Appendix: Hadron Masses

In a world with perfect flavor SU(3) symmetry we would expect that each of the representations exhibited in Lecture 10 would be characterized by a single mass, i.e., all of the states within a single representation would be degenerate. While the masses within each multiplet are approximately equal, there is clear evidence of flavor SU(3) breaking. When Gell-Mann first proposed this structure, he also provided a suggestion for the algebraic form of the breaking. [Recall that he was not allowed, by his own rules, to talk about the masses of the quarks themselves.] Motivated by the data, his suggestion was that the primary term in the Lagrangian which breaks flavor SU(3) should have the algebraic structure of the $\lambda_8$ ($= Y$) member of the 8 of SU(3) generators (these idea was referred to as octet dominance). The practical implication of this is much like our study of $I = 3/2$ dominance in the case of the $\Delta$ resonance. Assuming octet dominance we can work out the corresponding SU(3) Clebsch-Gordan coefficients to find the expected form of the masses, including breaking. [There is one important difference from our studies of angular moment and isospin where we considered representations of SU(2) (or SO(3)). Each member of a representation of SU(3) is characterized by 2 numbers rather than just one number, $J$, in SU(2) (the SU(3) representations are 2-dimensional rather than 1-dimensional as in SU(2)).] Also, since fermion mass terms in the Lagrangian appear linearly ($\sim m$), while boson mass terms appear quadratically ($\sim m^2$), Gell-Mann suggested a corresponding form for the breaking. Thus the so-called Gell-Mann - Okubo mass formulae express masses in terms of the isospin $I$ and hypercharge $Y$ and certain arbitrary constants (the reduced matrix elements) to be determined from the data. For the octets and decuplets of interest we have the general forms

$$m_F = M_0 + M_1 Y + M_2 \left( I(I+1) - \frac{Y^2}{4} \right),$$

$$m_B^2 = \tilde{M}_0^2 + \tilde{M}_1^2 Y + \tilde{M}_2^2 \left( I(I+1) - \frac{Y^2}{4} \right).$$ (10.A.1)

In each case the first term gives the SU(3) invariant mass of the specific representation, while the last two terms provide the suggested form of the SU(3) breaking. Note that this formula ignores the more subtle effects of isospin symmetry breaking, i.e., mass variation with $I_z$. Consider first applying this formula to the baryon octet. For the various states the quantum numbers and mass breaking formula yields the following.
Baryon | Y | I(I+1)-Y²/4 | Formula
--- | --- | --- | ---
n,p = N | 1 | ½ | $M_0 + M_1 + \frac{1}{2}M_2$
Σ | 0 | 2 | $M_0 + 2M_2$
Λ | 0 | 0 | $M_0$
Ξ | -1 | ½ | $M_0 - M_1 + \frac{1}{2}M_2$

With 4 relations and 3 unknowns we find one predicted constraint among the 4 masses. Plugging numbers into this relation we find

$$3m_\Delta + m_\Xi = 2(m_N + m_\Xi)$$
$$\Rightarrow 3(1116) + 1193 = 2(939 + 1318)$$
$$\Rightarrow 4541 \approx 4514.$$  

Thus the agreement (all numbers are in MeV) is better than 1 %. For the case of the decuplet, the $\mathbf{10}$, we have

| Baryon | Y | I(I+1)-Y²/4 | Formula
--- | --- | --- | ---
$\Delta$ | 1 | 7/2 | $M_0 + M_1 + \frac{1}{2}M_2$
$\Sigma^*$ | 0 | 2 | $M_0 + 2M_2$
$\Xi^*$ | -1 | 1/2 | $M_0 - M_1 + \frac{1}{2}M_2$
$\Omega$ | -2 | -1 | $M_0 - 2M_1 - M_2$

The content of this table is the “equal spacing rule”, the mass differences between each of the iso-multiplets should be equal,

$$m_\Delta - m_\Sigma^* = m_\Sigma^* - m_\Xi^* = m_\Xi^* - m_\Omega = M_1 + \frac{1}{2}M_2.$$  

On the other hand, observations yield

$$m_\Delta - m_\Sigma^* = 1232 - 1385 = -153,$$
$$m_\Sigma^* - m_\Xi^* = 1385 - 1533 = -148,$$
$$m_\Xi^* - m_\Omega = 1533 - 1672 = -139,$$

so agreement with the prediction is again good to ~ 14 MeV out of total masses of ~ 1400 MeV.
Now consider the octet of pseudoscalar mesons.

