Lecture 24: Solving 2\textsuperscript{nd} Order Partial Differential Equations I - Homogeneous Equations with No Time Dependence (Chapter 13 in Boas)

Our goal in the final Lectures is to review and synthesize what we have learned about solving the 2\textsuperscript{nd} order partial differential equations characteristic of most of physics, including satisfying the associated boundary conditions. As outlined in previous lectures we start with (the simple forms of) Laplace’s equation with no time dependence,

\[ \nabla^2 \Psi = 0, \] \hspace{1cm} (24.1)

which applies to temperatures, potentials (gravitational and electrical), etc., in regions without sources. The specific form of the solution depends on the number of (relevant) dimensions and the geometric structure of the problem’s boundary conditions. We proceed by separating variables in the appropriate number of dimensions and the required geometry. In particular, we want to choose the coordinate system (typically rectangular, cylindrical or spherical) so that the boundaries corresponds to one of the coordinates being constant, e.g., \( z = z_0 \) or \( r = R_0 \). Then we pick a representation of the solution to Laplace’s equation that has complete/orthogonal functions (solutions of a Sturm-Liouville problem) that vary along the boundary. We then use these appropriate special functions to match the boundary conditions. The essential feature of all such problems is that the solution to the differential equation that matches the (fully specified) boundary conditions is unique. Once we find one such solution, by any method, we are done!

1-D: First consider the one-dimensional problem (the trivial case where the boundaries are points), \( \Psi \rightarrow y(x) \) (or \( \Psi \rightarrow x(t) \)),

\[ \frac{d^2}{dx^2} y(x) = 0 \Rightarrow y(x) = a + bx. \] \hspace{1cm} (24.2)

The constants \( a \) and \( b \) are chosen to fit boundary conditions of the form, e.g., \( y(0) = c, \ y'(0) = 0 \), which yields \( a = c, \ b = 0 \).

2-D: More interesting is the 2-D problem, where we can choose either rectangular coordinates, \( x, y \), or polar coordinates, \( \rho, \phi \). Separation of variables in the former case yields \( \Psi \rightarrow X(x)Y(y) \) and Laplace’s equation can be written
\[
\frac{1}{X} \frac{d^2}{dx^2} X(x) = -\frac{1}{Y} \frac{d^2}{dy^2} Y(y) = -k^2. \tag{24.3}
\]

As usual this makes sense only if both sides of the above equation are (the same) constant, written here as \(-k^2\) (where \(k\) has units of 1/distance). From our experience with the relevant special functions we know that the explicit forms of the solutions are products of the form

\[
\Psi \propto e^{\pm ikx} e^{\pm iky} = \begin{pmatrix} \cos kx \\ \sin kx \end{pmatrix} \begin{pmatrix} \cosh ky \\ \sinh ky \end{pmatrix} : [k^2 > 0]
\]

or

\[
\Psi \propto e^{\pm ikx} e^{\pm iky} = \begin{pmatrix} \cosh kx \\ \sinh kx \end{pmatrix} \begin{pmatrix} \cos ky \\ \sin ky \end{pmatrix} : [k^2 < 0]. \tag{24.4}
\]

Note that the behavior in each case is sinusoidal (complete/orthogonal) in one direction and hyperbolic (not complete/orthogonal) in the other. The specific choice depends on the boundary conditions to be matched. As an example consider a 2-D space (rectangle) defined by \(0 \le x \le L_x, 0 \le y \le L_y\). We assume for the moment that the boundary condition is \(\Psi = 0\) except on the boundary at \(y = L_y\), where we require \(\Psi(x, y = L_y) = \Psi_1(x)\), which, as noted, may be a function of \(x\). This problem might describe the temperature distribution in a thin plate with 3 edges held at \(0^\circ\) and the 4th edge held at \(T = \Psi_1\). This last boundary condition demands that we choose the case \(k^2 > 0\) with the sine/cosine functions in the \(x\) coordinate. It is in this coordinate, \(i.e.,\) the coordinate along the boundary with the nontrivial boundary condition, that we find an eigenvalue/eigenvector problem and a complete set of orthogonal functions. This is just where we need them and this feature will remain true in all of our examples.

