Lecture 16: Inhomogeneous 2nd order, linear, ordinary differential equations with non-periodic driving functions – Fourier Integral Transform (See Section 7.12 in Boas)

As in the last lecture we want to consider equations with the general form

\[ ax'' + bx' + cx = F(t) \]

where we start by thinking of the right-hand-side as periodic,

\[ F(t) = F(t + T) \]

(and typically satisfying the smoothness constraints of Dirichlet). In this lecture we want to extend this discussion to include the case where the period becomes arbitrarily long, \( T \to \infty \), i.e., where the function is not actually periodic. The resulting analysis will follow directly from our previous one. The major change will be that, instead of expressing \( F(t) \) in terms of an infinite sum of discrete Fourier components (labeled by integer), we will need to consider an integral over an infinite number of continuously related components (labeled by a continuous variable).

We start with the usual exponential expansion of Eq. (15.16) written in symmetric form,

\[
F(t) = \sum_{n=-\infty}^{\infty} c_n e^{in(2\pi/T)t},
\]

\[
c_n = \frac{1}{T} \int_{-T/2}^{T/2} dt \ F(t) e^{-in(2\pi/T)t}.
\]

Now we want to carefully take the \( T \to \infty \) limit while identifying a new (continuous instead of discrete) variable \( \omega = n2\pi/T \), which becomes continuous in this limit, i.e.,

\[
\Delta \omega = \Delta n 2\pi/T \quad \overset{T\to\infty}{\longrightarrow} \quad d\omega.
\]

Thus we make the identifications (with a symmetric choice of normalization)

\[
F(t) = \sum_{n=-\infty}^{\infty} c_n e^{in(2\pi/T)t} = \sum_{n=-\infty}^{\infty} c_n e^{in(2\pi/T)t} \Delta n \overset{T\to\infty}{\longrightarrow} \int_{-\infty}^{+\infty} \left( \frac{T}{2\pi} \right) d\omega \ e^{i\omega t} c_n(\omega)
\]

\[
\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \ G(\omega) e^{i\omega t},
\]

and
\[
\frac{T}{\sqrt{2\pi}} c_n \xrightarrow{T \to \infty} G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \ F(t) e^{-i\omega t},
\]

\[i.e., \text{the basis functions are now } e^{i\omega t}, e^{-i\omega t}. \text{ Note that we have chosen the normalization of the Fourier transform function } G(\omega) \text{ so that a factor of } 1/\sqrt{2\pi} \text{ appears (symmetrically) in both the transform and the inverse transform,}
\]

\[
G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \ F(t) e^{-i\omega t},
\]

\[
F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \ G(\omega) e^{i\omega t}.
\]

This is clearly just one possible choice. In Boas the other simple choice is made where there is a factor of $1/2\pi$ in the first equation and a factor of 1 in the second equation (see Eq. 7.12.2 in Boas). You will see both choices in the literature. Clearly, in practice, either choice will result in the same solution to the initial differential equation.

The properties of the Fourier integral expansion, \textit{i.e.}, the Fourier transform, include the following.

- In this formalism orthogonality is expressed by

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \ e^{i(\omega - \omega') t} = \delta(\omega - \omega').
\]

- Completeness is expressed by the very similar expression

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{i\omega(t-t')} = \delta(t - t').
\]

- Special cases:
• \( F(t) \) is an odd function - \( F(t) = -F(-t) \), as in the periodic case only the sine contributes and we have

\[
G_{\text{odd}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, F_{\text{odd}}(t) e^{-i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, F_{\text{odd}}(t) \left( \cos \omega t - i \sin \omega t \right)
\]

\[
= -i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, F_{\text{odd}}(t) \sin \omega t = -i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} dt \, F_{\text{odd}}(t) \sin \omega t
\]

\[
\Rightarrow G_{\text{odd}}(-\omega) = -G_{\text{odd}}(\omega),
\]

(16.7)

\[
F_{\text{odd}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, G_{\text{odd}}(\omega) e^{i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega \, G_{\text{odd}}(\omega) \left( e^{i\omega t} - e^{-i\omega t} \right)
\]

\[
= i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} d\omega \, G_{\text{odd}}(\omega) \sin \omega t.
\]

These results suggest that for odd functions we define the Fourier sine transform,

\[
G_s(\omega) \left[ = iG_{\text{odd}}(\omega) \right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} dt \, F_{\text{odd}}(t) \sin \omega t, \quad (16.8)
\]

\[
F_{\text{odd}}(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} d\omega G_s(\omega) \sin \omega t.
\]

• \( F(t) \) is an even function - \( F(t) = F(-t) \), only the cosine contributes and we have
\( G_{\text{even}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, F_{\text{even}}(t) e^{-i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, F_{\text{even}}(t) \left( \cos \omega t - i \sin \omega t \right) \)

