Lecture 23: Frobenius and Bessel (More of Chapter 12 in Boas)

In our previous discussions we have focused on the case where we solved a differential equation via a Taylor series expansion about a regular point of the equation, typically the origin. Now we want to consider the case where we expand about the origin when it is a regular singular point. The general technique, due to Frobenius (see 12.11 in Boas), essentially corresponds to defining a Laurent expansion about the origin, although the singularity in the resulting solution may be a branch point rather than a simple pole. The technique involves replacing our Ansatz of a simple power series with a powers series times a power,

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n \rightarrow \sum_{n=0}^{\infty} a_n x^{n+s}. \]  

(23.1)

The exponent \( s \) is then chosen so that the \( n = 0 \) term yields a solution of the equation. Since we are considering second order equations, there will typically be two values for \( s \) corresponding to different behavior at the origin. As suggested in our earlier discussions, typically one solution is well behaved at the origin, while the other solution is not well behaved. The former solution is the one of physical interest for problems that include the origin in the region where the solution is expected to be finite. As an interesting example of this behavior we now focus on Bessel’s equation (see 12.12 – 12.17 in Boas).

The differential equation studied by Bessel arises in several contexts in physics and the solutions display a variety of useful properties, of which we will make only a brief survey. Consider first our friend Laplace’s equation but now in cylindrical coordinates. Recall that in this case we are assuming that the underlying physics has the symmetry structure of a cylinder. We keep the \( z \)-axis, but re-express the \( x \) and \( y \) coordinates in terms of a radial distance (perpendicular to the \( z \)-axis) and azimuthal angle around the \( z \)-axis. In contrast to the notation of Boas, we will use a more standard notation for these variables, which will serve to distinguish them from spherical coordinates (in our notation the angle \( 0 \leq \phi < 2\pi \) is an azimuthal [periodic, \( \phi = 0 \) is the same as \( \phi = 2\pi \)] angle in both spherical and cylindrical coordinates, while \( 0 \leq \theta \leq \pi \) is a polar angle and appears only in spherical coordinates). As usual we define
\[ x^2 + y^2 = \rho^2, \tan \phi = \frac{x}{y}, \quad \frac{d}{dx} \tan \phi \frac{d}{dy} \tan \phi = \frac{\rho}{\rho^2} \frac{d}{d\rho} \tan \phi \frac{d}{d\rho} \tan \phi \]

In this notation Laplace’s equation is

\[ \nabla^2 \Psi(\rho, \phi, z) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \Psi \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \Psi + \frac{\partial^2}{\partial z^2} \Psi = 0. \] (23.3)

As in the spherical case we assume that separation of variables is possible, \( \Psi = R(\rho)\Phi(\phi)Z(z) \). As before this is only possible if, after the separation,

\[ \frac{1}{R} \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} R \right) + \frac{1}{\rho^2} \frac{d^2}{d\phi^2} \Phi + \frac{1}{Z} \frac{d^2}{dz^2} Z = 0, \] (23.4)

the various terms, properly normalized, are all equal to constants. In particular, the third term, which depends only on \( z \) and not the other variables, must be a constant. We can also infer the separation constants from the expected forms of the various functions. For infinite range of the \( z \) variable, \(-\infty < z < \infty\), we think in terms of real exponentials, \( Z(z) \sim e^{\pm kz} \) (Laplace style behavior, but not orthogonal functions), and write

\[ \frac{1}{Z} \frac{d^2}{dz^2} Z(z) = k^2, \] (23.5)

where the parameter \( k \) has units of inverse distance, but is otherwise unconstrained. Similarly we expect to describe the (periodic) \( \phi \) dependence in terms of the complex exponentials \( \Phi(\phi) \sim e^{\pm ip\phi} \) (or \( \sin p\phi, \cos p\phi \), orthogonal functions) so that

\[ \frac{1}{\Phi} \frac{d^2}{d\phi^2} \Phi = -p^2, \] (23.6)

where the periodic boundary conditions require that \( p \) is an integer. Thus, after separating out the \( z \) and \( \phi \) dependence, the (cylindrical) radial equation looks like (after multiplying through by \( \rho^2 \))
\[ \rho \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} R(\rho) \right) + \left( k^2 \rho^2 - p^2 \right) R(\rho) = 0. \]  