Overall the logic proceeds as follows. As noted above we choose \(k^2 > 0\) (the upper line in Eq. (24.4)) in order to obtain complete functions in \(x\). The boundary condition at \(y = 0\), \(\Psi(x, 0) = 0\), means that we select the \(\sinh ky\) form (and not \(\cosh ky\)). The boundary condition at \(x = 0\), \(\Psi(0, y) = 0\), means that we select the \(\sin kx\) form (and not \(\cos kx\)). Finally the boundary condition at \(x = L_x\), \(\Psi(L_x, y) = 0\), provides the eigenvalue condition, \(\sin kL_x = 0 \Rightarrow k_n = n\pi/L_x\). Thus, via linear superposition, we
have the general form (i.e., this form satisfies Laplace’s equation and the 3 vanishing boundary conditions at \( x = 0, x = L_x \) and \( y = 0 \))

\[
\Psi(x, y) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L_x} \right) \sinh \left( \frac{n\pi y}{L_x} \right). \tag{24.5}
\]

The final, nontrivial boundary condition says

\[
\Psi(x, L_y) = \Psi_1(x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L_x} \right) \sinh \left( \frac{n\pi L_y}{L_x} \right). \tag{24.6}
\]

Note that, as required on the physics side (and automatically respected by the mathematics side), the arguments of the sines and cosines (both harmonic and hyperbolic, i.e., the special functions of mathematics) are dimensionless quantities even in a “real” physics problem where \( x, y \) and \( L_x, L_y \) all have dimensions of length (meters). As usual the relevant eigenfunctions (here the \( \sin \left( \frac{n\pi x}{L_x} \right) \) functions) are orthogonal,

\[
\int_0^{L_x} dx \sin \left( \frac{m\pi x}{L_x} \right) \sin \left( \frac{n\pi x}{L_x} \right) = \frac{L_x}{2} \delta_{nm}, \tag{24.7}
\]

and complete so that we can fit the boundary conditions on \( y = L_y \). Thus we can solve for the coefficients (at least implicitly) from Eq. (24.6) in the usual way by “projecting” onto the basis functions, i.e., multiply through by the function on both sides of the equation, integrate and use Eq. (24.7) ,

\[
\int_0^{L_x} dx \sin \left( \frac{m\pi x}{L_x} \right) \Psi_1(x) = \sum_{n=1}^{\infty} a_n \sinh \left( \frac{n\pi L_y}{L_x} \right) \frac{L_x}{2} \delta_{nm} = a_m \sinh \left( \frac{m\pi L_y}{L_x} \right) \frac{L_x}{2} \tag{24.8}
\]

\[
\Rightarrow a_m = \frac{1}{\sinh \left( \frac{m\pi L_y}{L_x} \right)} 2 \int_0^{L_x} dx \sin \left( \frac{m\pi x}{L_x} \right) \Psi_1(x).
\]
The resulting solution, expressed as the sum in Eq. (24.5) with the coefficients of Eq. (24.8), vanishes on 3 sides of the square and matches the nonzero boundary condition on the fourth side.

Note that, if the boundary condition is zero on all 4 edges, there is no nontrivial choice of functions (we cannot have sines and cosines for both the \( x \) and \( y \) behavior), which is the right answer, \( i.e., \Psi = 0 \) everywhere. If \( \nabla^2 \Psi = 0 \) is valid everywhere in some region (no sources anywhere) and \( \Psi = 0 \) on the boundary of the region, then \( \Psi = 0 \) everywhere in the region.

