and chose to pursue a career in mathematics at a time in Germany when women typically were not allowed to study at universities nor to join their faculties.

Recall from your studies of currents in classical electromagnetism or in classical mechanics that a 4-D current can be written as

$$J^\mu = (\rho, \vec{J}), \quad (13.24)$$

where the $0^{th}$ component is the spatial density of whatever “charge” is flowing in the current, while the 3-vector part is the flux of whatever is flowing ($\vec{J} \sim \rho \vec{v}$). The
conservation (or continuity) equation says that the time rate of change of the density, \( \frac{\partial \rho}{\partial t} \), is the flux into the region of interest, \( -\nabla \cdot \mathbf{J} \), or

\[
\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.
\]

(13.25)

It may be helpful to think about the charge \( Q \) inside of a closed surface \( S \), defining a volume \( V \),

\[
\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t} \iiint_V d^3x \rho = -\iiint_V d^3x \nabla \cdot \mathbf{J} = -\iiint_S d\sigma \cdot \mathbf{J},
\]

(13.26)

i.e., the charge inside changes as minus the charge flowing out through the surface.

Consider next the simplest nontrivial example of the above structure, which is provided by a complex scalar field (so that the charge conjugation operator can yield a nontrivial result, i.e., the field can have nontrivial quantum numbers). The Lagrangian (density) is

\[
\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi = \left| \partial_\mu \phi \right|^2 - m^2 |\phi|^2,
\]

(13.27)

where there are now two degrees of freedom, \( \phi \) and \( \phi^* \) (or \( \text{Re} \, \phi \) and \( \text{Im} \, \phi \)). Both fields satisfy the Klein-Gordon equation as before, i.e., the particle and antiparticle have the same mass. This Lagrangian has a continuous \( U(1) \) symmetry, i.e., invariance under a change in the phase of the fields (with \( \alpha \) a real, continuous parameter)

\[
\phi \rightarrow e^{i\alpha} \phi, \quad \phi^* \rightarrow e^{-i\alpha} \phi^*,
\]

(13.28)

which we can think of as simply a “rotation” of the 2-component vector (\( \text{Re} \, \phi \), \( \text{Im} \, \phi \)) where the physics depends only on the “length” of the vector \( |\phi|^2 \). The corresponding Noether current is easily derived,

\[
J^\mu = i\left[ (\partial_\mu \phi^*) \phi - \phi^* \left( \partial_\mu \phi \right) \right].
\]

(13.29)

When this \( U(1), \) i.e., phase invariance, is identified with electromagnetism (see the next lecture), the current will contain a multiplicative factor of the electric charge of the field.
Here we continue with the discussion of classical E&M (see the textbook by J. D. Jackson). This will allow us to “naturally” introduce vector fields. Recall that (in Heavyside-Lorentz notation with no magnetic monopoles) Maxwell’s equations look like

\[
\nabla \cdot \mathbf{E} = \rho, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{13.30}
\]

The 6 degrees of freedom represented by the 3-vector electric and magnetic fields are most simply represented in terms of the 4-vector field (or potential) \( A^\mu = (\phi, \vec{A}) \) with engineering dimension L\(^{-1}\) or E\(^1\), where

\[
\tilde{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi, \quad \tilde{B} = \nabla \times \vec{A}. \tag{13.31}
\]

The form of the expression for \( \tilde{B} \) (a cross product) reminds us that, while the electric field is an ordinary vector (odd under parity transformations), the magnetic field is a pseudovector (even under parity transformations).

We next note that the last two of Maxwell’s equations (the homogeneous ones) are satisfied automatically by the (new) definitions of these 3-vectors, while the inhomogeneous equations can be easily expressed in terms of a 4-curl, the field-strength tensor, defined by

\[
F^{\mu \nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{13.32}
\]

This antisymmetric 4-tensor has 6 degrees of freedom that are just the components of the electric and magnetic fields (with our choice of metric)

\[
F^{\mu \nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}. \tag{13.33}
\]

The inhomogeneous Maxwell equations are now contained in a single 4-vector equation

\[
\nabla \cdot \tilde{F} = 4\pi \rho, \quad \nabla \times \tilde{F} = 4\pi \mathbf{J}. \tag{13.34}
\]
\[ \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \left( \partial_\mu A^\mu \right) = J^\nu. \] (13.34)

After a little thought we recognize that this equation is the Euler-Lagrange Equation, with respect to the 4-vector field \( A^\nu \), arising from the following Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\nu J^\nu. \] (13.35)

We can interpret the first term as the “kinetic energy” for a vector field and the second term as describing the coupling of that field to an external current \( J^\nu \).

