Lecture 14: Introduction to (Review of) Differential Equations [See Chapter 8, Sections 1 to 7, in Boas. We will return for the rest of Chapter 8 shortly.]

As we have noted several times, most of the physical systems that we are interested in can be described by (systems of) differential equations. Here we want to begin to develop the tool set we need to systematically consider the issue of solving differential equations. To begin let’s review some terminology. Differential equations are labeled ordinary or partial depending on whether the derivatives that appear in the equation are ordinary, e.g., \( (d/dx) \), implying a single independent variable, or partial, e.g., \( (\partial/\partial x) \), implying more than 1 independent variable (the issue is – who is being varied and who is held fixed – we’ll return to this issue). The order of a differential equation defines the highest derivative present in the equation. For example, a first order equation involves only first derivatives, \( y'(x) = dy/dx \) (note the fairly standard notation that a derivative with respect to \( x \) is represented by a prime, while derivatives with respect to time are represented by dots), and a 2\(^{nd} \) order equation involves \( y''(x) = d^2y/dx^2 \) (and possibly \( y' \)). A linear differential equation is one where each term in the equation involves the dependent variable, or its derivatives, to no more than the first power, e.g., \( y'' + by' + cy = f(x) \) is a linear second order ordinary differential equation. (Note that nonlinear differential equations have the special feature of exhibiting isolated singular solutions that are not contained in the usual general solutions with arbitrary constants. Nonlinear equations can also exhibit chaotic behavior.) If every term in the linear equation has a single power of the dependent variable, or its derivatives, the equation is said to be homogeneous (if you multiply \( y \) by a constant, you are multiplying every term in the equation by the same constant – but note that the label homogeneous is used in several related ways, see Eq. 8.4.11 and exercise 4.13.1 in Boas). Thus \( y'' + by' + cy = 0 \) is a linear second order homogeneous ordinary differential equation.

A first order linear ordinary (but inhomogeneous) differential equation can often be solved simply by integration

\[
\frac{dy}{dx} = y'(x) = f(x) \Rightarrow dy = y'(x) \, dx
\]

\[
\Rightarrow y(x) = \int dx \, y'(x)
\]

\[
\Rightarrow y(x) = \int dx \, f(x) + C = \int_{x_0}^{x} d\bar{x} \, f(\bar{x}) + y(x_0).
\]
This equation illustrates the expected result that, at least in principle, we solve differential equations by doing integrals. Note also that in the last expression we have been careful to use a “dummy” variable of integration that is explicitly different from the independent variable $x$, which now appears as the upper limit of the integral. We have also made explicit the point that the constant of integration $C$ is just the desired function evaluated at the lower limit of the integral. Since the derivative of an integral with respect to the upper limit is just the integrand evaluated at the upper limit (see Leibniz’ rule in Eq. 4.12.13 of Boas),

$$\frac{d}{dx} \left[ \int_{x_0}^{x} f(\bar{x}) + y(x_0) d\bar{x} \right] = f(x)\bigg|_{x=x_0} = f(x),$$

we are guaranteed to have a solution of the original differential equation satisfying the boundary condition $y = y(x_0)$ at $x = x_0$. Thus the middle expression in the last line of Eq. (14.1) can be thought of as the general solution, while the last expression is the particular solution satisfying the specific boundary conditions (and, perhaps, other conditions).

Note that, to solve a differential equation by simple integration, it is necessary that we can separate all the explicit dependence on the independent variable from the dependence on the dependent variable. Thus for a first order linear homogeneous equation of the form $y' = -\alpha y$ (where $\alpha$ is a constant), we must first divide by $y$ before integrating, putting all of the $y$ dependence on the LHS,

$$y'(x) = -\alpha y(x) \Rightarrow \frac{dy}{y} = -\alpha dx$$

$$\Rightarrow \int \frac{dy}{y} = -\alpha \int dx \Rightarrow \ln(y(x)) = -\alpha x + C$$

$$\Rightarrow y(x) = y(0)e^{-\alpha x}. \quad (14.3)$$

The final expression has used the boundary value at $x = 0$. This (necessary) procedure of separating the variables on the two sides of the equation is called, no surprise, separation of variables and equations that can be solved this way are called separable. The most general separable linear first order equation looks like
\[ y' + P(x) y = Q(x), \] which we can solve using the above ideas. For the homogenous case \( Q = 0 \) we can proceed as above (but now with \( \alpha \) not a constant),

\[ y' + P(x) y = 0 \implies y(x) = C e^{-\int_{x_0}^{x} P(x) \, dx} = y(0) e^{-\int_{x_0}^{x} P(x) \, dx} \equiv y(0) e^{-I(x)}. \] (14.4)