(23.7)

We obtain the usual form of Bessel’s equation if we define a variable \( x = k\rho \) (which is a dimensionless variable, unlike \( \rho \)) and replace \( R(\rho) \to y(x) \) to obtain

\[
 x^2 y'' + xy' + \left( x^2 - p^2 \right) y = 0
\]

\[
 \Rightarrow y'' + \frac{1}{x} y' + \left( 1 - \frac{p^2}{x^2} \right) y = 0.
\]

(23.8)

From the second (canonical) form of the equation we see, as expected, that the origin is a regular singular point. Note that, from the way we derived this equation using cylindrical coordinates, we expect that the periodicity in \( \phi \) will require that \( p \) is an integer. Indeed the original study by Bessel, defining the usual Bessel function, does correspond to integer values of \( p \). However, the equation is thoroughly studied for general values of \( p \) and we will show now that half-integer values of \( p \) are also of immediate interest. In particular, before proceeding to solve Bessel’s equation, let us take a brief detour to see how this same equation arises in the case of spherical coordinates.

Consider again Laplace’s equation in spherical coordinates but now allow a non-zero right-hand-side. This typically arises from some non-zero time dependence in either a diffusion problem (see, \( \text{e.g.} \), 13.3.3 in Boas), \( \nabla^2 \Psi = \left(1/\alpha^2\right) d^2 \Psi / dt \) or the wave equation \( \nabla^2 \Psi = \left(1/v^2\right) d^2 \Psi / dr^2 \). Again we assume that separation of variables applies, \( \Psi = R(r)\Phi(\phi)\Theta(\theta)T(t) \). With the assumption that the time dependence is exponential, \( \text{e.g.}, T \sim e^{iot} \) in the wave equation case \( T \sim e^{-k^2 \alpha^2} \) in the diffusion case, we can write the right-hand-side of the equation as \( \nabla^2 \Psi = -k^2 \Psi \) (the Helmholtz equation), where, as above, \( k \) has the units of inverse distance (in the wave equation case \( k \) is the wave number, \( k = \omega/v \), with \( v \) the velocity of the wave). Assuming that we treat the \( \theta \) and \( \phi \) dependence as in the discussion of the Associated Legendre equation in the previous Lecture, the radial part of the spherical equation becomes,

\[
 \frac{d}{dr} \left( r^2 \frac{d}{dr} R(r) \right) = \left[ l(l+1) - k^2 r^2 \right] R(r).
\]

(23.9)
As above we define the dimensionless variable \( x = kr \) and (since we know the answer we want) make the replacement \( R(r) \rightarrow y(x)/\sqrt{x} \). Now the spherical radial equation becomes

\[
\frac{d}{dx} \left( x^2 \frac{d}{dx} \left[ \frac{y}{\sqrt{x}} \right] \right) + \left[ x^2 - l(l+1) \right] \frac{y}{\sqrt{x}} = 0
\]

\[
\Rightarrow x^2 \left[ \frac{y''}{\sqrt{x}} - \frac{y'}{(\sqrt{x})^3} + \frac{3}{4} \frac{y}{(\sqrt{x})^5} \right] + 2x \left[ \frac{y'}{\sqrt{x}} - \frac{1}{2} \frac{y}{(\sqrt{x})^3} \right] + \left[ x^2 - l(l+1) \right] \frac{y'}{\sqrt{x}} = 0
\]

\[
\Rightarrow x^2 y'' + xy' + \left[ x^2 - \left( l + \frac{1}{2} \right)^2 \right] y = 0.
\]

We recognize this final expression as Bessel’s equation but with \( p \rightarrow l + 1/2 \), a half-integer value. The (spherical) radial behavior of the inhomogeneous Laplace equation is given by a Bessel function of half-integer order divided by \( \sqrt{x} \). This form is called the spherical Bessel function. We will return to this point below.