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, F_{\text{even}}(t) \cos \omega t = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} dt \, F_{\text{even}}(t) \cos \omega t
\]

\[\Rightarrow G_{\text{even}}(-\omega) = G_{\text{even}}(\omega), \quad (16.9)\]

\[
F_{\text{even}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, G_{\text{even}}(\omega) e^{i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega \, G_{\text{even}}(\omega) \left( e^{i\omega t} + e^{-i\omega t} \right)
\]

\[= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} d\omega \, G_{\text{even}}(\omega) \cos \omega t. \]

Thus we define the Fourier cosine transform,

\[G_{c}(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} dt \, F_{\text{even}}(t) \cos \omega t, \quad (16.10)\]

\[F_{\text{even}}(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} d\omega \, G_{c}(\omega) \cos \omega t. \]

As an example consider the (non-periodic) function in the figure defined by

\[F(t) = \begin{cases} 
1, & -1 < t < 1 \\
0, & |t| > 1
\end{cases}. \quad (16.11)\]

The function is non-zero only in the interval of length 2 symmetric about the origin. Substituting into Eq. (16.4) we find the Fourier transform to be

\[G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, F(t) e^{-i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} dt \, e^{-i\omega t} \]

\[= \frac{1}{\sqrt{2\pi}} \left[ e^{-i\omega t} \right]_{-1}^{1} = \frac{e^{-i\omega} - e^{i\omega}}{-i\omega \sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \sin \omega \frac{\sin \omega}{\omega}. \quad (16.12)\]

The inverse transform looks like
\[ F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega G(\omega) e^{i \omega t} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\sin \omega}{\omega} e^{i \omega t} \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\sin \omega \cos \omega t + i \sin \omega \sin \omega t}{\omega} \]

\[ = \frac{2}{\pi} \int_{0}^{\infty} d\omega \frac{\sin \omega \cos \omega t}{\omega}. \]  (16.13)

This leads us to the interesting result for the last integral

\[ \int_{0}^{\infty} d\omega \frac{\sin \omega \cos \omega t}{\omega} = \begin{cases} \frac{\pi}{2} |t| < 1 \\ 0, |t| > 1 \\ \frac{\pi}{4}, |t| = 1 \end{cases} \]  (16.14)

where the last result follows from the fact that the expansion converges to the midpoint of any jump discontinuity.

Next we will list and briefly discuss further special relations for the Fourier transform. First recall that the Fourier transform is a linear operation. Thus it follows that (introducing the symbol \( \mathcal{F}[F(t)] \) for the transform)

\[ \mathcal{F}[F_1(t) + F_2(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i \omega t} \{ F_1(t) + F_2(t) \} = G_1(\omega) + G_2(\omega), \]

\[ \mathcal{F}[cF(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i \omega t} \{ cF(t) \} = cG(\omega). \]  (16.15)

- If two functions of time are just translations (in time by \( t_0 \)) of each other,

\[ F_2(t) = F_1(t + t_0) , \]
then the Fourier transforms are simply related by a (complex) exponential factor

\[
G_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} F_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} F_1(t + t_0)
= e^{+i\omega t_0} \int_{-\infty}^{\infty} dt' \, e^{-i\omega t'} F_1(t') \bigg|_{t'=t+t_0} = e^{+i\omega t_0} G_1(\omega).
\] (16.16)

- If two functions of time are related by a (real) exponential factor,

\[
F_2(t) = e^{-\gamma t} F_1(t),
\] (16.17)

then the Fourier transforms are simply related by a (complex) translation (in \(\omega\) space)

\[
G_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} F_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} e^{-\gamma t} F_1(t)
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} F_1(t) \bigg|_{\omega'=\omega-i\gamma} = G_1(\omega') = G_1(\omega - i\gamma).
\] (16.18)

- If two functions of time are related by a factor of time,

\[
F_2(t) = t F_1(t),
\] (16.19)

then the Fourier transforms are simply related by a derivative

\[
G_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} F_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} t F_1(t)
= \frac{1}{\sqrt{2\pi}} \frac{1}{-i} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} F_1(t) = i \frac{d}{d\omega} G_1(\omega).
\] (16.20)
Finally consider a function of time defined as a convolution of two other functions of time (this idea of a convolution will have many uses),

\[ F_3(t) \equiv \int_{-\infty}^{\infty} dt' F_1(t') F_2(t-t'). \] (16.21)

The corresponding Fourier transform is just a product,

\[ G_3(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} F_3(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' e^{-i\omega t'} F_1(t') F_2(t-t') \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt'' F_2(t'') e^{-i\omega t'} \bigg|_{t'' = t-t'} \int_{-\infty}^{\infty} dt' e^{-i\omega t'} F_1(t') \] (16.22)

\[ = \sqrt{2\pi} G_1(\omega) G_2(\omega). \]