As expected there is a single arbitrary constant (of integration), which we use here to fit the boundary condition at \( x = 0 \). Next we return to the original inhomogeneous problem. This is most easily addressed if we consider the quantity \( y(x) e^{I(x)} \) and perform some manipulations. In particular, we find

\[
\frac{d}{dx} \left( y(x) e^{I(x)} \right) = e^{I(x)} \left( y' + yP(x) \right) = e^{I(x)} \left( y' + yP(x) \right) = e^{I(x)} Q(x)
\]

\[ \implies y(x) e^{I(x)} = \int_{x_0}^{x} dx e^{I(x)} Q(x) + C \]

\[ \implies y(x) = e^{-I(x)} \int_{x_0}^{x} dx e^{I(x)} Q(x) + y(0) e^{-I(x)} \left( I(x) = \int_{x_0}^{x} d\bar{x} P(\bar{x}) \right). \] (14.5)

We recognize the second term on the right-hand-side as the (previously found) solution to the homogeneous equation (including the boundary condition), while the first term is a particular solution to the inhomogeneous problem.

A general non(easily)-separable version of the first order equation can be written in the form (note the dependence on both \( x \) and \( y \), \textit{i.e.}, the \( P \) and \( Q \) functions are different from above)

\[ y'(x) = - \frac{P(x, y)}{Q(x, y)} \iff P(x, y) \frac{dx}{dy} + Q(x, y) \frac{dy}{dx} = 0. \] (14.6)

This expression is written in a form so as to remind us of the 2-D Green’s theorem and the curl-free theorems of Chapter 6 in Boas. In the case that \( \partial P/\partial y = \partial Q/\partial x \) (the curl-free case) we must be able to find another function \( F(x, y) \) such that

\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \iff P(x, y) = \frac{\partial F(x, y)}{\partial x}, Q(x, y) = \frac{\partial F(x, y)}{\partial y}
\]

\[ \implies Pdx + Qdy = dF = 0. \] (14.7)
In this case the original differential equation is (at least implicitly) solved by

\[ F(x, y) = \text{constant.} \quad (14.8) \]

This is a more powerful result than it may first seem because, even if it is not initially true that \( \partial P/\partial y = \partial Q/\partial x \), we may be able to find a factor to multiply by in Eq. (14.6) so that the new functions, \( \overline{P} \) and \( \overline{Q} \), do satisfy this relation. For example, consider

\[ P(x, y) \, dx + Q(x, y) \, dy = 0, \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \]

\[ \Rightarrow f(x, y) \, P(x, y) \, dx + f(x, y) \, Q(x, y) \, dy = P(x, y) \, dx + Q(x, y) \, dy = 0, \quad (14.9) \]

The factor \( f(x, y) \) is called an integrating factor, which is just the role played by \( e^{I(x)} \) above.

**ASIDE**: To see this connection explicitly we put the equation from Eq. (14.5) in the language of Eq. (14.6). We have

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]

\[ \Rightarrow dy + \left( yP(x) - Q(x) \right) \, dx = 0 \Rightarrow \begin{cases} P(x, y) = yP(x) - Q(x) \\ Q(x, y) = 1 \end{cases}, \]

\[ \Rightarrow \frac{\partial P(x, y)}{\partial y} = P(x) \neq \frac{\partial Q(x, y)}{\partial x} = 0. \]