Now we return to the discussion of the original equation of Bessel, Eq. (23.8), and we substitute the form suggested by Frobenius (Eq. (23.1)) to find

\[
\sum_{n=0}^{\infty} a_n \left( (n+s)(n+s-1)x^{n+s} + a_n (n+s)x^{n+s} + \left( x^2 - p^2 \right) a_n x^{n+s} \right) = 0
\]

\[
\Rightarrow \sum_{n=0}^{\infty} \left\{ a_n \left[ (n+s)^2 - p^2 \right] + a_{n-2} \right\} x^{n+s} = 0 \quad [a_{-2} = a_{-1} = 0],
\]

where the second term, \( a_{n-2} \), only contributes for \( n \geq 2 \) as noted. As with our study of the Legendre equation, the solutions split into two forms, one based on even values of \( n \) and one based on odd values. Consider the special case \( n = 0 \), \( i.e., \) the coefficient of \( x^s \), which, since there is no second term, requires that

\[
s^2 = p^2 \quad \Rightarrow \quad s = \pm p.
\]
Since we are (initially) interested in solutions that are well behaved at the origin, \( x = 0 \), we focus on \( s = p \geq 0 \). Note from the discussion above that \( p \) is not determined by this radial equation, but rather is determined by an eigenvalue problem based on the angular behavior. Here we are simply looking for the solutions of the radial equation, which are characterized by their behavior at the origin and at infinity. The previous equation provides us with a recursion relation for the coefficients. Choosing as noted, \( s = p \), we have

\[
a_n = -\frac{a_{n-2}}{(n+p)^2 - p^2} = -\frac{a_{n-2}}{n(n+2p)} = \frac{a_{n-4}}{n(n-2)(n+2p)(n+2p-2)}
\]

\[
\Rightarrow a_{2m} = (-1)^m \frac{a_0}{2^{2m} m! \Gamma(m+p+1)}. \tag{23.13}
\]

With the (arbitrary but conventional) choice \( a_0 = 1/(2^p \Gamma(1+p)) \) we obtain the Bessel function of the first kind of order \( p \),

\[
J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m+p}. \tag{23.14}
\]

Clearly the Bessel function of order \( p \) behaves as \( i.e., \) vanishes as \( x^p \) as \( x \to 0 \). Note that changing the order by one, \( p \to p + 1 \), corresponds to keeping the terms with odd powers of \( x \) in the original sum, \( i.e., \) we have included both types of solutions in this expression when we include all integer values of \( p \). Due to the alternating signs in this series the Bessel function is an oscillating function, much like the sine and cosine. However, in this case the distance between successive zeroes is not precisely a constant (but the zeroes are tabulated). Here are plots of \( J_0(x) \) and \( J_1(x) \) that clearly exhibit the oscillatory behavior.
The two ends of these plots (the asymptotic behavior) can be characterized simply as

\[
\lim_{x \to 0} J_p(x) = \frac{1}{\Gamma(1 + p)} \left( \frac{x}{2} \right)^p \left( 1 + \mathcal{O}(x^2) \right),
\]

\[
\lim_{x \to \infty} J_p(x) = \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{2p + 1}{4} \pi \right) \left( 1 + \mathcal{O} \left( \frac{1}{x} \right) \right). \tag{23.15}
\]

From this last result we see that at least for large \( x \) the spacing between the zeros of \( J_p(x) \) approaches \( \pi \).

The second solution of the Bessel equation can be constructed from the case \( s = -p \). For non-integer values of \( p \), \( J_{-p}(x) \) is an independent function and, as suggested above, is singular at the origin, \( J_{-p}(x) \xrightarrow{x \to 0} x^{-p} \). For the case of integer \( p \) values we have, due to the magic of the Gamma function, that

\[
J_{-p}(x) = (-1)^p J_p(x) \quad [p \text{ integer}] \tag{23.16}
\]

and more care is needed to define a second independent solution. The standard form found in the literature is (named after Neumann and/or Weber)

\[
N_p(x) = Y_p(x) = \frac{\cos(\pi p)J_p(x) - J_{-p}(x)}{\sin(\pi p)}, \tag{23.17}
\]

where the form for integer \( p \) values is obtained by a careful limiting process. The expression for integer \( p \) values exhibits a logarithmic singularity at the origin.