Of course, the discussion above corresponds to a special case with just one nonzero boundary condition. In general, the boundary conditions will be nonzero on all four sides, \( e.g., \)

\[
\Psi(x, L_y) = \Psi_1(x), \\
\Psi(x, 0) = \Psi_2(x), \\
\Psi(L_x, y) = \Psi_3(y), \\
\Psi(0, y) = \Psi_4(y). \\
\tag{24.9}
\]

The essential point is that these conditions can be treated as 4 separate problems like the one above with zero boundary conditions on 3 sides and appropriate functions to fit the behavior on the fourth side. Then we use linear superposition to obtain the solution that matches the full set of nonzero boundary conditions. In a shorthand notation the individual functional forms for the 4 separate boundary condition problems are (note the complete, orthogonal functions along the boundary in each case)

\[
\Psi(x, L_y) = \Psi_1(x): \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L_x} \right) \sinh \left( \frac{n\pi y}{L_y} \right), \\
\Psi(x, 0) = \Psi_2(x): \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L_x} \right) \sin \left( \frac{n\pi \{L_y - y\}}{L_x} \right),
\]

\[
\Psi(L_x, y) = \Psi_3(y): \sum_{n=1}^{\infty} c_n \sinh \left( \frac{n\pi x}{L_x} \right) \sin \left( \frac{n\pi y}{L_y} \right), \\
\Psi(0, y) = \Psi_4(y): \sum_{n=1}^{\infty} d_n \sin \left( \frac{n\pi \{L_y - y\}}{L_x} \right) \sinh \left( \frac{n\pi x}{L_x} \right).
\]
\[ \Psi(L_x, y) = \Psi_3(y) \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi y}{L_y} \right) \sinh \left( \frac{n\pi x}{L_x} \right), \]

\[ \Psi(0, y) = \Psi_4(y) \sum_{n=1}^{\infty} d_n \sin \left( \frac{n\pi y}{L_y} \right) \sinh \left( \frac{n\pi (L_x - x)}{L_x} \right). \]

(24.10)

We solve for the individual coefficients, \( a_n, b_n, c_n, d_n \), by matching the 4 separate nonzero boundary conditions yielding expressions like Eq. (24.8),

\[ a_n = \frac{1}{\sinh \left( \frac{n\pi L_y}{L_x} \right)} \frac{2}{L_x} \int_{0}^{L_x} dx \sin \left( \frac{n\pi x}{L_x} \right) \Psi_1(x), \]

\[ b_n = \frac{1}{\sinh \left( \frac{n\pi L_y}{L_x} \right)} \frac{2}{L_x} \int_{0}^{L_x} dx \sin \left( \frac{n\pi x}{L_x} \right) \Psi_2(x), \]

\[ c_n = \frac{1}{\sinh \left( \frac{n\pi L_y}{L_x} \right)} \frac{2}{L_y} \int_{0}^{L_y} dy \sin \left( \frac{n\pi y}{L_y} \right) \Psi_3(y), \]

\[ d_n = \frac{1}{\sinh \left( \frac{n\pi L_y}{L_x} \right)} \frac{2}{L_y} \int_{0}^{L_y} dy \sin \left( \frac{n\pi y}{L_y} \right) \Psi_4(y). \]

(24.11)

Finally we simply sum the 4 series to obtain the full result. On each edge we are summing 3 zeros plus the 1 correct, nonzero behavior, which is just what we want. Since the solution is unique, we are done!

If the boundary conditions are specified in polar form, \( i.e., \) in terms of \( \rho \) and \( \phi \), we should separate in those variables, \( \Psi \rightarrow R(\rho)\Phi(\phi) \). Here we can think of describing the time independent temperature distribution in a thin (so no variation in \( z \)) disk. Laplace’s equation now takes the (separated) form
\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial}{\partial \rho} \Psi \right] + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \Psi = 0
\]

(24.12)

\[
\Rightarrow \frac{\rho}{R} \frac{d}{d\rho} \left[ \rho \frac{d}{d\rho} R(\rho) \right] = -\frac{1}{\Phi} \frac{d^2}{d\phi^2} \Phi(\phi) = m^2.
\]

If we have a complete disk \((0 \leq \phi < 2\pi)\), then the boundary condition is that \(\Psi\) should be periodic in \(\phi\). This results in eigenfunctions that are \(\Phi(\phi) \propto e^{\pm im\phi}\) or \(\Phi(\phi) \propto \sin m\phi, \cos m\phi\) with eigenvalues \(m = \text{integers}\). We expect the nonzero boundary conditions to be specified at a fixed value of \(\rho\), \(\Psi(\rho = \rho_0, \phi) = \Psi_0(\phi)\).