<table>
<thead>
<tr>
<th>Meson</th>
<th>Y</th>
<th>I(I+1)-Y^2/4</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\tilde{M}_0^2 + \tilde{M}_1^2 + \frac{1}{2}\tilde{M}_2^2$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
<td>2</td>
<td>$\tilde{M}_0^2 + 2\tilde{M}_2^2$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0</td>
<td>0</td>
<td>$\tilde{M}_0^2$</td>
</tr>
<tr>
<td>$\bar{K}$</td>
<td>-1</td>
<td>$\frac{1}{2}$</td>
<td>$\tilde{M}_0^2 - \tilde{M}_1^2 + \frac{1}{2}\tilde{M}_2^2$</td>
</tr>
</tbody>
</table>

And note that, although $S$ and $B$ are individually different, the $Y$ dependence is the same as the baryon octet. Let us use the same constraint as we found above to solve for the mass of the isoscalar and plug in numbers

$$m^2_\eta = \frac{1}{3} \left[ 2 \left( m^K_0^2 + m^K_1^2 \right) - m^2_\pi \right]$$

$$\Rightarrow 0.299 \approx 0.321,$$

where the units for the experimental data in the last line are GeV$^2$. Thus the agreement here looks more like 10%. For the vector mesons we have the identical table except for the particle names.

<table>
<thead>
<tr>
<th>Meson</th>
<th>Y</th>
<th>I(I+1)-Y^2/4</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^*$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\tilde{M}_0^2 + \tilde{M}_1^2 + \frac{1}{2}\tilde{M}_2^2$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0</td>
<td>2</td>
<td>$\tilde{M}_0^2 + 2\tilde{M}_2^2$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0</td>
<td>0</td>
<td>$\tilde{M}_0^2$</td>
</tr>
<tr>
<td>$\bar{K}^*$</td>
<td>-1</td>
<td>$\frac{1}{2}$</td>
<td>$\tilde{M}_0^2 - \tilde{M}_1^2 + \frac{1}{2}\tilde{M}_2^2$</td>
</tr>
</tbody>
</table>

This time the constraint is

$$m^2_\omega = \frac{1}{3} \left[ 2 \left( m^K_0^2 + m^K_1^2 \right) - m^2_\rho \right]$$

$$\Rightarrow 0.613 \sim 0.859.$$

Here the agreement is quite poor and we should turn to the point of this exercise, mixing between the isoscalar in the octet and the SU(3) singlet. We express the mixing, which is not included in the Gell-Mann Okubo formula, as
\[
\begin{pmatrix}
\Psi_\omega \\
\Psi_\phi
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\Psi_8 \\
\Psi_1
\end{pmatrix}.
\] (10.A.7)

In the basis of the SU(3) eigenstates we expect a “mixed” mass matrix

\[
M^2 = \begin{pmatrix}
    m_{88}^2 & m_{81}^2 \\
    m_{18}^2 & m_{11}^2
\end{pmatrix},
\] (10.A.8)

while in the “physical” basis we expect a diagonal form. We can express this as

\[
M^2 = \begin{pmatrix}
    m_{88}^2 & m_{81}^2 \\
    m_{18}^2 & m_{11}^2
\end{pmatrix}
= \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
    m_\omega^2 & 0 \\
    0 & m_\phi^2
\end{pmatrix}
\begin{pmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{pmatrix}.
\] (10.A.9)

Solving these equations for the mixing, we find

\[
\sin \theta = \sqrt{\frac{m_{88}^2 - m_\omega^2}{m_\phi^2 - m_\omega^2}}.
\] (10.A.10)

We can now interpret our earlier result as an estimate for \(m_{88}\) rather than a constraint on \(m_\omega\). Substituting for the physical masses we find

\[
\sin \theta_{1^-} = \sqrt{\frac{0.859 - (0.783)^2}{(1.020)^2 - (0.783)^2}} = 0.76, \theta_{1^-} \sim 50^\circ,
\] (10.A.11)

as noted earlier. Thus the “real” world for the vector mesons is near ideal mixing.

