Here are some solutions to the sample problems assigned for Chapter 14.

§14.2: 18

Solution: We want to learn about the analyticity properties of the function

\[ f(z) = \sqrt{z} = \sqrt{r e^{i\theta/2}} = \sqrt{r} \cos \frac{\theta}{2} + i \sqrt{r} \sin \frac{\theta}{2} \]

where

\[ \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \]

and

\[ \sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \]

Recall that the square root function has “2 sheets” connected by a branch cut running from the origin to infinity. In polar coordinates (see the first line above) the phase \( \theta \) must vary from 0 to \( 4\pi \) before \( \sqrt{z} \) returns to its starting point. In the last expressions the \( \pm \) signs account for this sheet structure. We proceed by considering the Cauchy-Riemann conditions and perform some simple manipulations to find

\[ U(x, y) = \pm \sqrt{x^2 + y^2} \sqrt{\frac{x^2 + y^2 + x}{2x^2 + y^2}} = \pm \sqrt{\frac{x^2 + y^2 + x}{2}} \]

and

\[ V(x, y) = \pm \sqrt{x^2 + y^2} \sqrt{\frac{x^2 + y^2 - x}{2x^2 + y^2}} = \pm \sqrt{\frac{x^2 + y^2 - x}{2}} \]
\[
\frac{\partial U}{\partial x} = \pm \frac{1}{2\sqrt{2}} \frac{1 + x\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + x}} = \pm \frac{1}{2\sqrt{2}} \frac{\sqrt{x^2 + y^2 + x}}{\sqrt{x^2 + y^2}} \\
\frac{\partial V}{\partial y} = \pm \frac{1}{2\sqrt{2}} \frac{y}{\sqrt{x^2 + y^2}} = \pm \frac{1}{2\sqrt{2}} \frac{\sqrt{(x^2 + y^2)^2 - x^2}}{\sqrt{x^2 + y^2}} \\
\frac{\partial V}{\partial x} = \pm \frac{1}{2\sqrt{2}} \frac{-1 + x\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 - x}} = \mp \frac{1}{2\sqrt{2}} \frac{\sqrt{x^2 + y^2 - x}}{\sqrt{x^2 + y^2}} \\
\frac{\partial U}{\partial y} = \pm \frac{1}{2\sqrt{2}} \frac{y\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + x}} = \pm \frac{1}{2\sqrt{2}} \frac{\sqrt{(x^2 + y^2)^2 - x^2}}{\sqrt{x^2 + y^2 + x}} \\

\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \cdot \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} : z \neq 0.
\]

The square root is analytic (on a single sheet) everywhere in the complex plane except the origin, which is a branch point.
§14.2: 19

**Solution:** We want to learn about the analyticity properties of the logarithm function

\[ f(z) = \ln z = \ln \rho + i\phi = \frac{1}{2} \ln \left(x^2 + y^2\right) + i \tan^{-1} \frac{y}{x} \]

\[ \Rightarrow U(x, y) = \frac{1}{2} \ln \left(x^2 + y^2\right), \quad V(x, y) = \tan^{-1} \frac{y}{x}. \]

Again there is a branch cut from the origin to infinity, but now there are an infinite number of branches (due to the multi-valued arctan function). Next consider the Cauchy-Riemann conditions

\[ \frac{\partial U}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial V}{\partial y} = \frac{x}{x^2 + y^2}, \]
\[ \frac{\partial U}{\partial y} = \frac{y}{x^2 + y^2}, \quad \frac{\partial V}{\partial x} = \frac{-y}{x^2 + y^2}, \]
\[ \Rightarrow \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}: z \neq 0. \]

The logarithm is analytic (on a single sheet) everywhere in the (finite) complex plane except the origin, which is a branch point.

§14.2: 23

**Solution:** We want to learn about the analyticity properties of the function

\[ f(z) = \frac{x - iy}{x^2 + y^2} = \frac{z^*}{|z|^2} = \frac{1}{z} \]
\[ \Rightarrow U(x, y) = \frac{x}{x^2 + y^2}, \quad V(x, y) = -\frac{y}{x^2 + y^2}. \]

So we consider the Cauchy-Riemann conditions
Thus this function is analytic everywhere in the complex plane except the origin, where the function has a simple pole. In fact, that is all this function is – just the pole $1/z$.

