A basic and very powerful (if pedestrian, recall we are lazy AND smart) way to solve any differential (or integral) equation is via a series expansion of the corresponding solution (or integrand). (This is always possible in regions where the expansion exists.) Since this process can be carried out term-by-term, it is ideal for numerical solutions by computer to arbitrary, but specified, accuracy, which is especially useful in situations where we have no other way to proceed. (This is also how your hand calculator works.) Recall, in particular, the familiar Taylor series expansion of a function about the point \( x = a \) (also called a Maclaurin series for \( a = 0 \))

\[
f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots (2.1)
\]

where the “primes” signify derivatives with respect to the variable \( x \). We will return to the subject of power series after we first consider the properties of plain old numerical series. Consider an infinite sequence of numbers labeled by an index \( n \)

\[
a_1, a_2, a_3, \ldots, a_n, \ldots ; n = 1, \ldots, \infty,
\]

We define the series to be the corresponding sum,

\[
S \equiv \sum_{n=1}^{\infty} a_n.
\]

The individual terms, \( a_n \), are just numbers or are perhaps expressed as a function of \( n \) (and any other parameters), e.g., \( a_n = cx^n \). This last expression takes us back to the power series expansion mentioned above. In particular, this expression, \( a_n = cx^n \), defines a geometric series where the ratio between subsequent terms is a single factor, \( x \), i.e., the same factor for each pair of contiguous terms.

Do such series arise in physics? The answer is yes, all the time! Consider the simple example of dropping a ball from a height \( h \). For a real ball the collisions with the floor result in the loss of energy, i.e., the collisions are inelastic (some of the kinetic energy of the CM of the ball is converted to heat, i.e., random kinetic energy of the individual molecules in the ball, each time the ball collides with the floor). If we define the fraction of kinetic energy lost in each bounce to be \( 1-x \), then the height
after one bounce is \( xh \) (recall that the height of the bounce is determined by when all the kinetic energy of the CM is converted into gravitational potential energy)

\[
\frac{E_1}{E_0} = \frac{mgh_1}{mgh} = \frac{mghx}{mgh} = x. \tag{2.4}
\]

After \( n \) bounces the height is \( h_n = x^n h = a_n \). Such sequences of numbers that “progress” by a power of a single factor correspond to “geometric progressions”. The geometric progression is the simplest progression. Now we can ask how far does the ball travel before all kinetic energy is lost? (Note that, in principle, this takes infinite time.) In equations we have

\[
D(x) = h + 2h_1 + 2h_2 + \ldots + 2h_n + \ldots
= h\left[1 + 2\left(x + x^2 + \ldots + x^n + \ldots\right)\right] \tag{2.5}
= h\left[1 + 2\sum_{n=1}^{\infty} x^n\right] \equiv h\left[1 + 2S(x)\right].
\]

In these expressions the ellipsis (the 3 dots, \( \ldots \)) represent the implied missing terms. For a fairly elastic ball with \( x = 4/5 \) we find

\[
D\left(\frac{4}{5}\right) = h + 2h\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n \equiv h + 2hS\left(\frac{4}{5}\right). \tag{2.6}
\]

So the question is, can we give meaning (\( i.e., \) a numerical value) to this infinite geometric series \( S(4/5) = \sum_{n=1}^{\infty} (4/5)^n \)? If the sum of the infinite number of terms exists, then that sum is the series, \( i.e., \) the value of the series. But how can we tell if the sum exists? Even if we know that the sum exists, how do we evaluate it? Note that it would take an infinite time for a computer to literally add up an infinite number of terms. We must be smarter and use more powerful techniques as we will see below.