The same exercise for the pseudoscalar mesons yields

\[
\sin \theta_{0^-} = \sqrt{\frac{m_{88}^2 - m_\eta^2}{m_{\eta'}^2 - m_\eta^2}} = \sqrt{\frac{0.321 - (0.547)^2}{(0.958)^2 - (0.547)^2}} = 0.19, \theta_{0^-} \sim 11^\circ.
\] (10.A.12)
As we discussed earlier, the mixing is much smaller for the pseudoscalar mesons. As we shall see shortly it is actually this small mixing that is the more surprising.

At this point we should be asking how does this structure look in a world where we can talk about quarks as “real” degrees of freedom. Clearly the primary breaking of SU(3) symmetry arises from the fact that the mass of the strange quark is considerably larger than the average of the up and down quarks. Thus we should look for a breaking that is proportional to the strange quark content. This idea immediately gives the “equal spacing” rule for the baryon decuplet as the number of strange quarks changes by 1 as we move from one iso-multiplet to the next. The constraint equation for the baryon octet is also consistent with counting the number of strange quarks. This is perhaps no surprise as the idea of octet dominance said that the breaking goes like the hypercharge, which simply counts the number of strange quarks.

For the meson octets the situation is more interesting as the $I = 1$ mesons have the same hypercharge as the $I = 0$ meson. Again assuming the breaking of $m^2$ goes as the average number of strange quarks times $m_s$, we find ($\mu$ is some overall factor with units of mass and $\hat{m}$ is the average up/down quark mass)

$$m_s^2 = \mu(2\hat{m}),$$
$$m_k^2 = \mu(\hat{m} + m_s),$$
$$m_\eta^2 = \frac{1}{3} \mu(2\hat{m} + 4m_s)$$

$$\Rightarrow 3m_\eta^2 + m_\pi^2 = 4m_k^2. \tag{10.A.13}$$

The last line comes from eliminating the 2 parameters in the previous 3 relations and is just the constraint relationship found by Gell-Mann (without mixing). On the other hand it is natural in this language to expect the true eigenstates to be driven by the strange quark mass, i.e., approach the “ideal” mixing case. Thus, as seen from the prospective of quarks, the situation observed earlier for the vector mesons is more natural than that of the pseudoscalars. We will return to this issue when we study the strong interactions in more detail, i.e., discuss chiral symmetry. In fact, in the context of chiral symmetry the explicit relationships suggested in the expressions above between the pseudoscalar meson masses and the quark masses are correct. The pseudoscalar masses (squared) should vanish (linearly) as the quark masses go to zero. In the limit of zero quark masses the pseudoscalar mesons are the massless Goldstone bosons required when QCD dynamically breaks the underlying chiral symmetry. These symmetry considerations do not apply to the vector bosons.
Another way to address the question of hadronic masses that has proved useful (e.g., see the HW) is to think in terms of essentially static “constituent” quarks rather than “current” quarks used earlier. The nearly equal “constituent” quark masses for the up and down quarks are defined to be 1/3 of the nucleon mass. We can think of the constituent quark mass as the sum of the mass of the current quark plus the energy of an associated “cloud” of virtual gluons and quark-antiquark pairs,

$$m_{\text{constituent}} = m_{\text{current}} + c\Lambda_{\text{QCD}},$$

(10.A.14)

where $\Lambda_{\text{QCD}}$ is the dimensionfull scale of the strong interactions ($\sim 200$ MeV) and $c$ is a constant of order 1. In some sense the constituent quark is the “long” distance object ($d \sim 1$ fm), while the current quark is the short distance object. The naïve picture is that there are 3, largely uncorrelated, such objects in a nucleon. Mesons are composed of two such objects. This simple picture is indicated in the figures below and is remarkably useful (even though we know it must be violated by confinement)! The quarks that carry the quantum numbers of the hadron and a substantial fraction of the hadron’s momentum, i.e., the ones we have been talking about above, are called “valence” quarks, while the quarks in the virtual pairs are called “sea” (or sometimes “ocean”) quarks.