§14.2: 34

**Solution:** We consider the function $\ln(1-z)$ expanded about the origin. We know from last quarter that this function has a series expansion

$$\ln(1-z) = -\sum_{n=1}^{\infty} \frac{(z)^n}{n} = -z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots,$$

which we further know converges for $|z|<1$. Let’s consider this result in our new language. The Cauchy-Riemann conditions tell us

$$\frac{\partial U}{\partial x} = \frac{1}{2} \ln\left((1-x)^2 + y^2\right) + i \tan^{-1} \frac{-y}{1-x},$$

$$\Rightarrow \frac{\partial U}{\partial x} = \frac{1-x}{(1-x)^2 + y^2}, \quad \frac{\partial V}{\partial y} = \frac{-y}{(1-x)^2 + y^2},$$

$$\frac{\partial V}{\partial x} = \frac{y}{(1-x)^2 + y^2}, \quad \frac{\partial U}{\partial y} = \frac{1-x}{(1-x)^2 + y^2},$$

$$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}, \quad z \neq (1, 0).$$
This logarithm is analytic (on a single sheet) everywhere in the (finite) complex plane except the point (on the real axis) \( x = 1, y = 0 \), which is a branch point. This singular point explains why the power series expansion about the origin cannot converge for \( |z| \geq 1 \).

§14.2: 48

Solution: In this exercise we want to think about the Cauchy-Riemann conditions in polar coordinates. In Boas notation we have \( z = re^{i\theta} \) and thus \( \partial z/\partial r = e^{i\theta} = z/|z| \) and \( \partial z/\partial \theta = iz \). Hence a unique derivative requires (equating real and imaginary parts in the last step and finding C-R in polar coordinates)

\[
\frac{\partial f}{\partial r} = \frac{df}{dz} \frac{\partial z}{\partial r} = \frac{df}{dz} \left| z \right|, \quad \frac{\partial f}{\partial \theta} = \frac{df}{dz} \frac{\partial z}{\partial \theta} = \frac{df}{dz} iz
\]

\[
\Rightarrow \frac{df}{dz} = \left| z \right| \frac{\partial f}{\partial r} = \left| z \right| \left( \frac{\partial U}{\partial r} + i \frac{\partial V}{\partial r} \right) = -i \frac{\partial f}{z \partial \theta} = -i \left( \frac{\partial U}{\partial \theta} + i \frac{\partial V}{\partial \theta} \right)
\]

\[
\Rightarrow \frac{\partial U}{\partial r} = \frac{1}{r} \frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial r} = -\frac{1}{r} \frac{\partial U}{\partial \theta}.
\]

Now we can apply this form to the square root function (see 14.2:18 above). We find

\[
f(z) = \sqrt{z} = \sqrt{re^{i\theta/2}} = \sqrt{r} \cos \frac{\theta}{2} + i \sqrt{r} \sin \frac{\theta}{2}
\]

\[
\Rightarrow U(r, \theta) = \sqrt{r} \cos \frac{\theta}{2}, \quad V(r, \theta) = \sqrt{r} \sin \frac{\theta}{2},
\]

\[
\Rightarrow \frac{\partial U}{\partial r} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2}, \quad \frac{\partial V}{\partial \theta} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2},
\]

\[
\frac{\partial V}{\partial r} = \frac{1}{2\sqrt{r}} \sin \frac{\theta}{2}, \quad \frac{\partial U}{\partial \theta} = -\frac{1}{2\sqrt{r}} \sin \frac{\theta}{2},
\]

\[
\Rightarrow \frac{\partial U}{\partial r} = \frac{1}{r} \frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial r} = -\frac{1}{r} \frac{\partial U}{\partial \theta} : r \neq 0.
\]
So we obtain our previous result that the square root is analytic in the entire complex plane except the origin, where there is a branch point. However, the calculation is clearly simpler in polar coordinates.

§14.3: 17

**Solution:** We want to consider the following (closed) contour integrals

\[
\oint_C \frac{\sin z}{2z-\pi} \, dz = \frac{1}{2} \oint_C \frac{\sin z}{z-\pi/2} \, dz; \quad a) \ C \Rightarrow |z| = 1
\]

\[
\oint_C \frac{\sin z}{z^2 - \pi} \, dz = \oint_C \frac{\sin z}{z^2 - \pi^2} \, dz; \quad b) \ C \Rightarrow |z| = 2.
\]

To proceed we must recognize two features of the integrand \( \sin z/(2z-\pi) \). First the sine function is an entire function, analytic in the entire complex plane (recall from last quarter that we know that its power series expansion is convergent for all \( z \)). Second we see that there is a simple pole at \( z_0 = \pi/2 \) \([x_0 = \pi/2 = 1.5, y_0 = 0]\). Now we are ready to apply Cauchy’s integral theorem. For the first contour the pole is outside of the contour and the integral vanishes, while the second contour encircles the pole and we find

\[
\oint_{C_\theta} \frac{\sin z}{2z-\pi} \, dz = 0,
\]

\[
\oint_{C_\varphi} \frac{\sin z}{2z-\pi} \, dz = 2\pi i \left( \frac{1}{2} \sin \frac{\pi}{2} \right) = \pi i.
\]