First let’s define some terminology. If the sum exists (\( i.e., \) it is defined) and is less than infinity, we say the series converges. If the sum does not exist (\( i.e., \) it is not defined) or is infinite, we say the series diverges.
Now consider how to evaluate the sum. Since humans, like computers, do best considering finite things, consider only the first $N$ terms,

$$ S_N = \sum_{n=1}^{N} a_n = x + x^2 + \ldots + x^N = \sum_{n=1}^{N} x^n = \sum_{n=1}^{N} \left( \frac{4}{5} \right)^n. \quad (2.7) $$

In an obvious notation, this is called a partial sum. Since this quantity is guaranteed to be finite (assuming $a_n < \infty, n \leq N < \infty$), we can use normal algebraic manipulations (see below for the non-guaranteed, infinite case). Consider the following (true but unmotivated) manipulations,

$$ xS_N = x^2 + x^3 + \ldots + x^{N+1} = \sum_{n=1}^{N} x^{n+1} = S_N + x^{N+1} - x $$

$$ \Rightarrow S_N \left(1 - x\right) = x \left(1 - x^N\right) \quad (2.8) $$

$$ \Rightarrow S_N = x \frac{1 - x^N}{1 - x}. $$

Note that this last expression implies the special and very useful relation (see below)

$$ \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots. \quad (2.9) $$

Now consider the limit

$$ \lim_{N \to \infty} S_N = \frac{x}{1 - x} \left[ 1 - \lim_{N \to \infty} x^N \right]. \quad (2.10) $$

Evaluating series is closely related to taking limits – we must be able to do both! For the general case the possibilities are

1) $\lim_{N \to \infty} S_N$ exists and is finite, in which case the series is said to converge and $S = \lim_{N \to \infty} S_N$;
2) $\lim_{N \to \infty} S_N$ does not exist, because either it exhibits a limit cycle, e.g., 
$S_{10^5} = 1, S_{10^5+1} = -1, S_{10^5+2} = 1, \ldots$, or we have $\lim_{N \to \infty} S_N \to \pm \infty$; in either case the series is said to diverge.

It is often helpful to consider the remainder $R_N \equiv S - S_N$. The series converges iff (if and only if) $\lim_{N \to \infty} R_N = 0$. The series diverges for $\lim_{N \to \infty} R_N \neq 0$, independent of the precise form of the nonzero right-hand-side. Another way to express convergence is to say that, if you pick a small parameter, $\epsilon \ll 1$, I can always find a large integer $M$ such that $|R_N| < \epsilon$ for all $N > M$. (Note that, if you are not comfortable taking these limits, $N \to \infty$, you should practice doing so. It will be very useful below.)

In our simple case above we have the following possibilities (note that $x > 0$ here, the fraction of energy lost in each bounce is less than 100%)

1) $x < 1$, $\lim_{N \to \infty} x^N = 0$ and thus the series converges, with $S = \frac{x}{1-x}$,

2) $x > 1$, $\lim_{N \to \infty} x^N = \infty$ and thus the series diverges,

3) $x = 1$, $\lim_{N \to \infty} x^N = 1$, $S = 1 \frac{1-1}{1-1}$, which is ill-defined, but using the original form we find $S = \sum_{n=1}^{\infty} 1 = \infty$ and the series diverges.

Clearly, we must be careful in this last case. For our specific example we have

$$D = h + 2h \frac{\sqrt{5}}{1-\frac{\sqrt{5}}{4}} = h + 8h = 9h.$$ (2.11)

The distance traveled is finite, even though there are an infinite number of steps. Of course, when the height of the bounce is less than an Angstrom there is little contribution (and the motion is difficult to detect). This series is said to converge quickly. The contribution from the bounces after bounce 100 is

$$D_{N>100} = 2h \sum_{n=101}^{\infty} \left(\frac{4}{5}\right)^n = 2h \left(\frac{4}{5}\right)^{100} S \left(\frac{4}{5}\right) = 1.6 \times 10^{-9} h.$$ (2.12)
Note that in (realistic) physics applications, where the uncertainty in the measurements is always greater than zero, we are typically also interested in comparing to a theoretical calculation, also with finite accuracy, \( i.e., \) the value of \( R_N \) for large \( N \).