Note that, as promised, the complete set of eigenfunctions, \(e^{\pm im\phi}\), are functions of the coordinate along the boundary, \(i.e., \phi\). The radial equation has solutions of the form \(R(\rho) \propto \rho^\alpha\), which yield (note that this is not Bessel’s equation)

\[
\frac{\rho}{\rho^\alpha} \frac{d}{d\rho} \left[ \rho \frac{d}{d\rho} \rho^\alpha \right] = \alpha^2 = m^2 \Rightarrow \alpha = \pm m.
\]

(24.13)

As we have now come to expect, one solution is well behaved at \(\rho = 0\) but poorly behaved as \(\rho \to \infty\), while the other solution has the opposite behavior. In the special case \(m = 0\), where the obvious regular (at the origin) solution is \(R = \text{constant}\), the irregular solution is \(R(\rho) \propto \ln \rho\) (the reader should verify that this is a solution of the above equation when \(m = 0\)). For our consideration of the interior of a disk (including the origin) we choose the regular solutions. We now have a typical Fourier series expansion \((T = 2\pi)\)

\[
\Psi(\rho, \phi) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \rho^m \cos m\phi + \sum_{m=1}^{\infty} b_m \rho^m \sin m\phi,
\]

(24.14)

with the coefficients matched to the boundary condition at \(\rho = \rho_0\)
\[ \Psi(\rho, \phi) = \Psi_0(\phi) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \rho_0^m \cos m\phi + \sum_{m=1}^{\infty} b_m \rho_0^m \sin m\phi \]

\[ a_m = \frac{1}{\pi \rho_0^m} \int_0^{2\pi} d\phi \cos (m\phi) \Psi_0(\phi), \]  
\[ b_m = \frac{1}{\pi \rho_0^m} \int_0^{2\pi} d\phi \sin (m\phi) \Psi_0(\phi). \]  
\[ \text{(24.15)} \]

The factors of \( \rho_0^{-m} \) in the coefficients, which arise automatically in the mathematics, serve to remind us that a) in a physics problem \( \rho \) and \( \rho_0 \) will generally both have dimensions of length (meters) and b) we cannot sum up terms with differing dimensions, i.e., the real expansion parameter must be the dimensionless ratio \( \rho/\rho_0 \) (we must divide \( \rho \) by some dimensionful quantity and \( \rho_0 \) is the only choice),

\[ \Psi(\rho, \phi) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \overline{a}_m \left( \frac{\rho}{\rho_0} \right)^m \cos m\phi + \sum_{m=1}^{\infty} \overline{b}_m \left( \frac{\rho}{\rho_0} \right)^m \sin m\phi \]

\[ \Psi(\rho_0, \phi) = \Psi_0(\phi) = \frac{\overline{a}_0}{2} + \sum_{m=1}^{\infty} \overline{a}_m \cos m\phi + \sum_{m=1}^{\infty} \overline{b}_m \sin m\phi \]  
\[ \overline{a}_m = \frac{1}{\pi} \int_0^{2\pi} d\phi \cos (m\phi) \Psi_0(\phi), \]  
\[ \overline{b}_m = \frac{1}{\pi} \int_0^{2\pi} d\phi \sin (m\phi) \Psi_0(\phi). \]  
\[ \text{(24.16)} \]

Note that the temperature at the center of the disk can be nonzero, \( a_0 \neq 0 \), only if the boundary condition at \( \rho = \rho_0 \) has a nonzero average value, \[ \int_0^{2\pi} d\phi \Psi_0(\phi) \neq 0. \]

The reader is encouraged to consider how the form of the solution changes if the disk becomes an annulus (not including the origin). Now there are two boundaries in \( \rho \) and we must keep both solutions for \( R(\rho) \), \( \rho^m \) and \( \rho^{-m} \).