§14.3: 23

**Solution:** Now consider a contour integral around the square with vertices at \( z = \pm 1 \pm i \) of the following integrand

\[
\oint_C \frac{e^{3z}}{(z - \ln 2)^4} \, dz = \oint_C \frac{e^{3\ln 2} \cdot e^{3(z - \ln 2)}}{(z - \ln 2)^4} \, dz = 8 \sum_{n=0}^{\infty} \oint_C \frac{3^n (z - \ln 2)^{n-4}}{n!} \, dz.
\]

Cauchy tells us that we are only interested in the simple pole term corresponding \( n = 3 \) in the sum. Thus we have
\[ \oint_C \frac{e^{3z}}{(z - \ln 2)^4} \, dz = 8 \oint_C \frac{3^3 (z - \ln 2)^{-1}}{3!} \, dz = 2\pi i \frac{8}{6} 27 = 72\pi i, \]

Note that, for this nonzero result, it is essential that \( \ln 2 = 0.693\ldots < 1 \) (so that the pole is inside of the contour).

§14.6: 3

Solution: We want to expand the function \( \frac{\sin z}{z^4} \) in a Laurent series around the singular point at the origin. This is just the familiar series for the sine function divided by \( z^4 \). We have

\[ \frac{\sin z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-3}}{(2n+1)!}. \]

Hence the residue (from the simple pole term, \( n = 1 \)) is then

\[ R = \frac{(-1)^n}{(2n+1)!} \bigg|_{n=1} = -\frac{1}{3!} = -\frac{1}{6}. \]

§14.6: 5

Solution: Now consider \( e^z / z^2 - 1 \) about the point \( z = 1 \). Expressing everything in terms of \( z - 1 \) we find

\[ \frac{e^z}{z^2 - 1} = \frac{e^{(z-1)}}{(z-1)(2 + z - 1)} = \frac{e}{2} \frac{1}{z - 1} \left\{ \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \right\} \left\{ \sum_{m=0}^{\infty} (-1)^m \left( \frac{z-1}{2} \right)^m \right\} \]

\[ = \frac{e}{2} \frac{1}{z - 1} \left\{ 1 + (z - 1) + \frac{(z - 1)^2}{2} + \cdots \right\} \left\{ 1 - \left( \frac{z-1}{2} \right)^2 + \cdots \right\} \]

\[ = \frac{e}{2} \left[ \frac{1}{z - 1} + \frac{1}{2} + \frac{1}{4} (z - 1) + \cdots \right]. \]
Thus the residue (the first term) is just \( R = e/2 \).

§14.7: 1

Solution: Finally we want to practice evaluating integrals using Cauchy. Consider first an integral around a circle, which we can think of as an integral in the complex plane using the polar representation, \( z = e^{i\theta}, \ dz = id\theta e^{i\theta} = i\,dz\,d\theta. \) Thus we can write

\[
\oint_{|z|=1} \frac{d\theta}{13 + 5\sin \theta} = \oint_{|z|=1} \frac{dz}{iz} \frac{1}{13 + \frac{5}{2i}(z - \frac{1}{z})} = \oint_{|z|=1} \frac{2dz}{5z^2 + 26iz - 5}
\]

Thus the one simple pole inside the contour is the one at \( z = -i/5 \), while the other pole (at \( z = -5i \)) is outside of the contour. So finally we obtain

\[
\oint_{|z|=1} \frac{2dz}{5(z + \frac{i}{5})(z + 5i)} = 2\pi i \left( \frac{2}{5\left(\frac{i}{5} + 5i\right)} \right) = \frac{4\pi}{24} = \frac{\pi}{6}.
\]

§14.7: 13

Solution: Now consider an integral along \( \frac{1}{2} \) of the real axis, which we can extend to the full axis by using the even symmetry of the integrand,

\[
I = \int_{-\infty}^{\infty} \frac{x^2}{x^2 + 4} \frac{dx}{x^2 + 9} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{(x + 2i)(x - 2i)(x + 3i)(x - 3i)} \, dx.
\]

We see that there are 4 simple poles that we can use, via Cauchy, to evaluate the integral, 2 above the real axis (+2i, +3i) and 2 below (-2i, -3i). As usual we want to
add an arc at infinity, which we are allowed to do (by Jordan) since the integral along
the arc behaves like

$$\lim_{R \to \infty} \int_{\text{arc at } R} \frac{R^2 e^{2i\theta} i d\theta R e^{i\theta}}{(R^2 e^{2i\theta} + 4)(R^2 e^{2i\theta} + 9)} = \lim_{R \to \infty} \frac{1}{R} \int_0^{\pm \pi} i e^{-i\theta} d\theta = 0.$$