Hence we need an integrating factor, which is just the familiar exponential factor. We then have

\[ I(x) \equiv \int_0^x d\overline{x} P(\overline{x}) \Rightarrow \begin{cases} \overline{P}(x, y) = e^{I(x)} \left( yP(x) - Q(x) \right) \\ \overline{Q}(x, y) = e^{I(x)} \end{cases} \]

\[ \frac{\partial \overline{P}}{\partial y} = P(x) e^{I(x)} = \frac{\partial \overline{Q}}{\partial x}. \]
Thus the corresponding (potential) function $F(x,y)$ is given by

$$F(x, y) = e^{I(x)} y - \int_0^x d\bar{x}Q(\bar{x}) e^{I(\bar{x})} \rightleftharpoons \begin{cases} \frac{\partial F}{\partial x} = \bar{P} = e^{I(x)} \left( yP(x) - Q(x) \right) \\ \frac{\partial F}{\partial y} = \bar{Q} = e^{I(x)} \end{cases}$$

$$F(x, y) = \text{constant} = y(0) \Rightarrow y(x) = y(0) e^{-I(x)} + \int_0^x d\bar{x}Q(\bar{x}) e^{I(\bar{x})}.$$ 

As expected, we obtain again our previous result.

As in Boas consider the example equation

$$xdy - ydx = 0 \Rightarrow P = -y, Q = x \Rightarrow \frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1. \quad (14.10)$$

If we choose an integrating factor $f = 1/x^2$, we have

$$\bar{P} = -\frac{y}{x^2}, \bar{Q} = \frac{1}{x} \Rightarrow \frac{\partial \bar{P}}{\partial y} = -\frac{1}{x^2} = \frac{\partial \bar{Q}}{\partial x}$$

$$\Rightarrow F(x, y) = \frac{y}{x} = c \Rightarrow y(x) = cx, \quad (14.11)$$

$$xdy - ydx = xcdx - cx dx = 0.$$ 

As discussed in Boas, equations which do not fall into one of the above simple forms can often be put in separable, integrable form by an appropriate change of variable.

To illustrate these techniques let us consider familiar problems from mechanics. Note that, while we typically think of mechanics as described by 2nd order equations (i.e., Newton), we can often use conserved quantities, i.e., symmetries, to obtain a 1st order differential equation. Consider (non-relativistic) 1-D motion in a uniform (vertical) gravitational field (motion near the surface of the earth), which is a conservative system with constant (conserved) total energy ($z$ is the dependent variable and $t$ is the independent one),
\[ E_{\text{TOT}} = T + V = \frac{m}{2} z^2 + mgz \Rightarrow \dot{z} = \frac{dz}{dt} = \pm \frac{2}{m} \sqrt{E_{\text{TOT}} - mgz} \]

(14.12)

For a given constant total energy we can solve this problem (as above) by separation of variables. We have

\[ \pm \frac{dz}{\sqrt{\frac{2}{m}(E_{\text{TOT}} - mgz)}} = dt \Rightarrow \pm \int_{z_0}^{\dot{z}} \frac{d\tilde{z}}{\sqrt{\frac{2}{m}(E_{\text{TOT}} - m\tilde{g}\tilde{z})}} = t - t_0, \]

(14.13)

where we have defined \( z(t_0) = z_0 \). Integrating the LHS we find

\[ \pm \left\{ \frac{2}{m} \sqrt{E_{\text{TOT}} - mgz_0} - \frac{2}{m} \sqrt{E_{\text{TOT}} - m\tilde{g}\tilde{z}} \right\} = t - t_0 \]

(14.14)

We can simplify this expression if we evaluate the total energy at \( t_0 \) and define the velocity at that time to be \( \dot{z}(t_0) = \dot{z}_0 \), \( E_{\text{TOT}} = m\dot{z}_0^2/2 + mgz_0 \). Thus we have

\[ \pm \left\{ \sqrt{\dot{z}_0^2 + 2mgz_0 - 2mgz_0} - \frac{2}{m} \sqrt{E_{\text{TOT}} - mgz} \right\} = \left\{ \dot{z}_0 \pm \sqrt{\frac{2}{m}(E_{\text{TOT}} - mgz)} \right\} = g(t - t_0) \]

\[ \Rightarrow \frac{2}{m} (E_{\text{TOT}} - mgz) = \dot{z}_0^2 - 2g(z - z_0) = \left( \dot{z}_0 - g(t - t_0) \right)^2 \]

\[ = \dot{z}_0^2 - 2g\dot{z}_0(t - t_0) + g^2(t - t_0)^2 \]