Is there any experimental evidence for this picture? Consider the total cross section for $\pi p$ scattering versus $pp$ scattering. A naïve prediction of the picture above is that the scattering cross section is proportional to the number of constituent quarks, i.e.,

$$\frac{\sigma_{\pi p}}{\sigma_{pp}} \approx \frac{2}{3}.$$  

(10.A.15)

The data (see the PDG summary) yield
\[
\frac{\sigma_{\pi^+ p} + \sigma_{\pi^- p}}{2\sigma_{pp}} = 0.62 \left[ 10 \text{ GeV} < p_{\text{LAB}} < 300 \text{ GeV} \right] \tag{10.A.16}
\]

in remarkable agreement with our naïve prediction.

A second interesting example is the inclusive production of lepton pairs as in

\[
\pi N \rightarrow \gamma^* X \rightarrow \mu^+ \mu^- X, \tag{10.A.17}
\]

where the \(X\) implies that we sum over all possible states. This process is referred to as the Drell-Yan process, having been first defined by S. Drell and T.-M. Yan. It is pictured as occurring through the annihilation of the antiquark in the pion with a quark from the nucleon. As long as the \(Q^2\) of the virtual photon is a substantial fraction of the total \(s\), this annihilation process will involve a valence antiquark from the pion and a valence quark from the nucleon, as suggested by the quark diagram shown here. Using our naïve picture and knowing the specific valence quark content of the pions, we predict

\[
\frac{\sigma_{\pi^+ c \rightarrow \mu^+ \mu^- X}}{\sigma_{\pi^- c \rightarrow \mu^+ \mu^- X}} \approx \frac{Q_d^2}{Q_u^2} = \frac{1}{4}, \tag{10.A.18}
\]

in good agreement with the experimental data. Note that we also used the fact that, for the isoscalar nucleus \(^{12}\text{C}\), the number of \(u\) quarks is equal to the number of \(d\) quarks. We will study the corrections to this naïve picture when we study perturbative application of QCD, the non-Abelian gauge theory.

In the context of the “static” constituent quark picture of QCD we can go further to understand the mass differences of the hadrons, although we will not pursue the details here (except, perhaps, in the HW). The difference in the masses of the baryon \(J = 3/2\) decuplet and the \(J = 1/2\) octet can be largely understood in terms of the point interaction between the color magnetic dipole moments of the individual constituent quarks. This is exactly analogous to the hyperfine splitting of the ground state of the hydrogen atom, which, of course, involves the usual electromagnetic rather than color magnetic dipole moments. In both cases the dipole moments are aligned with the spin and we can relate the strength of the interaction, \(i.e.,\) the mass splitting, to the spin wave function (and the isospin wave function, since the strength of the
individual quark dipole moment depends on the inverse of the quark mass). This is understood to produce a splitting of order 100 MeV. There is also a true electromagnetic dipole-dipole interaction, but this effect is on the scale of 1 MeV. Similar considerations can be used to understand the splitting between the masses of $0^-$ and $1^-$ mesons, although in this picture we are ignoring the underlying issues of chiral symmetry. The finer details that break isospin symmetry arise from the mass difference between the up and down quarks, from the coulomb energy differences arising from “assembling” differently charged hadrons and from the magnetic dipole interaction already mentioned. All of these last effects make mass contributions of order a few MeV.

So far we have not really discussed the bulk of the hadronic mass, which is independent of the symmetry-breaking differences. This contribution is the analog of the parameter $M_0$ used earlier, which we can think of as the energy of the gluon “cloud” in the figures above. This cannot be evaluated by simple means, as it is an intrinsic part of confinement (i.e., difficult nonperturbative physics). However, lattice QCD analyses are beginning to provide reliable results for the masses of the hadrons, including this contribution.