Related to \( F^{\mu\nu} \) is a second tensor, “\( F \) Dual”, defined by
\[ \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu
u\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \] (13.36)

In terms of this new tensor the homogeneous Maxwell equations take the form
\[ \partial_\mu \tilde{F}^{\mu\nu} = 0. \] (13.37)

You may recall from your studies of classical E&M that the above equations do not fully specify the vector field. The “physical fields”, \textit{i.e.}, the electric and magnetic fields or the field-strength tensor \( F^{\mu\nu} \), are invariant under the following “local gauge transformation” depending on the scalar function \( \lambda(x^\mu) \)
\[ \phi \rightarrow \phi - \frac{\partial \lambda}{\partial t} \Rightarrow A^\mu \rightarrow A^\mu - \partial^\mu \lambda. \] (13.38)

Thus, without changing the physics, we are free to apply another condition on \( A^\mu \), the “gauge condition”. As usual with such freedom, the specific “choice of gauge” will be made based on making the problem at hand as simple as possible. An example is the “Lorentz gauge” where we require

\[ \tilde{A} \rightarrow \tilde{A} + \nabla \lambda \]
\[ \partial_\mu A^\mu = 0, \quad (13.39) \]

\( i.e., A^\mu \) is polarized orthogonal to the 4-direction of its variation. In this case the Euler-Lagrange Equation for the vector field has the simple form

\[ \partial_\mu F^{\mu\nu} = \partial_\nu \partial^\mu A^\nu = J^\nu, \quad (13.40) \]

or, in a region with no sources,

\[ \partial_\mu \partial^\mu A^\nu = 0. \quad (13.41) \]

This tells us that the vector field (the photon) satisfies a massless Klein-Gordon equation. The momentum, \( q^\mu \), of a freely propagating photon must satisfy \( q_\mu q^\mu = 0 \) (the photon is massless). In the free case we can easily write down the explicit form as a plane wave with a polarization vector

\[ A^\nu = \varepsilon^{\nu} (q) e^{-iq \cdot x}. \quad (13.42) \]

The equation of motion (13.41) requires that \( q^\mu \) is a “light-like” vector, \( q_\mu q^\mu = 0 \) (it is precisely light after all), while the Lorentz gauge condition requires

\[ q_\nu \varepsilon^{\nu} (q) = 0, \quad (13.43) \]

\( i.e., \) the wave is transversely polarized, as a 4-vector, and seems to have 3 degrees of freedom. But we are not done yet! Recall that we defined the gauge transformation in terms of the function \( \lambda(x) \) in order to get to the Lorentz gauge. But this constraint does not fully specify \( \lambda(x) \). In particular, we can add to \( \lambda(x) \) any scalar function \( \Lambda(x) \), which satisfies the free, massless wave equation

\[ \partial_\mu \partial^\mu \Lambda = 0, \quad (13.44) \]

and still satisfy the gauge condition,

\[ \partial_\nu A^\nu = 0 \Rightarrow \partial_\nu \left( A^\nu - \partial^\nu \Lambda \right) = 0. \quad (13.45) \]
If \( \Lambda(x) \) is a function of the scalar variable \( x, q^\mu \), \( \partial^\nu \Lambda \) will be proportional to \( q^\nu \) and \( \partial_\mu \partial^\nu \Lambda \) will be proportional to \( q^2 = 0 \). In particular, for a plane wave form we have

\[
\Lambda(x \cdot q) = ce^{-iq \cdot x},
A^\nu \rightarrow (\epsilon^\nu + icq^\nu) e^{-iq \cdot x}.
\] (13.46)

Thus we can always choose the coefficient \( c \) so as to cancel the 0\textsuperscript{th} component of \( A^\nu \), i.e.,

\[
A^\nu \rightarrow A^\nu - \partial^\nu \Lambda,
\epsilon^\nu e^{-ix \cdot q} \rightarrow \tilde{\epsilon}^\nu e^{-ix \cdot q}, \tilde{\epsilon}^\nu \cdot \tilde{q} = 0.
\] (13.47)