(14.15)

\[ \Rightarrow z = z_0 + \dot{z}_0(t - t_0) - \frac{g}{2}(t - t_0)^2. \]
This final result should be familiar from introductory physics. We can, of course, obtain it directly by integrating the second order Newton equation using separation of variables twice,

\[
m\ddot{z} = -mg \Rightarrow \int_{z_0}^{\dot{z}} dz = -\int_{t_0}^{t} dt \Rightarrow \dot{z} = \dot{z}_0 = -g(t - t_0)
\]

\[
\Rightarrow \int_{z_0}^{z} dz = z - z_0 = \int_{t_0}^{t} dt \{ \dot{z}_0 - g(t - t_0) \} = \dot{z}_0(t - t_0) - \frac{g}{2}(t - t_0)^2.
\]

(14.16)

ASIDE: We can also use similar methods for motion in the 3-D, conservative gravitation potential of the form \(V(r) = -\frac{GMm}{r}\) in spherical coordinates (where \(G\) is Newton’s gravitational constant, \(M\) is the mass of the earth and \(m\) is the mass of the test particle). For purely radial motion (\(\Theta = \Phi = 0\)) conservation of energy now yields

\[
E_{TOT} = \frac{m}{2} \dot{r}^2 - \frac{GMm}{r}
\]

\[
\Rightarrow \dot{r} = \pm \sqrt{\frac{2GM}{r} + \frac{2E_{TOT}}{m}}.
\]

Again this equation can be separated and integrated (although with a bit more effort than in 1-D). But note that we can already see the general structure of the results. If \(E_{TOT} > 0\), then any trajectory with \(\dot{r}(t = 0) > 0\) (the plus sign above) will continue to have \(\dot{r}(t) > 0\) even as \(r \to \infty\), i.e., the mass escapes from the earth. For \(E_{TOT} < 0\) (which is now possible) and \(\dot{r}(t = 0) > 0\) there will be a turning point where \(\dot{r} = 0\), \(r = GMm/|E_{TOT}|\), and the mass subsequently falls back towards the earth. For the intermediate situation \(E_{TOT} = 0\), the mass “just” escapes from the earth with \(\dot{r} \to 0\) as \(t\) and \(r \to \infty\). For more general motion (non-zero angular motion) we can still simplify the problem by using the fact that angular momentum is conserved in a central potential.

As a final comment on first order equations, we note that nonlinear equations like \(y' = \sqrt{1 - y^2}\) can exhibit both general solutions with an arbitrary constant, e.g.,
\( y = \sin(x - x_0) \), and also singular (isolated) solutions like \( y = \pm 1 \) that do not correspond to a value of the constant of integration \( x_0 \). The singular solutions form the tangent (boundary) to the general solutions.

Returning to the general issue of differential equations in physics, as noted above we are typically interested in 2\(^{\text{nd}}\) order linear differential equations, including those that cannot be reduced to a 1\(^{\text{st}}\) order form. We will focus first on systems with only a single independent variable, usually time, where only ordinary derivatives occur (with respect to the independent variable). Further we will deal first with the general behavior of solutions to 2\(^{\text{nd}}\) order linear ordinary differential equations with coefficients that are \textit{constants}. A simple example is 1-D motion in gravity as in Eq. (14.16). This situation becomes somewhat more interesting if we add viscous (velocity-dependent) damping (due to air resistance),

\[
m\ddot{z} + b\dot{z} = -mg. \tag{14.17}
\]

We can often proceed by treating the 2\(^{\text{nd}}\) order equation as a 1\(^{\text{st}}\) order equation. With the change of variables \( \dot{z} = v \) we obtain

\[
\dot{v} + \frac{b}{m} v = -g \tag{14.18}
\]