Thus, modulo the constant factor \( \sqrt{2\pi} \), the Fourier transform of a convolution (often written as \( F_1 \ast F_2(t) \)) is the product of the individual transforms. This result is symmetrical in the sense that, if the Fourier transform of some function can be expressed as the convolution (in \( \omega \) space) of 2 known Fourier transforms, the inverse transform is just the (ordinary) product of the corresponding functions of time,

\[ G_4(\omega) = \int_{-\infty}^{\infty} d\omega' G_1(\omega') G_2(\omega - \omega'), \] (16.23)

\[ \Rightarrow F_4(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{i\omega t} G_4(\omega) = \sqrt{2\pi} F_1(t) F_2(t). \]

We can use this last result to obtain the Fourier transform version of Parseval’s Theorem. We start by noting that simple substitution says that (note no change in the exponential)

\[ G_1^*(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} F_1^*(t). \] (16.24)
We can use this result to derive the conclusion that the Fourier transform of the product of a function (of time) with its complex conjugate is the convolution of the Fourier transform with the conjugate transform with a minus sign in the argument (and with no $\sqrt{2\pi}$ factors). Starting with Eq. (16.23) and making the indicated substitutions and taking the limit $\omega \to 0$ we find

$$
\int_{-\infty}^{\infty} dt e^{-i\omega t} F_1(t) F_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} \left( \sqrt{2\pi} F_1(t) F_2(t) \right)
$$

$$
= G_4(\omega) = \int_{-\infty}^{\infty} d\omega' G_1(\omega')G_2(\omega - \omega')
$$

substitute $F_2(t) \to F_1^*(t)$, $G_2(\omega - \omega') \to G_1^*(-(\omega - \omega'))$ (16.25)

$$
\Rightarrow \int_{-\infty}^{\infty} dt e^{-i\omega t} |F_1(t)|^2 = \int_{-\infty}^{\infty} d\omega' G_1(\omega')G_1^*(-(\omega - \omega'))
$$

$$
\xrightarrow{\omega = 0} \int_{-\infty}^{\infty} dt |F_1(t)|^2 = \int_{-\infty}^{\infty} d\omega' |G_1(\omega')|^2.
$$

In words, the power ($\sim |F_1(t)|^2$) integrated over all time is equal to the energy spectrum versus frequency ($\sim |G_1(\omega)|^2$) integrated over all frequencies. The total energy is the total energy.

So now we can apply this expansion to the original differential equation, $a\ddot{x} + b\dot{x} + cx = F(t)$, where we expect to again turn the differential equation into an algebraic equation. We label the Fourier transform of the variable $x(t)$ by $\tilde{x}(\omega)$,

$$
\mathcal{F}[x(t)] \equiv \tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt x(t)e^{-i\omega t}. \tag{16.26}
$$

Next we need the transform of the derivative,
\[ x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \tilde{x}(\omega) e^{i\omega t} \]

\[ \Rightarrow \dot{x}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \tilde{x}(\omega) i\omega e^{i\omega t} \]

\[ \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \dot{x}(t) e^{-i\omega t} = i\omega \tilde{x}(\omega) \]

\[ \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \ddot{x}(t) e^{-i\omega t} = (i\omega)^2 \tilde{x}(\omega). \]

So transforming the original differential equation yields

\[ a\dddot{x}_p + b\dot{x}_p + cx_p = F(t) \]

\[ \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \left\{ a\dddot{x}_p + b\dot{x}_p + cx_p = F(t) \right\} e^{-i\omega t} \]

\[ \Rightarrow \left( -a\omega^2 + ib\omega + c \right) \tilde{x}_p(\omega) = G(\omega) \]

\[ \Rightarrow \tilde{x}_p(\omega) = \frac{G(\omega)}{-a\omega^2 + ib\omega + c} = \frac{G(\omega)}{a(i\omega - \alpha_1)(i\omega - \alpha_2)} \]

\[ \Rightarrow x_p(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega e^{i\omega t} \tilde{x}_p(\omega) \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d\omega G(\omega)e^{i\omega t}}{-a\omega^2 + ib\omega + c} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d\omega G(\omega)e^{i\omega t}}{a(i\omega - \alpha_1)(i\omega - \alpha_2)}. \]

This last expression is the particular solution for a general time dependent (and integrable) driving function. The remaining, perhaps challenging, step in this analysis is to learn how to perform the inverse transform, \(i.e.,\) to evaluate the above integral. We turn to that subject next as it allows us to discuss the especially interesting subject of analytic functions of complex variables.