Explicitly we choose \( c = i\epsilon^0/q^0 \) and find \( \tilde{\epsilon}^\nu = \epsilon^\nu - \tilde{q}(\tilde{\epsilon} \cdot \tilde{q} / |\tilde{q}|^2) \). Thus we are always free to select a gauge in which the polarization of the vector field is transverse (as a 3-vector) to its direction of motion in 3-space. Thus it must be the case that there are only 2 degrees of freedom for a freely propagating (on-shell) photon. If the photon (or classical E&M plane wave) is moving in the \( z \) direction, these degrees of freedom correspond to linear polarization in either the \( x \) or \( y \) directions. As we have discussed earlier, it is sometimes more illuminating to use the circular polarization or helicity basis

\[
\epsilon_\pm = \frac{\hat{\epsilon}_x \pm i\hat{\epsilon}_y}{\sqrt{2}},
\] (13.48)

which are also called right-handed (positive helicity) and left-handed (negative helicity).

We now want to contrast the U(1) symmetric case above to the case of an explicitly massive vector field \( A^\mu \), where there is no gauge invariance and the Lagrangian has the form (here \( F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \))

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\nu A_\nu.
\] (13.49)

Note that the mass term is \textit{not} invariant under a gauge transformation defined in Eq. (13.38). We can, however, still determine the equation of motion, \( i.e., \) the Euler-Lagrange Equation, as before to find

\[
\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0.
\] (13.50)
If we operate on this equation with $\partial_v$ and recall that $F^{\mu\nu}$ is antisymmetric, we have

$$\partial_v \partial_\mu F^{\mu\nu} + m^2 \partial_v A^\nu = m^2 \partial_v A^\nu = 0.$$  \hspace{1cm} (13.51)

Thus, as long as $m \neq 0$, the massive vector field must satisfy

$$\partial_v A^\nu = 0.$$  \hspace{1cm} (13.52)

This looks like the Lorentz gauge condition above, but it arises in quite a different fashion. Above we had gauge invariance and no mass, while here we have a mass and no gauge invariance. It is still true that this equation means that the massive vector field has only 3 degrees of freedom. However, there is no further gauge symmetry to reduce that number to 2. A massive vector field can be represented by 3 linear polarizations, $x, y, z$ (2 transverse and 1 longitudinal polarization) or by $+, -$ and 0 helicities. For motion in the $z$ direction we have (where the symbol $\lambda$ now represents the helicity)

$$\varepsilon^{\mu} (\lambda = \pm) = \frac{(0, \pm i, 0)}{\sqrt{2}},$$

$$\varepsilon^{\mu} (\lambda = 0) = \frac{\left(|\bar{q}| = q_z, 0, 0, E = \sqrt{q^2 + m^2}\right)}{m}.$$  \hspace{1cm} (13.53)

These forms have the required properties

$$q_\mu \varepsilon^{\mu} = 0, \sum_\lambda \varepsilon^{\mu*} (\lambda) \varepsilon^{\nu} (\lambda) = -g^{\mu\nu} + \frac{q_\mu q_\nu}{m^2}.$$  \hspace{1cm} (13.54)

So we have introduced notation for scalar and vector fields. The final type of fields we need are fermions. In the 4-component notation of Dirac,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$  \hspace{1cm} (13.55)
we have the following equation of motion (the Dirac equation) for a free massive fermion (recall Lecture 9)

\[(\not{\partial} - m)\psi \equiv (i\not{\partial} - m)\psi(x) = (i\gamma^\mu \partial_\mu - m\mathbf{1}_{\alpha\beta})\psi_\beta(x) = 0.\] (13.56)

In the last expression the indices have been made explicit (\(\mu = 0,1,2,3\) – the usual Lorentz index; \(\alpha,\beta = 1,2,3,4\) – the spinor index). Recall that the 4x4 Dirac matrices satisfy the anticommutation relations

\[\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\text{Diag}(1,-1,-1,-1).\] (13.57)