This is an equation amenable to an integrating factor as in Eq. (14.5) above. We identify \( P = b/m \) and \( Q = -g \). Thus we find

\[
I(t) = \frac{b}{m} (t - t_0),
\]

\[
\frac{d}{dt} (ve^l) = -ge^l
\]

\[
\Rightarrow v(t) = -ge^{-l(t)} \int_{t_0}^{t} d\bar{t} e^{(\bar{t} - t_0)b/m} + v_0 e^{-l(t)} \left[ v(t_0) \equiv v_0 \right]
\]

\[
= -\frac{mg}{b} e^{-l(t)} \left[ e^{(\bar{t} - t_0)b/m} \right]_{t_0}^{t} + v_0 e^{-l(t)}
\]

\[
= -\frac{mg}{b} \left(1 - e^{-l(t)}\right) + v_0 e^{-l(t)}. \tag{14.19}
\]
Hence, if we release an object at rest, \( v_0 = 0 \), the large time (\( t \to t_0, e^{-t_0} \to 0 \)) or terminal velocity is just \( v_\infty = -mg/b \neq -\infty \) and the 2 exponentials have gone to zero. Due to the damping the falling object reaches a maximum velocity. The result is, in fact, true for any initial velocity, \( v(t) = v_\infty + (v_0 - v_\infty) e^{-t(t)} \). To find \( z(t) \) we can integrate once more.

As mentioned in previous lectures (recall especially Lecture 5), more familiar examples of 2\textsuperscript{nd} order linear differential equations (with constant coefficients) arise in the study of electric circuits and the motion of a mass on a spring in 1-D (the harmonic oscillator), including damping. By the rule of Feynman, that the same equations have the same solutions, we can study the general case and then customize the coefficients to match the specific system. Due to the importance of these systems we will (once more) discuss the solutions systematically and with full generality. The simplest and most familiar case is the \textit{homogeneous} case with no “right-hand-side” for the differential equation, \textit{i.e.}, no external driving force or voltage. Taking \( x \) to label the dependent variable or coordinate (the location of the mass, or the charge on a capacitor), the general form of the equation of interest is

\[
ad\ddot{x} + b\dot{x} + cx = 0, \quad (14.20)
\]

where \( a, b, c \) are know constants (> 0), which we take to be real for now, and \( \dot{x} = dx/dt, etc. \) As we have previously discussed, we make use of the simple behavior of exponentials under the derivative operation and try the Ansatz \( x(t) = x_0 e^{at} \). Then Eq. (14.20) becomes

\[
\left( a\alpha^2 + b\alpha + c \right) x_0 e^{at} = 0
\]

\[
\Rightarrow a\alpha^2 + b\alpha + c = 0 \quad (14.21)
\]

\[
\Rightarrow \alpha_{1,2} = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}.
\]

With \( D \) representing the derivative with respect to the independent variable we can also write this equation in the notation of Boas,

\[
ad\ddot{x} + b\dot{x} + cx = a\left( D - \alpha_1 \right)\left( D - \alpha_2 \right) x = 0. \quad (14.22)
\]
Thus the general solution to the homogenous equation, also called the complementary solution, can be written

\[ x_c(t) = x_1 e^{\alpha_1 t} + x_2 e^{\alpha_2 t}, \quad (14.23) \]

where \( x_1, x_2 \) are constants to be fit to the initial conditions, \( x(0), \dot{x}(0) \). Note that the exponents \( \alpha_{1,2} \) can be complex, and that the \( \alpha_{1,2} \) will be pure imaginary if there is no damping, \( b = 0 \). To make a connection to our previous discussions let us go through this same argument with an explicitly complex Ansatz \( x(t) = \text{Re} \left[ z_0 e^{i\beta t} \right] \). The complex analogue of Eq. (14.21) is

\[
\begin{align*}
(-a\beta^2 + ib\beta + c)z_0 e^{i\beta t} &= 0 \\
\Rightarrow -a\beta^2 + ib\beta + c &= 0 \\
\Rightarrow \beta_{1,2} &= i \frac{b}{2a} \pm \sqrt{c - \frac{b^2}{4a^2}} \equiv i\gamma \pm \omega_0'
\end{align*}
\]

(14.24)

In this case the complementary solution is written in the familiar form

\[
\begin{align*}
x_c(t) &= \text{Re} \left[ z_0 e^{-\gamma t} e^{i\omega_0' t} \right] = |z_0| e^{-\gamma t} \cos(\omega_0' t + \phi_0) \\
&= |z_0| e^{-\gamma t} \left( \cos \omega_0' t \cos \phi_0 - \sin \omega_0' t \sin \phi_0 \right).
\end{align*}
\]