Returning to the more general expression for the geometric series, we define

\[
S = \sum_{n=0}^{\infty} ar^n
\]

\[
\Rightarrow S_N = \sum_{n=0}^{N-1} ar^n = \frac{a}{r} \sum_{n=1}^{N} r^n = a \frac{1-r^N}{1-r},
\]

which matches Eq. (1.4) in Boas. For \( |r|<1 \), we can take the limit \( N \to \infty \) as above to find

\[
S = \lim_{N \to \infty} S_N = \frac{a}{1-r}.
\]  

(2.13)

It is clearly important to be able to tell if a series converges or not. Even though divergent series are sometimes useful (see below), we must be careful when using them! In particular, we cannot reliably use the same algebraic manipulations on full sums as we did on the partial sum. For example, we might consider

\[
S = 1 + 2 + 4 + 8 + \ldots = \sum_{n=0}^{\infty} 2^n : [x = 2]
\]

\[
\Rightarrow 2S = 2 + 4 + 8 + \ldots = S - 1 \Rightarrow S = -1,
\]

which is clearly nonsense \((S = \infty)\).

Next we observe the importance of signs! Consider the following similar series (the Harmonic series)

\[
1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n},
\]

(2.16)
which diverges (as we demonstrate below), and

\[ 1 - \frac{1}{2} + \frac{1}{3} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}, \]  

(2.17)

which converges. It is clearly essential that we define convergence tests that we can generally apply, including applying to cases with alternating signs.

Consider again the general expression in Eq. (2.3), \( S = \sum_{n=1}^{\infty} a_n \) with all terms finite, \( |a_n| < \infty, n < \infty \). To this series/sum we can apply the following set of tests.

0) The Preliminary Test or Divergence Test: if \( \lim_{n \to \infty} a_n \neq 0 \), \( S \) diverges, if \( \lim_{n \to \infty} a_n = 0 \), we must test further, i.e., this test is necessary but not sufficient. For example, \( a_n = 1/n \), \( \lim_{n \to \infty} a_n = 0 \) so we test further, but for \( a_n = 2^n \), \( \lim_{n \to \infty} a_n = \infty \), or for \( a_n = 1^n \), \( \lim_{n \to \infty} a_n = 1 \), we know that the series diverges.

**ASIDE:** We can apply this test directly only if all \( a_n \geq 0 \). Otherwise consider \( \bar{S} = \sum_{n=1}^{\infty} |a_n| \). If \( \bar{S} \) converges, then we say \( S \) converges absolutely. If the sum is not absolutely convergent (i.e., \( \bar{S} \) does not converge), \( S \) may still converge conditionally, if there is enough cancellation. We will discuss this case in more detail below.

1) The Comparison Test: compare the series of interest to a series whose convergence properties you already know. This test has two parts.

a) Assume we know that the series \( C = \sum_{n=1}^{\infty} c_n \), with all \( c_n > 0 \), converges. If

\[ |a_n| \leq c_n \text{ for all } n, n \geq N \]  

(with \( N \) some large integer), then \( S \) converges absolutely. Note that the first \( N-1 \) (finite) terms cannot affect convergence, only the numerical value of \( S \) (if it convergences).
b) Assume we know that the series \( D = \sum_{n=1}^{\infty} d_n \), with all \( d_n > 0 \), diverges. If \( |a_n| \geq d_n \) for all \( n, n \geq N \), then \( \overline{S} \) diverges. The series \( S \) may still converge if there are varying signs and enough cancellation.

\textbf{ASIDE:} If \( |a_n| > c_n \) or \( |a_n| < d_n \), we learn nothing from these comparisons.

Consider the example \( S = \sum_{n=1}^{\infty} \frac{1}{n!} \), \( a_n = \frac{1}{n!} \), \( n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \) and compare to the geometric series with \( x = \frac{1}{2} \), \( c_n = \frac{1}{2^n} \), which we know converges. We have

\[
S = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots,
\]

\[
C = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots.
\]

Thus \( a_n < c_n \), \( n > 3 \) and \( S \) converges (very rapidly!).

The downside with the comparison test is that to use it we must first know the convergence properties of many series. So let's develop some other tests.

2) The Integral Test: consider the expression \( \int a_n \, dn \) with the integrand treated as a (smooth) function of \( n \). This will be useful for the case \( 0 \leq a_{n+1} \leq a_n \) for \( n \geq N \) (some large \( N \)). The series converges or diverges depending on whether the integral \( \int a_n \, dn \) is finite or infinite (independent of the finite lower limit). You can see this from the following figures.
For the example (above) of the Harmonic series, \( a_n = \frac{1}{n} \), which is illustrated in the above figures, we learn that

\[
\int_1^\infty \frac{dn}{n} = \ln n \bigg|_1^\infty = \infty.
\]

(2.18)

The Harmonic series (with + signs) diverges as noted earlier in Eq. (2.16). The limitation with this test is that we must be able to perform the integral.