Returning to the spherical coordinate case with half-integer \( p \) values (\( p = l + 1/2 \)), the standard expression for the spherical Bessel function that is finite at the origin is (note the remarkable form in terms of \( \sin x \), which we will not derive here)

\[
j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) = x^l \left( -\frac{1}{x} \frac{d}{dx} \right)^l \sin x. \tag{23.18}
\]

The second spherical function, singular at the origin, is given by
As outlined in Chapter 12 in Boas many different forms for the Bessel function have been studied and are often useful in special cases in physics, including various recursion relations for the Bessel functions, much as we saw for the Legendre polynomials. A sample relation is

\[ \frac{d}{dx} \left[ x^n J_p (x) \right] = x^n J_{p-1} (x). \]  \hspace{1cm} (23.20)

We do not have the time to fully pursue these relations here.

As described in Chapter 13 of Boas we can put together the various functions we have studied to define solutions of the various 2nd order differential equations we meet in physics and fit the relevant boundary conditions. For example (see Chapter 13.5) consider the temperature inside a (semi-infinite) cylinder of uniform material with boundary conditions such that the temperature is a constant $T_0$ on the end of the cylinder at $z = 0$, but the temperature is held to zero on the boundary at $\rho = R_0$ and as $z \to \infty$. Since, away from heat sources, the temperature satisfies Laplace’s equation, we must be able to express the temperature distribution as a sum over the basis functions that satisfy Laplace. Further, since we are lazy but smart, we choose to use cylindrical functions that will facilitate matching boundary conditions in terms of cylindrical coordinates. Our Ansatz for the most general temperature distribution inside the cylinder takes the form (for positive $z$ with constant coefficients we label $T_{k,p}$)

\[ T(\rho, \phi, z) \sim \sum_{k,p} T_{k,p} J_p (k \rho) e^{\pm ip \phi} e^{-kz}. \]  \hspace{1cm} (23.21)

Each term in the sum satisfies Laplace’s equation in cylindrical coordinates, is periodic in $\phi$ and is well behaved as $z \to \infty$. Here $p$ must be integer valued to ensure periodic behavior in $\phi$. The terms also constitute a complete set of functions with these properties inside the cylinder. Since the specific boundary conditions in our case have no variation in $\phi$, we choose $p = 0$. The eigenvalues for $k$ are fixed by the requirement that $J_0 (k R_0) = 0$ on the boundary of the cylinder. If the $m^{th}$ zero of $J_0$ is
\( x_{0,m}, J_0(x_{0,m}) = 0 \) (recall the plots of the Bessel functions), we have the eigenvalues for \( k \) given by \( k_m = x_{0,m}/R_0 \). The \( x_{0,m} \) (the \( x_{p,m} \) for \( J_p(x), J_p(x_{p,m}) = 0 \)) play the same role as the constants \( m\pi \) do for the sine function. Note that from the large argument form of the Bessel function in Eq. (23.15) it follows that for \( m \gg 1, x_{p,m+1} - x_{p,m} = \pi \). The combined functions \( J_p(k_m \rho)e^{ip\phi} \), with \( m \) and \( p \) integer valued, are complete and orthogonal on the 2-D surface of a disk of radius \( R_0 \) analogously to the behavior of the \( Y_{lm} \) on the surface of a sphere. Thus the form of the temperature distribution as been reduced to

\[
T(\rho, \phi, z) = \sum_{m=1}^{\infty} c_m J_0(k_m \rho)e^{-k_m z},
\]

with the coefficients specified by the boundary condition

\[
T(\rho < R_0, \phi, z = 0) = T_0 = \sum_{m=1}^{\infty} c_m J_0(k_m \rho)
\]

\[
\Rightarrow c_m = \frac{2T_0}{k_m R_0 J_1(k_m R_0)}.
\]