This property is required so that a solution of the (linear) Dirac equation is also a solution of the (quadratic) Klein-Gordon equation. There is also a fifth matrix conventionally defined (with some variation in the literature),

\[\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,\] (13.58)

that anticommutes with the others,

\[\{\gamma^5, \gamma^\mu\} = 0.\] (13.59)

The above relations specify that \((\gamma^0)^2 = (\gamma^5)^2 = +1, (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1\) and we generally desire a unitary representation where \(\gamma^{0\dagger} = \gamma^0, \gamma^{k\dagger} = -\gamma^k (k = 1,2,3), \gamma^{5\dagger} = \gamma^5.\) Thus we can write \(\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0.\) In Lecture 9 we displayed the following explicit representation
If we take the Hermitian conjugate of the Dirac Equation, we find

$$(-i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} - m\psi^\dagger = 0). \ (13.60)$$

With the standard definition $\bar{\psi} \equiv \psi^\dagger \gamma^0$ we have

$$-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0. \ (13.62)$$

If we multiply this expression on the right by $\psi$ and the original Dirac equation on the left by $\bar{\psi}$, we obtain

$$(-i\partial_\mu \bar{\psi}\gamma^\mu)\psi - m\bar{\psi}\psi = 0,$$

$$\bar{\psi}(i\partial_\mu \gamma^\mu\psi) - m\bar{\psi}\psi = 0,$$

$$\Rightarrow \partial_\mu (\bar{\psi}\gamma^\mu\psi) = 0,$$

where the first two expressions were subtracted to obtain the last. Thus the vector quantity $\bar{\psi}\gamma^\mu\psi$ represents a conserved current if $\psi$ is a free Dirac field, independent of the size of its mass. Analogously to our earlier discussion of the of the complex scalar field $\phi$, this is precisely the Noether current corresponding to the continuous...
global invariance under the transformation
\[ \psi \rightarrow e^{ia}\psi, \quad \bar{\psi} \rightarrow e^{-ia}\bar{\psi}. \] (13.64)

The conserved charge in this case is just the fermion number carried by \( \psi \). The Lagrangian corresponding to the Dirac Equation in Eq. (13.56) is
\[ \mathcal{L}_{\text{Dirac}} = i\bar{\psi}\gamma^\mu \sigma_{\mu\nu} \partial_\nu \psi, \] (13.65)
where we note that the dimension of the fermion field is \( L^{-1.5}, E^{1.5} \) (\([\psi] = E^{1.5}\)). This Lagrangian is clearly invariant under the U(1) symmetry transformation in Eq. (13.64). Applying the Euler-Lagrange equation for variations with respect to \( \bar{\psi} \) yields the Dirac equation, while varying \( \psi \) yields the equation of motion for \( \bar{\psi} \).

Recall that we define the field for the antiparticle as
\[ \psi^C = C\bar{\psi}^T = C\gamma^0\psi^*. \] (13.66)

The matrix representing the \( C \) operator satisfies
\[ C^{-1}\gamma^\mu C = -\gamma^{\mu T}. \] (13.67)

Thus, for a given operator \( \Gamma \), we can relate the matrix element for antiparticles to that for particles via
\[ \bar{\psi}^C_1 \Gamma \psi^C_2 = -\psi^T_1 (C^{-1}\Gamma C)\psi^T_2 = -\psi^T_1 (\eta\Gamma^T)\psi^T_2 \]
\[ = \eta \bar{\psi}_2 \Gamma \psi_1, \] (13.68)

where we included a minus sign in the last step from commuting the 2 fermions. The phases, \( \eta \), for various operators, are given by the following table.

<table>
<thead>
<tr>
<th>( \Gamma )</th>
<th>1</th>
<th>( \gamma^5 )</th>
<th>( \gamma^\mu )</th>
<th>( \gamma^\mu \gamma^5 )</th>
<th>( \sigma^{\mu\nu} \equiv i \left[ \gamma^\mu, \gamma^\nu \right]/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
Note that the conserved current defined by $\bar{\psi} \gamma^\mu \psi$ in Eq. (13.63) is odd under $C$ $\left[ \bar{\psi} \gamma^\mu \psi = -\bar{\psi}^C \gamma^\mu \psi^C \right]$ and thus acts like an electric current. We will see that it is just that. An explicit representation for $C$ is $C = i \gamma^5 \gamma^0$ and $C^T = C^\dagger = C^{-1} = -C$.