What if we consider a wedge (a piece of the pie), \( 0 \leq \phi \leq \phi_0 < 2\pi \), instead of a full disk? In this case there are three boundaries, \( \{ \rho = \rho_0, 0 \leq \phi \leq \phi_0 \} \), \( \{ 0 \leq \rho \leq \rho_0, \phi = 0 \} \)
and \( \{0 \leq \rho \leq \rho_0, \phi = \phi_0\} \). As with the rectangle above we treat this situation as three separate problems with a nonzero boundary condition on only one boundary at a time and then use linear superposition. The motivated student is encouraged to consider which sets of complete, orthogonal functions are required for each configuration. An example is exercise 13.5.13 in Boas where \( \phi_0 = \pi/4 \) and only the boundary \( \{\rho = \rho_0, 0 \leq \phi \leq \phi_0\} \) has a nonzero value. In this case we need the complete set of angular function obeying \( f(\phi = 0) = f(\phi = \pi/4) = 0 \). By inspection these are just the “even” sine functions, \( f_n(\phi) = \sin(n \phi) \).

3-D: Finally we can consider the case for structure in all 3 spatial dimensions. If the boundary conditions are specified in rectangular coordinates, we consider \( \Psi \rightarrow X(x)Y(y)Z(z) \) and Laplace’s equation looks like

\[
\frac{1}{X} \frac{d^2}{dx^2}X(x) + \frac{1}{Y} \frac{d^2}{dy^2}Y(y) + \frac{1}{Z} \frac{d^2}{dz^2}Z(z) = 0. \tag{24.17}
\]

As before, we solve this equation by assuming that each term is a constant, say \( k_x^2, k_y^2, k_z^2 \), and that the sum of the constants is equal to zero, \( k_x^2 + k_y^2 + k_z^2 = 0 \).

Typically two of the constants are negative corresponding to sinusoidal (complete/orthogonal) behavior in 2 dimensions, while the third is positive corresponding to hyperbolic behavior. As in the 2-D case we think of the problem with boundary conditions specified on each of the 6 sides of the cube \( 0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq L_z \),

\[
\Psi(x, y, L_z) = \Psi_1(x, y),
\Psi(x, y, 0) = \Psi_2(x, y),
\Psi(L_x, y, z) = \Psi_3(y, z),
\Psi(0, y, z) = \Psi_4(y, z),
\Psi(x, L_y, z) = \Psi_5(x, z),
\Psi(x, 0, z) = \Psi_6(x, z). \tag{24.18}
\]

We treat each case as a separate problem with zero boundary conditions on 5 sides and the specified function on the sixth side. For example, for the first case we take
\[ k_x^2 = -(m \pi / L_x)^2 < 0, \quad k_y^2 = -(n \pi / L_y)^2 < 0 \] (sinusoidal behavior in \( x \) and \( y \)) and
\[ k_z^2 = -k_x^2 - k_y^2 = \left( m \pi / L_x \right)^2 + \left( n \pi / L_y \right)^2 \] (hyperbolic in the \( z \) direction). Similarly to the 2-D example above we take the \( \sinh k_z \) function to match the vanishing boundary condition at \( z = 0 \). We choose the sine functions in \( x \) and \( y \) to match the vanishing boundary conditions at \( x = 0 \) and \( y = 0 \). Finally the vanishing boundary conditions at \( x = L_x \) and \( y = L_y \) provide the eigenvalue constraints as above. In the same shorthand as in Eq. (24.10) above, the forms of the 6 solutions look like

\[
\Psi(x, y, L_z) = \Psi_1(x, y) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin \left( \frac{m \pi x}{L_x} \right) \sin \left( \frac{n \pi y}{L_y} \right) \sinh \left( z \sqrt{\left( \frac{m \pi}{L_x} \right)^2 + \left( \frac{n \pi}{L_y} \right)^2} \right),
\]

\[
\Psi(x, y, 0) = \Psi_2(x, y) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} \sin \left( \frac{m \pi x}{L_x} \right) \sin \left( \frac{n \pi y}{L_y} \right) \sin \left( [L_z - z] \sqrt{\left( \frac{m \pi}{L_x} \right)^2 + \left( \frac{n \pi}{L_y} \right)^2} \right),
\]

\[
\Psi(L_x, y, z) = \Psi_3(y, z) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m,n} \sin \left( \frac{m \pi y}{L_y} \right) \sin \left( \frac{n \pi z}{L_z} \right) \sinh \left( x \sqrt{\left( \frac{m \pi}{L_y} \right)^2 + \left( \frac{n \pi}{L_z} \right)^2} \right),
\]