3) The Ratio Test: first define the ratio of adjacent terms in the series and then the asymptotic value of the ratio,

\[
\rho_n \equiv \left| \frac{a_{n+1}}{a_n} \right| \Rightarrow \rho \equiv \lim_{n \to \infty} \rho_n.
\]

(2.19)

Since for the geometric series \( \rho \) is just the fixed ratio of terms, we learn from our analysis above of the geometric series coupled with the comparison test that

\[
< 1, S \text{ converges absolutely} \\
= 1, \text{ may converge, use further test.} \\
> 1, S \text{ diverges}
\]

(2.20)

As familiar examples consider

\[
a_n = \frac{1}{n!} \Rightarrow \rho_n = \frac{n!}{(n+1)!} = \frac{1}{n+1} \Rightarrow \rho = 0 \Rightarrow \text{converges,}
\]

\[
a_n = \frac{1}{n} \Rightarrow \rho_n = \frac{n}{(n+1)} = \frac{1}{1 + \frac{1}{n}} \Rightarrow \rho = 1 \Rightarrow ?
\]

(2.21)
Clearly the case Ρ=1 requires further study. In particular, we need to specify and understand the implications of how Ρₙ approaches 1 as \( n \to \infty \), which we can analyze a step at a time. To see what the possible behavior looks like we consider some examples. Consider first the Riemann Zeta function defined by the series

\[
S(p) = \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}
\]  

(2.22)

and apply the integral test. We have that

\[
I(p) = \int \frac{dn}{n^p} = \left. \frac{1}{1-p} n^{1-p} \right|_1^\infty \Rightarrow \begin{cases} < \infty, & p > 1 \\ = \infty, & p < 1 \end{cases}
\]

\[
= \ln n \bigg|_1^\infty = \infty, \quad p = 1,
\]

(2.23)

where the special case \( p = 1 \) is just the Harmonic series. Thus the Zeta function exists, \( i.e. \), the series converges, for \( p > 1 \) (strictly for \( \text{Re}[p] > 1 \)) and diverges otherwise.

Back to the ratio test, we have for the Zeta function expanding in powers of \( 1/n \ll 1 \) (think about a Taylor series expansion of \( (1+x)^{-p} \) for small \( x \))

\[
\rho_n = \left( \frac{n}{n+1} \right)^p = (1 + \frac{1}{n})^{-p} \xrightarrow[n \to \infty]{} 1 - \frac{p}{n} + \frac{p(p+1)}{2n^2} - \ldots
\]

(2.24)

Thus we conclude, via the comparison test, that for a general series \( S \),

\[
\text{if } \rho_n = 1 - \frac{p}{n} + \ldots \text{ as } n \to \infty \Rightarrow \quad S \text{ converges for } p > 1
\]

\[
S \text{ diverges for } p < 1
\]

(2.25)

while for \( p = 1 \) we must work harder still. Next consider the similar series

\[
S(s) = \sum_{n=2}^{\infty} \frac{1}{n(ln n)^s},
\]

(2.26)
for which the integral test tells us that

\[ \int_{n}^{\infty} \frac{dn}{n (\ln n)^s} \rightarrow \int_{m=\ln n}^{\infty} \frac{dm}{m^s} = \begin{cases} < \infty, & s > 1 \\ \infty, & s \leq 1 \end{cases}, \]  

(2.27)

\[ \frac{\ln n}{(1 + n) \ln (1 + n)^s} \rightarrow \left( \frac{1}{1 + \frac{1}{n}} \right) \left( \frac{1}{1 + \frac{1}{n \ln n}} \right)^s \]  

(2.28)

i.e., the form of the final integral is just as for the Zeta function. On the other hand

the ratio test for this series has the form (think about how to perform the indicated expansions for large \( n \))

\[ \rho_n = \left( \frac{1}{1 + \frac{1}{n}} \right) \left( \frac{\ln n}{(1 + n) \ln (1 + n)^s} \right) \]  

\[ \rightarrow 1 - \frac{1}{n} - \frac{s}{n \ln n} + \ldots, \]

where we saw from the integral test that this corresponds to a convergent sum iff (if and only if) \( s > 1 \). We can summarize these insights in the following general result for the ratio test. Keep the largest terms as \( n \rightarrow \infty \) and define the following sequence of ever more detailed results in this limit.