(14.25)

where, as usual, \( \phi_0 \) is the phase of the complex constant \( z_0 \). Note, in particular, that as long as \( \omega_0 \) is real (and the exponential is complex), the operation of taking the Real part includes the second solution with the \(-\omega_0 \) exponent and we don’t need to explicitly include it to find the general solution. On the other hand, if \( \omega_0 \) is imaginary (and the exponentials are real) as in Eq. (14.23), we do need to include both exponentials. In any case, we still have 2 (real) constants to be determined by the initial conditions. We are just writing the same result in a different notation. We can make the identifications
\[ \gamma = \frac{b}{2a} = -\frac{\alpha_1 + \alpha_2}{2}, \]
\[ \omega_0 = \sqrt{\frac{c - \frac{b^2}{4a^2}}{a}} = i \frac{\alpha_1 - \alpha_2}{2}, \]
\[ \alpha_1 = -\gamma - i\omega_0, \alpha_2 = -\gamma + i\omega_0. \]  

In either notation the explicit form of the solution will depend on the values of the 
(physics dependent) constants \( a, b, c \). So let us list the various possibilities.

- \( \frac{b^2}{4a^2} > \frac{c}{a} \) - with the damping dominant, the \( \alpha_{1,2} \) are both real and negative, 
while the “frequency” \( \omega_0 \) is imaginary and smaller in magnitude than \( \gamma \). 
This is the “over-damped” case with the complementary solution exhibiting 
only exponentially damped behavior as in Eq. (14.23).

- \( \frac{b^2}{4a^2} = \frac{c}{a} \), \( \alpha_1 = \alpha_2 = -\frac{b}{2a} = -\gamma \equiv \alpha_0 \) - the critically 
damped case where the 
complementary solution looks like \( x_c(t) = (x_1 t + x_2) e^{-\gamma t} \) (the reader is 
encouraged to check that this is a solution for any value of the arbitrary 
constants).

- \( \frac{b^2}{4a^2} < \frac{c}{a} \), \( \alpha_{1,2} = -\frac{b}{2a} \pm i \sqrt{\frac{c}{a} - \frac{b^2}{4a^2}} = -\gamma \pm i\omega_0 \) - complex conjugates, 
the under-damped, oscillatory case with \( x_c(t) = d_0 e^{-\gamma t} \cos(\omega_0 t + \phi_0) \), where the 
constants to fit the initial conditions are now \( d_0, \phi_0 \). Note that 
\( \omega_0 < \bar{\omega} = \sqrt{c/a}, \omega_0^2 = \bar{\omega}^2 - \gamma^2 \), where \( \bar{\omega} \) is the natural frequency of the system 
with no damping.

As a first look at the inhomogeneous problem with a “driving term”, consider the 
general form \( a\ddot{x} + b\dot{x} + cx = F(t) \). We base the general solution on the 
(complementary) solution of the homogeneous problem plus a(ny) particular solution 
of the inhomogeneous equation, \( x(t) = x_c(t) + x_p(t) \). Thus our goal is now to 
find particular solutions for the various possible forms of \( F(t) \). Note these particular
solutions are not unique, we can always add any amount of the solution of the homogenous problem. So we consider some familiar forms for the driving function:

- \( F(t) = \sum_{n=0}^{N} \beta_n t^n \), a power series in \( t \). Here we use the method of undetermined coefficients to write the particular solution as a corresponding power series, \( x_p(t) = \sum_{n=0}^{N} b_n t^n \), and solve for the \( b_n \) in terms of the \( \beta_n \) and \( a, b, c \) by equating the coefficients of the powers of \( t \) on the two sides of the initial equation. For example,
  
  - \( N = 0 \), \( b_0 = \beta_0 / c \);
  
  - \( N = 1 \), \( b_1 = \beta_1 / c, b_0 = (c\beta_0 - b\beta_1) / c^2 \);
  
  - \( N = 2 \), \( b_2 = \beta_2 / c, b_1 = (c\beta_1 - 2b\beta_2) / c^2, b_0 = (c^2\beta_0 - 2ac\beta_2 - b(d\beta_1 - 2t\beta_3)) / c^3 \);
  
  - etc.