\[
\Psi(0, y, z) = \Psi_4(y, z) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_{m,n} \sin \left( \frac{m \pi y}{L_y} \right) \sin \left( \frac{n \pi z}{L_z} \right) \sin \left( [L_x - x] \sqrt{\left( \frac{m \pi}{L_y} \right)^2 + \left( \frac{n \pi}{L_z} \right)^2} \right),
\]

\[
\Psi(x, L_y, z) = \Psi_5(x, z) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{m,n} \sin \left( \frac{m \pi x}{L_x} \right) \sin \left( \frac{n \pi z}{L_z} \right) \sinh \left( y \sqrt{\left( \frac{m \pi}{L_x} \right)^2 + \left( \frac{n \pi}{L_z} \right)^2} \right),
\]

\[
\Psi(x, 0, z) = \Psi_6(x, z) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{m,n} \sin \left( \frac{m \pi x}{L_x} \right) \sin \left( \frac{n \pi z}{L_z} \right) \sin \left( [L_y - y] \sqrt{\left( \frac{m \pi}{L_x} \right)^2 + \left( \frac{n \pi}{L_z} \right)^2} \right).
\]

Expanding each of the boundary distributions \( \Psi_j \) in the appropriate double Fourier expansion yields the 6 sets of coefficients. For example, we have
\[
a_{m,n} = \frac{4}{\sinh \left( L_z \sqrt{\left( \frac{m\pi}{L_x} \right)^2 + \left( \frac{n\pi}{L_y} \right)^2} \right)}
\]

(24.19)

\[
\times \int_0^L dx \int_0^{L_y} dy \sin \left( \frac{m\pi x}{L_x} \right) \sin \left( \frac{n\pi y}{L_y} \right) \Psi_1(x, y).
\]

After finding each of the 6 sets of coefficients, we simply sum the resulting 6 series to find the overall solution. A specific example where only one of the sides of the cube has a non-zero boundary condition is exercise 13.5.9 in Boas. Here \( L_x = L_y = L_z = 10 \) and \( \Psi_2 = 100^\circ \) with all the other \( \Psi_k \)'s (and corresponding coefficients) vanishing. We have

\[
\Psi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} \sin \left( \frac{m\pi x}{10} \right) \sin \left( \frac{n\pi y}{10} \right) \sinh \left( \frac{10 - z}{10} \right) \sqrt{m^2 + n^2},
\]

(24.20)

\[
\Psi(0, y, z) = \Psi(10, y, z) = \Psi(x, 0, z) = \Psi(x, 10, z) = \Psi(x, y, 10) = 0,
\]

\[
\Psi(x, y, 0) = 100^\circ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} \sin \left( \frac{m\pi x}{10} \right) \sin \left( \frac{n\pi y}{10} \right) \sinh \left( \pi \sqrt{m^2 + n^2} \right).
\]

Using the orthogonality of the sine functions,

\[
\int_0^{10} \int_0^{10} dx dy \sin \left( \frac{m\pi x}{10} \right) \sin \left( \frac{m'\pi x}{10} \right) \sin \left( \frac{n\pi y}{10} \right) \sin \left( \frac{n'\pi y}{10} \right) = \left( \frac{10}{2} \right)^2 \delta_{m,m'} \delta_{n,n'},
\]

(24.21)

we have
\[
\begin{align*}
    b_{m,n} \left( \frac{10}{2} \right)^2 \sinh \left( \pi \sqrt{m^2 + n^2} \right) &= 100 \int_0^{10} \int_0^{10} dx \, dy \sin \left( \frac{m \pi x}{10} \right) \sin \left( \frac{n \pi y}{10} \right) \\
    &= 100 \left( \frac{20}{\pi} \right)^2 \left[ \begin{array}{ll}
        m \text{ and } n \text{ odd} \\
        0 \text{ otherwise}
    \end{array} \right] \\
    \Rightarrow b_{m,n} &= \frac{100 \left( \frac{20}{\pi} \right)^2}{\sinh \left( \pi \sqrt{m^2 + n^2} \right)} \left( \frac{4}{\pi} \right)^2 \left[ \begin{array}{ll}
        m \text{ and } n \text{ odd} \\
        0 \text{ otherwise}
    \end{array} \right].
\end{align*}
\]