If \( \rho > 1 \), \( S \) diverges

If \( \rho < 1 \), \( S \) converges

else if \( \rho = 1 \) and \( \rho_n = 1 - \frac{p}{n} + \ldots \) as \( n \rightarrow \infty \)

\[ p < 1, \; S \] diverges

\[ p > 1, \; S \] converges

else if \( \rho = 1 \), \( p = 1 \) and \( \rho_n = 1 - \frac{1}{n} - \frac{s}{n \ln n} + \ldots \) as \( n \rightarrow \infty \)

\[ s < 1, \; S \] diverges

\[ s > 1, \; S \] converges

\[ \Rightarrow \]

\[ s > 1, \; S \] converges

\textit{etc.}

As an example consider again our friend the Harmonic series \( a_n = 1/n \). From Eq. (2.21) we have
\[ a_n = \frac{1}{n} \Rightarrow \rho_n = \frac{1}{1 + \frac{1}{n}} \xrightarrow{n \to \infty} 1 - \frac{1}{n} + \frac{1}{n^2} - \ldots, \quad (2.30) \]

where the last step used Eq. (2.9), but with \( x \to -x \) (a useful result in its own right),

\[ \frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \ldots \left[ |x| < 1 \right]. \quad (2.31) \]

In Eq. (2.30) we see that the coefficient of \(-1/n\) is 1 corresponding to \( p = 1 \) and that the coefficient of \( 1/n \ln n \) is zero (since the next term is \( 1/n^2 \)), \( s = 0 \). Hence by the rules of Eq. (2.29) the Harmonic series diverges as advertised.

4) The Combined Test: combine test 1) and 3) (and 2)), again in two steps.

a) Pick a convergent series \( C = \sum c_n \), \( c_n > 0 \) and take the ratio to (the absolute value of ) terms in \( S = \sum a_n \),

\[ \text{if } \lim_{n \to \infty} \left| \frac{a_n}{c_n} \right| < \infty, \text{ then } S \text{ is absolutely convergent.} \quad (2.32) \]

This means that the terms in \( S \), for large \( n \), are proportional to those in \( C \) and both series converge if one does. In principle, we don’t care if this proportionality constant is large, while in test 1) we were really checking for proportionality constant 1.

b) Pick a divergent series \( D = \sum d_n \), \( d_n > 0 \), and again consider the ratio,

\[ \text{if } a_n > 0 \text{ and } \lim_{n \to \infty} \frac{a_n}{d_n} > 0, \text{ then } S \text{ diverges.} \quad (2.33) \]

Again we must be careful in the case of alternating signs (see below).
This last test allows us to simplify the series we are considering by choosing a simpler comparison series with the same asymptotic behavior, which we illustrate with some examples. Consider the series defined by

\[ a_n = \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}, n \geq 3. \]  

(2.34)

For large \( n \) we have

\[ a_n \xrightarrow[n \to \infty]{} \frac{\sqrt{2n^2}}{4n^3} = \frac{1}{2\sqrt{2}n^2}. \]  

(2.35)

So we don’t need to consider the full complexity of \( a_n \), but can consider instead the simpler series given by \( B = \sum b_n \), \( b_n = 1/n^2 \). Both the integral test and the ratio test tell us that \( B \) converges,

\[ \int_{n=1}^{\infty} \frac{dn}{n^2} = -\frac{1}{n} \bigg|_{n=1}^{\infty} < \infty, \]

(2.36)

\[ \rho_n = \frac{n^2}{(n+1)^2} \xrightarrow[n \to \infty]{} 1 - \frac{2}{n} + \ldots \quad \left[ p = 2 > 1 \right]. \]