- \( F(t) = \eta e^{\kappa t} \), an exponential, where \( \kappa \neq \alpha_1, \alpha_2 \). With the Ansatz \( x_p(t) = be^{\kappa t} \) we find easily that \( x_p(t) = \eta e^{\kappa t} / \left[ a(\kappa - \alpha_1)(\kappa - \alpha_2) \right] \). Note that this works also if \( \kappa = i\omega \), i.e., a sinusoidal driving function, \( F(t) = \text{Re}[\eta e^{i\omega t}] \). In this case the maximum response occurs when \( \omega \approx \omega = \sqrt{c/a} \), i.e., when you drive the system near its natural frequency. To see this we note that the particular solution for a sinusoidal driving function (in complex notation, \( z_p(t) = z_0 e^{i\omega t} \)) is given by

\[
z_p(t) = \frac{\eta e^{i\omega t}}{a(i\omega - \alpha_1)(i\omega - \alpha_2)} = \frac{\eta e^{i\omega t}}{a(i(\omega + \omega_0) + \gamma)(i(\omega - \omega_0) + \gamma)}
\]

\[
= \frac{\eta e^{i\omega t}}{a(\gamma^2 + \omega_0^2 - \omega^2 + 2i\gamma\omega)} = \frac{\eta e^{i\omega t}}{a(\omega^2 - \omega_0^2 + 2i\gamma\omega)}
\]

\[
= \frac{\eta e^{i(\omega \delta - i\delta)}}{a\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}}, \delta = \tan^{-1}\left(\frac{2\gamma\omega}{\omega^2 - \omega_0^2}\right).
\]

The general form of this solution (or more correctly its square) is called a
Lorentzian shape and it arises so commonly that web pages are dedicated to it. From this result we see that the driving force and the response are generally “out of phase”, i.e., $\delta \neq 0$.

The other question we want to answer is when is the magnitude of the response largest? To answer this accurately we need to be careful because there are really two questions. If we mean for a fixed system, i.e., fixed values of $\gamma, \omega_0$, we ask what value of the driving frequency $\omega$ yields the largest response, i.e., we want to find when $\frac{\partial |z_p|}{\partial \omega} = 0$. In this case the response is largest, i.e., the magnitude of the denominator is minimum, for $\omega^2 = \bar{\omega}^2 - 2\gamma^2 = \omega_0^2 - \gamma^2$.

However, we could also ask what value of $\bar{\omega}$ should we “tune” the system to in order to obtain the largest response from a given driving frequency $\omega$ (this is how radio tuners work). Here we want $\frac{\partial |z_p|}{\partial \bar{\omega}} = 0$ and the answer is $\bar{\omega} = \omega$. In both cases the large response is labeled a resonance.

- $F(t) = \eta e^{\kappa t}$, but with $\kappa = \alpha_1$ (or $\kappa = \alpha_2$). We must proceed carefully as we did for the degenerate homogeneous case. The trick again is to add an extra power of $t$ and try the Ansatz $x_p(t) = bt e^{\kappa t}$. Then we find $x_p(t) = \eta t e^{\alpha_1 t} / a(\alpha_1 - \alpha_2)$.

- $F(t) = \eta e^{\kappa t}$, but with $\kappa = \alpha_1 = \alpha_2$. We must proceed even more carefully and try $x_p(t) = bt^2 e^{\kappa t}$. Then we find $x_p(t) = \eta t^2 e^{\alpha t} / 2a$. The reader is encouraged to check this result.

- $F(t) = e^{\kappa t} \sum_{n=0}^{N} \beta_n t^n$, where we combine our previous results. We can write a particular solution in terms of undetermined coefficients, $x_p(t) = e^{\kappa t} \sum_{n=0}^{N} b_n t^n$, and solve for the coefficients term by term.

The underlying idea is this last example is the every important concept of linear superposition, which is always applicable to a linear equation. If the right-hand-side of the inhomogeneous equation is a sum of terms, we find the particular solution as a sum of particular solutions for each of the individual driving terms and sum them up. In the next lectures we will look at periodic driving terms in more detail, i.e., use Fourier series techniques.