Thus,

\[
\Psi(x, y, z) = 100 \left( \frac{4}{\pi} \right)^2 \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin \left( \frac{(2m+1) \pi x}{10} \right) \sin \left( \frac{(2n+1) \pi y}{10} \right) \sinh \left( \frac{10 - z}{10} \pi \sqrt{[2m+1]^2 + [2n+1]^2} \right)}{2m+1 \cdot 2n+1 \cdot \sinh \left( \pi \sqrt{[2m+1]^2 + [2n+1]^2} \right)}.
\]

If the boundary conditions are specified on cylindrical surfaces, then we should separate variables in cylindrical coordinates, \( \rho, \phi, z \). We think in terms of a (finite) cylinder as composed of 3 surfaces, the curved surface \( \rho = \rho_0, 0 \leq \phi < 2\pi, 0 \leq z \leq L_z \), and 2 ends, \( 0 \leq \rho \leq \rho_0, 0 \leq \phi < 2\pi, z = 0 \) and \( 0 \leq \rho \leq \rho_0, 0 \leq \phi < 2\pi, z = L_z \). As usual we split the problem up and treat only one non-zero boundary condition at a time. On the curved surface we have the 2-D orthogonal set of functions, \( \sin m \phi \) and \( \cos m \phi \), which can be used to match any behavior on that surface but vanishing on the ends of the cylinder \( z = 0, z = L_z \). The corresponding radial dependence is given by \( J_m \left( n \pi \rho / L_z \right) \), \textit{i.e.}, the separation constant \( k \) used earlier in our study of cylindrical coordinates has the value \( k_n = n \pi / L_z \) arising from the eigenvalue problem of the \( z \) dependence (vanishing at the ends). The form of the solution used to match the 2-D boundary conditions (in \( \rho \) and \( \phi \)) on the ends of the cylinder was discussed in Lecture 23. In the shorthand notation used earlier to specify the form of the solution for each
In each case the solution vanishes on 2 of the 3 surfaces and provides complete, orthogonal functions, in 2 dimensions, on the third surface. These orthogonal functions are in turn specified by two sets of eigenvalues. This allows us to satisfy any set of boundary conditions on the 3 surfaces and provides the full solution via linear superposition.

Finally consider the case of spherical boundary conditions as described in Lecture 22. If the boundary conditions are given on the surfaces of complete spheres, then we know the solutions have the form (again in our shorthand notation)

$$\Psi (\rho, \phi, L_z) = \Psi_3 (\rho, \phi) : \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{m,n} \cos (m\phi) \sin \left( \frac{n\pi z}{L_z} \right) J_m \left( \frac{n\pi \rho}{L_z} \right)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_{m,n} \sin (m\phi) \sinh \left( \frac{x_{m,n} L_z - z}{\rho_0} \right) J_m \left( \frac{x_{m,n} \rho_0}{\rho_0} \right),$$

$$\Psi (\rho, \phi, L_z) = \Psi_3 (\rho, \phi) : \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} f_{m,n} \cos (m\phi) \sin \left( \frac{x_{m,n} \rho_0}{\rho_0} \right) J_m \left( \frac{x_{m,n} \rho_0}{\rho_0} \right)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{m,n} \sin (m\phi) \sinh \left( \frac{x_{m,n} \rho_0}{\rho_0} \right) J_m \left( \frac{x_{m,n} \rho_0}{\rho_0} \right),$$

where the spherical harmonics, $Y_{l,m} (\cos \theta, \phi)$, provide complete, orthonormal functions on the surface of a sphere. If the origin is included in the physical region, then we know that all $d_{l,m} = 0$. The reader is encouraged to consider how the solutions might look if we have only a fraction of a sphere, i.e., when boundary conditions are specified on the section of a sphere.
In the next Lecture we will consider what changes when we include time dependence (the Helmholtz equation) and sources (Poisson’s equation).