Hence we use \( B \) as the convergent comparison series and conclude that \( S \) is convergent,

\[ \frac{a_n}{b_n} \xrightarrow[n \to \infty]{} \frac{1}{2\sqrt{2}} < \infty. \]  

(2.37)

As another example consider

\[ a_n = \frac{3^n - n^2}{n^5 - 5n^2}, n \geq 2 \]

(2.38)

\[ \Rightarrow a_n \xrightarrow[n \to \infty]{} \frac{e^{n\ln 3} - e^{2\ln n}}{n^5} \xrightarrow[n \to \infty]{} \frac{3^n}{n^5}. \]
So try the comparison series $d_n = 3^n/n^5$, which we know diverges from the ratio test,

$$
\rho_n = \frac{3^{n+1}/(n+1)^5}{3^n/n^5} \xrightarrow{n\to\infty} \frac{3}{(1+1/n)^5} > 3 > 1. \quad (2.39)
$$

So the original series diverges since

$$
ad_n = \frac{n^5}{3^n}, \quad \frac{3^n - n^2}{n^5 - 5n^2} = \frac{1 - n^3/3^n}{1 - 5/n^3} \xrightarrow{n\to\infty} 1 > 0. \quad (2.40)
$$

Now let’s return to consider the case of a series with terms alternating in sign, which does not converge absolutely. It can still converge, in which case we say that it exhibits conditional convergence. An interesting example is the alternating sign version of the Harmonic series, $a_n = (-1)^{n+1}/n$. The absolute value version of this series, the usual Harmonic series, we know to diverge. Such a situation is addressed by out last test.

5) The Alternating Sign (Not Absolutely Convergent) Test: a series with terms that alternate in sign, which is not absolutely convergent, is conditionally convergent if (and only if) the terms systematically shrink in magnitude and asymptotically vanish,

$$
|a_{n+1}| \leq |a_n| \quad [n > N], \quad \lim_{n\to\infty} a_n = 0. \quad (2.41)
$$

The trick with conditionally convergent series is determining the value to which they converge. It is easy to be misled on this issue, especially if one just looks at the series. The answer can depend on how the series is arranged. Note that the test in Eq. (2.41) depends on the order of the terms. From our knowledge of the (Maclaurin) series expansion of the logarithm we have

$$
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad (2.42)
$$

which we could obtain directly by integrating Eq. (2.31). Thus we know that the (standard order) alternating sign Harmonic series converges to
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2) = 0.693147\ldots
\] (2.43)

On the other hand, we could arrange the terms in the series in the following explicitly misleading fashion (the right-hand-side is the running sum),

\[
\begin{align*}
1 + \frac{1}{3} + \frac{1}{5} &= 1.5333\ldots \\
- \frac{1}{2} &= 1.0333\ldots \\
+ \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} &= 1.5218\ldots \quad (2.44) \\
- \frac{1}{4} &= 1.2718\ldots \\
+ \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25} &= 1.5144\ldots 
\end{align*}
\]

which could be interpreted to mean convergence to 1.5 if the series is defined in this way. The lesson is that you have to be very careful with infinite series, especially when the signs alternate. In the later case we are effectively subtracting two divergent series in order to obtain a conditionally convergent series. (In the case above we have the two series \( A = \sum_{n=0}^{\infty} \frac{1}{2n+1} \) and \( B = \sum_{n=1}^{\infty} \frac{1}{2n} \), \( S = A - B \).)

Clearly the order in which we combine the terms can matter.

We close this discussion with a summary of useful facts about infinite series, which follow from the above considerations.

1) Convergence of a series is not affected by overall multiplication (i.e., multiplying every term) by a constant, or by changing a finite number of terms.

2) Two convergent series, \( A = \sum a_n \), \( B = \sum b_n \), can be added or subtracted term by term to obtain another convergent series, \( C = \sum (a_n \pm b_n) = A \pm B \), i.e., convergent series can be treated like ordinary numbers. (This is not true of divergent series.)
3) The terms in an absolutely convergent series can be re-arranged without fear of changing the convergence or the value of the series. Conditionally convergent series depend on the order of the terms, both for the question of convergence and for the value of the series (i.e., the sum).