Lecture 5 – Appendix B: Some sample problems from Boas

Here are some solutions to the sample problems assigned for Chapter 2.5, 2.7, 2.9, 2.10 and 2.11.

§2.5: 46

Solution: We want to find all solutions of the complex equation

\[
\frac{x + iy}{x - iy} = -i \Rightarrow \frac{(x + iy)^2}{x^2 + y^2} = \frac{x^2 - y^2 + 2ixy}{x^2 + y^2} = -i
\]

\[
\Rightarrow x^2 - y^2 + 2ixy = -i\left(x^2 + y^2\right)
\]

\[
\Rightarrow \begin{cases} \text{Re: } x^2 = y^2 \\ \text{Im: } 2xy = -x^2 - y^2 \Rightarrow \begin{cases} x = \pm y \Rightarrow x = -y. \end{cases} \end{cases}
\]

So the solution is the indicated line, \( x = -y \).

§2.7: 16

Solution: Here we want to find the disk of convergence for the following complex series

\[
S = \sum_{n=1}^{\infty} 2^n (z + i - 3)^{2n}.
\]

Applying the ratio test we learn

\[
\rho_n = \frac{2^{n+1}}{2^n} \left| \frac{(z + i - 3)^{2n+2}}{(z + i - 3)^{2n}} \right| = 2 \left| z - 3 \right|^2 = \rho < 1
\]

\[
\Rightarrow |z + i - 3| = |x - 3 + iy + i| < \frac{1}{\sqrt{2}} \Rightarrow (x - 3)^2 + (y + 1)^2 < \frac{1}{2}.
\]

It is probably easiest to think about this result for a circle about a point in the form

\[
|z - z_0| < \frac{1}{\sqrt{2}}, \quad z_0 = 3 - i,
\]
as indicated in the figure.

§2.9: 24

**Solution:** We want to express the following complex number in rectangular form. Since there are powers, using the polar form as an intermediate step is easiest. We have

\[
\begin{align*}
    z &= \left(1 - i\sqrt{3}\right)^{21} \frac{1}{(i-1)^{38}} \\
    z &= \left(2e^{-i\pi/3}\right)^{21} \frac{1}{\left(\sqrt{2}e^{i\pi/4}\right)^{38}} \\
    &= 4e^{-i3\pi/2} = 4e^{i\pi/2} = 4i.
\end{align*}
\]

This calculation would be much messier in fully rectangular notation. On the other hand, it is a calculation that is trivial using Mathematica.

§2.10: 16

**Solution:** Here we want to practice finding complex roots. We want to find all solutions of the following equation and the easiest path is using polar forms
As expected the solutions lie equally spaced (by $\Delta \theta = \pi/3 = 2\pi/6$) on the unit circle.

§2.10: 22

Solution: Again we want to practice finding complex roots. We want to find all solutions of the following equation and the easiest path is using polar forms

$$z = \sqrt[6]{-1} = \begin{cases} 
\sqrt[6]{e^{i\pi}} = e^{i\pi/6} = \frac{\sqrt{3} + i}{2} \\
\sqrt[6]{e^{i3\pi}} = e^{i\pi/2} = i \\
\sqrt[6]{e^{i5\pi}} = e^{i5\pi/6} = \frac{-\sqrt{3} + i}{2} \\
\sqrt[6]{e^{i7\pi}} = e^{i7\pi/6} = \frac{-\sqrt{3} - i}{2} \\
\sqrt[6]{e^{i9\pi}} = e^{i3\pi/2} = -i \\
\sqrt[6]{e^{i11\pi}} = e^{i11\pi/6} = \frac{\sqrt{3} - i}{2} 
\end{cases}$$

$$z = \sqrt[3]{2i - 2} = \begin{cases} 
\sqrt[3]{8e^{i3\pi/4}} = \sqrt[3]{2e^{i\pi/4}} = 1 + i \\
\sqrt[3]{8e^{i11\pi/4}} = \sqrt[3]{2e^{i11\pi/12}} = \frac{-\sqrt{3} - 1 + i\left(\sqrt{3} - 1\right)}{2}, \\
\sqrt[3]{8e^{i9\pi/4}} = \sqrt[3]{2e^{i9\pi/12}} = \frac{\sqrt{3} - 1 - i\left(\sqrt{3} + 1\right)}{2} 
\end{cases}$$

$$\cos\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{3} - 1}{2\sqrt{2}}, \quad \sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{3} - 1}{2\sqrt{2}},$$

$$\cos\left(\frac{19\pi}{12}\right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}, \quad \sin\left(\frac{19\pi}{12}\right) = \frac{-\sqrt{3} - 1}{2\sqrt{2}}.$$
As expected the solutions lie equally spaced (by $\Delta \theta = 2\pi/3 = 8\pi/12$) on the circle of radius $\sqrt{2}$.

§2.11: 5

Solution: We want to practice converting from polar form to rectangular form. We simplify follow our noses to convert the various factors,

$$z = e^{i\theta/4} / e^{i(\ln 2)/2} = e^{i\pi/8} e^{i(\ln 2)/2} = \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \sqrt{2} = \frac{1+i}{\sqrt{2}} \sqrt{2} = 1+i.$$

$$= 1 + i.$$