Recall (again from the appendix to Lecture 9) that we can define fermions with specific helicities and handedness. Particularly useful are the eigenstates of $\gamma^5$, which are often called either chiral fermions or Weyl fermions and labeled with the “handed” notation (even though these definitions precisely match the helicity definition of handedness only in the massless case),

$$\gamma^5 \psi_{R,L} = \pm \psi_{R,L}.$$  \hfill (13.69)

As defined earlier, these states are obtained with the projection operators

$$\psi_{R,L} = \frac{1}{2} \left( 1 \pm \gamma^5 \right) \psi,$$
$$\bar{\psi}_{R,L} = \frac{1}{2} \bar{\psi} \left( 1 \mp \gamma^5 \right).$$  \hfill (13.70)

With these projection operators it is easy to verify that

$$\bar{\psi} \psi = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R,$$
$$\bar{\psi} \gamma^\mu \psi = \bar{\psi}_R \gamma^\mu \psi_R + \bar{\psi}_L \gamma^\mu \psi_L,$$  \hfill (13.71)

\textit{i.e.}, in the chiral basis the scalar expression is “off-diagonal” while the vector is “diagonal”. Thus, in order to have a mass term for a Dirac fermion in the Lagrangian, both helicities must be present. A pure chiral state \textit{(i.e., with only $\psi_L$ or $\psi_R$ present in the Lagrangian) must} be massless (in the Dirac language). Since only the mass term in the Lagrangian mixes the two chiral states, the massless case with both chiral states present exhibits chiral symmetry in the sense that we have two \textit{independent} $U(1)$ symmetries

$$U(1)_L : \psi_L \rightarrow e^{i \alpha_L} \psi_L, \bar{\psi}_L \rightarrow e^{-i \alpha_L} \bar{\psi}_L,$$
$$U(1)_R : \psi_R \rightarrow e^{i \alpha_R} \psi_R, \bar{\psi}_R \rightarrow e^{-i \alpha_R} \bar{\psi}_R.$$  \hfill (13.72)

A related but orthogonal concept is the Majorana fermion. This addresses the question of whether a fermion with zero additive quantum numbers can be its own
antiparticle (analogous to the photon). For a Majorana fermion, $\psi_M$, we have

$$\psi_M^C = C\gamma^0 \psi_M^* = \psi_M. \quad (13.73)$$

In contrast to the case of Dirac particles, which have 4-components corresponding to distinct particle and antiparticle states, each with two helicities, the Majorana particle has just 2 components corresponding to the two helicities of the identical particle and antiparticle. Note that this is not the same as a (2-component) Weyl or chiral particle. In the latter case, $C$ relates a right-handed particle to a left-handed antiparticle [recall the table above]

$$\gamma^5 \psi_w = \psi_w \Rightarrow \gamma^5 \psi_w^C = \gamma^5 \left( C\gamma^0 \psi_w^* \right) = C\gamma^{5T} \gamma^0 \psi_w^*$$

$$= -C\gamma^{0T} \gamma^5 \psi_w^* = -C\gamma^{0T} \psi_w^* = -\psi_w^C. \quad (13.74)$$

Thus the particle and antiparticle are distinct and not Majorana-like. Further, it is possible in the Majorana case, unlike the Weyl case, to include a mass term in the Lagrangian (but we will not work out the details here). It remains unclear whether the idea of a Majorana particle plays a role in neutrino physics. An important experiment test is the search for neutrinoless contributions to nuclear double beta-decay, e.g., in $^{82}$Se $\rightarrow$ $^{82}$Kr. In the usual Dirac (chiral weak interaction) case, the underlying process is

$$2n \rightarrow 2p + 2e^- + 2\overline{\nu}_{eR}. \quad (13.75)$$

If the neutrino is a Majorana particle and has a (small) mass (so that $L$ and $R$ mix), the antineutrino emitted in the “first” decay can be absorbed as a (left-handed) neutrino in the “second” decay. In this case we should observe the process

$$2n \rightarrow 2p + 2e^- \quad (13.76)$$

In the next lecture we will relate the above structure of classical E&M to the idea of a local U(1) gauge symmetry and see how a massless vector particle, the photon, is a necessary component.