See the discussion in 13.5 for details. The essential point for our current purpose is the orthogonality relation for the Bessel functions. This involves not Bessel functions of different order, but rather Bessel functions of the same order but involving different zeros, \( i.e. \) the sum above is over \( m \), not \( p \). Recall that \( p \) is determined by the \( \phi \) dependence while \( m \) is related to the \( \rho \) dependence. In our previous language we are working with a Sturm-Liouville problem for fixed \( p \) with the eigenfunctions characterized by different \( m \) values. [This is the analogue of the Associated Legendre problem where we observed orthogonal polynomials labeled by different values of \( l \) (\( \geq m \)) for a fixed value of \( m \).] The relevant relation is

\[
\int_0^{R_0} \rho d\rho J_p\left(\frac{x_{p,m}}{R_0} \rho\right) J_p\left(\frac{x_{p,n}}{R_0} \rho\right) = \delta_{mn} \frac{R_0^2}{2} J_{p+1}^2\left(x_{p,m}\right).
\]

Note that the right-hand-side is not zero for \( m = n \) because the zeros of \( J_p(x) \) are not
the zeros of $J_{p+1}(x)$ so $J_{p+1}(x_{p,m}) \neq 0$. Using this orthogonality relation above we have

$$
\int_0^{R_0} \rho d\rho T_0 J_0(k_n \rho) = \sum_{m=1}^{\infty} c_m \int_0^{R_0} \rho d\rho J_0(k_n \rho) J_0(k_m \rho) = \sum_{m=1}^{\infty} c_m \delta_{nn} \frac{R_0^2}{2} J_1^2(k_m R_0) = c_n \frac{R_0^2}{2} J_1^2(k_n R_0). \tag{23.25}
$$

Using the recursion relation in Eq. (23.20) above we also have (for the LHS)

$$
\int_0^{R_0} \rho d\rho T_0 J_0(k_n \rho) = T_0 \int_0^{R_0} d\rho \left( \rho J_0(k_n \rho) \right) = T_0 \frac{x_{0,n}}{k_n^2} \int_0^{R_0} dx \left( x J_0(x) \right)|_{x = k_n \rho} = T_0 \frac{x_{0,n}}{k_n^2} \left[ x J_1(x) \right]_{x = k_n \rho}^{x_{0,n}} = T_0 \frac{x_{0,n}}{k_n^2} \left[ k_n \rho J_1(k_n \rho) \right]_{0}^{R_0} = T_0 \frac{R_0}{k_n} J_1(k_n R_0) = c_n \frac{R_0^2}{2} J_1^2(k_n R_0).
$$

$$
\Rightarrow c_n = \frac{2T_0}{k_n R_0 J_1(k_n R_0)} = T_0 \frac{2}{x_{0,n} J_1(x_{0,n})}, \tag{23.26}
$$

the result claimed in Eq. (23.23). Pulling the pieces together the temperature distribution inside the cylinder is given by

$$
T(\rho \{ \leq R_0 \}, \phi, z \{ \geq 0 \}) = 2T_0 \sum_{n=1}^{\infty} \frac{J_0(k_n \rho)}{k_n R_0 J_1(k_n R_0)} e^{-k_n z} = 2T_0 \sum_{n=1}^{\infty} \frac{J_0(x_{0,n} \rho/R_0)}{x_{0,n} J_1(x_{0,n})} e^{-x_{0,n} z/R_0}. \tag{23.27}
$$

For our purposes the essential point is that we can write down the general solutions to diffusion equations, wave equations, etc. in terms of a linear combination of the appropriate orthogonal and complete “basis functions”. Then we can solve for the coefficients using the boundary conditions and the known (orthogonality) properties of these functions.
Summary: For physical systems in greater than 1 dimension and described by linear differential equations the method of separation of variables almost always serves to split the problem into simpler pieces, one for each dimension. The separate pieces in turn almost always correspond to one of the special equations we have studied with solutions that are the standard special functions we have discussed: powers, real exponentials (sinh’s and cosh’s), complex exponentials (sines and cosines), Legendre polynomials, spherical harmonics and Bessel functions. Employing linear superposition we can write the solution to the full problem as sums of products of these functions. Matching the boundary conditions (including time dependence) will result in eigenvalue problems for some of the separated equations (but typically not all), which serves to define the discrete eigenvalues over which we sum.