If we close the contour in the upper half plane (counter-clockwise), we encircle the
poles at $+2i$ and $+3i$ to find

$$I = \frac{1}{2} \oint_{\text{Closed Above}} \frac{z^2 dz}{(z + 2i)(z - 2i)(z + 3i)(z - 3i)}$$

$$= \frac{2\pi i}{2} \left( \frac{-4}{(4i)(-4 + 9)} + \frac{-9}{(-9 + 4)(6i)} \right) = \pi \left( \frac{1}{5} + \frac{3}{10} \right) = \frac{\pi}{10}.$$

If we choose to close the contour in the lower half plane (clockwise) we encircle the
poles at $-2i$ and $-3i$ with the opposite residues from above but with an extra minus
sign due to the opposite direction. Thus we obtain the same answer either way.

§14.7: 19

Solution: This integral is similar to the one we just did, but now with a cosine
function in the numerator. Again we can use the fact that the integrand is symmetric
to extend it along the entire real axis and then, due to the anti-symmetry of the sine
function, we can, for free, replace the cosine function with the complex exponential.
We have

$$I = \int_0^\infty \frac{\cos 2x dx}{(4x^2 + 9)^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos 2x dx}{(2x + 3i)^2 (2x - 3i)^2} = \frac{1}{32} \left[ \int_{-\infty}^\infty \frac{e^{i2x} dx}{\left( x + \frac{3}{2}i \right)^2 \left( x - \frac{3}{2}i \right)^2} \right].$$

Now we have to worry about the way the exponential behaves on the arc. We have
\[ z = R e^{i\theta}, \]
\[ e^{2iz} = e^{2iR(cos \theta + i \sin \theta)} = e^{-2R \sin \theta} e^{2iR \cos \theta}. \]

Thus the arc will be (exponentially) damped iff we choose \( \sin \theta > 0 \), \( 0 < \theta < \pi \). So in this case we choose to close in the upper half plane and find

\[
I = \frac{1}{32} \left[ \oint_{\text{Closed Above}} \frac{e^{2iz} \, dz}{(z + \frac{3}{2}i)^2 (z - \frac{3}{2}i)^2} \right].
\]

The final detail is that we must expand the integrand in order to find the appropriate residue at \( z = 3i/2 \) (simple poles). To this end we note that

\[
\frac{e^{2iz}}{32 \left( z + \frac{3}{2}i \right)^2 \left( z - \frac{3}{2}i \right)^2} = \frac{e^{-3} e^{2i(z-3i/2)}}{32 \left( z - \frac{3}{2}i \right)^2 \left( 3i + z - \frac{3}{2}i \right)^2}
\]

\[
= -\frac{e^{-3}}{288 \left( z - \frac{3}{2}i \right)^2} \left[ 1 + 2i \left( z - \frac{3}{2}i \right) + \cdots \right] \left[ 1 - \frac{2}{3i} \left( z - \frac{3}{2}i \right) + \cdots \right]
\]

\[
= -\frac{e^{-3}}{288 \left( z - \frac{3}{2}i \right)^2} \left[ 1 + i \left( \frac{8}{3} \right) \left( z - \frac{3}{2}i \right) + \cdots \right].
\]

Hence the desired residue and integral are given by

\[
R = -\frac{e^{-3} 8i}{288 \frac{3}{108}} = -i \frac{e^{-3}}{108}
\]

\[
\Rightarrow I = 2\pi i R = \frac{\pi e^{-3}}{54}.
\]
Solution: Here we want to consider an integrand with a branch cut as in example 5 in Boas. We start with the integral

$$I = \int_{0}^{\infty} \frac{\sqrt{x}}{1 + x^2} \, dx$$

where we choose to put the branch cut of the square root along the positive real axis. Next we consider a corresponding complex path integral around a contour as illustrated in Fig. 7.4 in Boas,

$$J = \oint_{C = \text{Fig. 7.4}} \frac{\sqrt{z}}{1 + z^2} \, dz = \oint_{C = \text{Fig. 7.4}} \frac{\sqrt{z}}{\left(z + i\right)\left(z - i\right)} \, dz$$

$$= \lim_{R \to \infty, r \to 0} \left\{ \int_{\text{arc, } z = R \, e^{i\theta}} \frac{\sqrt{Re^{i\theta/2}} i \theta e^{i\theta} R}{R^2 e^{2i\theta} + 1} + \int_{\text{arc, } z = r \, e^{i\theta}} \frac{\sqrt{r e^{i\theta/2}} i \theta e^{i\theta} r}{r^2 e^{2i\theta} + 1} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r