Lecture 24 2nd Order Partial Differential Equations I

In this Lecture we discuss solutions of differential equations expressed as expansions in terms of the appropriate functions we have learned about in the course. The specific choices of functions and arguments correspond to the specific boundaries, and especially the boundaries with zero boundary values. The coefficients in these expansions are then determined from fitting the non-zero boundary conditions. While Mathematica has some facility for directly solving partial differential equations, the most important application for us will be evaluating the integrals to find the coefficients and then making plots of the resulting solutions - the desired distributions, typically in more than 1 dimension.

Consider first the 1-D case were Mathematica can easily do everything, including matching the boundary conditions. For the example in the lecture we have

\[
\text{In[97]} = \text{DSolve}[y''[x] == 0, y[x], x]
\]

\[
\text{Out[97]} = \{(y[x] \to C[1] + x C[2])\}
\]

and with boundary conditions

\[
\text{In[98]} = \text{DSolve}[y''[x] == 0, y[0] == c, y'[0] == 0, y[x], x]
\]

\[
\text{Out[98]} = \{(y[x] \to c)\}
\]

\[
\text{In[99]} = \text{DSolve}[D[\Psi[x, y], \{x, 2\}] + D[\Psi[x, y], \{y, 2\}] == 0, \Psi[x, y], \{x, y\}]
\]

\[
\text{Out[99]} = \{\{\Psi[x, y] \to C[1] [i x + y] + C[2] [-i x + y]\}\}
\]

In this case Mathematica is telling us that we want solutions as functions of argument \(\pm i x + y\), which is true but not so useful. It is much easier to use the known expansions in Eq. (24.4) and use Mathematica to find the coefficients. So consider the example starting on page 2 of the Lecture with a non-zero boundary condition on the side \(y = Ly\), which here we take to be the constant \(\Psi[x, Ly] = 100\), with \(Lx = Ly = 10\).

\[
\text{In[100]} = Lx = 10; Ly = 10;
\]

\[
\text{In[101]} = \Psi[\_\_] := 100
\]

\[
\text{In[102]} = \frac{2}{\text{Sinh}[\frac{n \pi Ly}{Lx}]} \text{Integrate}[\text{Sin}[-\frac{n \pi x}{Lx}] \Psi[\_\_], \{x, 0, Lx\}]
\]

\[
\text{Out[102]} = -\frac{200 (-1 + \text{Cos}[n \pi]) \text{Csch}[n \pi]}{n \pi}
\]

\[
\text{In[103]} = a[\_\_] := -\frac{200 (-1 + \text{Cos}[n \pi]) \text{Csch}[n \pi]}{n \pi}
\]

Thus the (truncated) sum for the result distribution is

\[
\text{In[104]} = \Psi[\_\_, y\_, m\_] := \text{Sum}[a[n] \text{Sin}[\frac{n \pi x}{Lx}] \text{Sinh}[\frac{n \pi y}{Ly}], \{n, 1, m\}]
\]
In[105]:= \( \Phi(x, y, 10) \)

\[
\frac{400 \cosh(\pi) \sin(\frac{\pi x}{10}) \sinh(\frac{\pi y}{10})}{\pi} + \frac{400 \cosh(3 \pi) \sin(\frac{3\pi x}{10}) \sinh(\frac{3\pi y}{10})}{3 \pi} + \frac{80 \cosh(5 \pi) \sin(\frac{\pi x}{2}) \sinh(\frac{\pi y}{2})}{\pi} + \frac{400 \cosh(7 \pi) \sin(\frac{7\pi x}{10}) \sinh(\frac{7\pi y}{10})}{7 \pi} + \frac{400 \cosh(9 \pi) \sin(\frac{9\pi x}{10}) \sinh(\frac{9\pi y}{10})}{9 \pi}
\]

Out[105]=

Now make a plot in 2-D - we'll use a contour plot

In[106]:= ContourPlot[\( \Phi(x, y, 10) \), \( \{x, 0, Lx\} \), \( \{y, 0, Ly\} \), Contours \[\rightarrow\] Automatic, ContourLabels \[\rightarrow\] Automatic, FrameLabel \[\rightarrow\] \{"x", "y"\}, LabelStyle \[\rightarrow\] \{Large\}]

Clearly the function is near 100 at the top boundary and falls off into the interior, as expected. However, 10 terms is too few to get a smooth result. Try more terms and more contours.
In[107]:= \text{ContourPlot}@\nabla y @ x, y, 100 D, 8 x, 0, Lx, 8 y, 0, Ly, \text{Contours} \rightarrow \{(90), (80), (70), (60), (50), (40), (30), (20), (10)\}, \text{ContourLabels} \rightarrow \text{Automatic}, \text{FrameLabel} \rightarrow \{"x", "y"\}, \text{LabelStyle} \rightarrow \{\text{Large}\}

Out[107]=

Note the expected symmetries.

Now consider a similar situation where we add a similar nonzero boundary condition on \(x = Lx\), and thus another sum of terms with \(x\) and \(y\) switched. Define

\begin{align*}
\text{b}[n_\_] & := -\frac{200 (-1 + \cos[n \pi]) \text{Csch}[n \pi]}{n \pi} \\
\text{With a distribution}
\end{align*}

\begin{align*}
\text{\&2}[x\_, y\_, m\_] & := \\
& \text{Sum}[a[n] \text{Sin}[\frac{n \pi y}{Lx}] \text{Sinh}[\frac{n \pi x}{Ly}], \{n, 1, m\}] + \text{Sum}[\text{b}[n] \text{Sin}[\frac{n \pi x}{Lx}] \text{Sin}[\frac{n \pi y}{Ly}], \{n, 1, m\}]
\end{align*}
A distribution that should be intuitively reasonable.

Next consider a similar situation but in cylindrical coordinates. The relevant functions are given in Eq. (24.14). Here we will define a radius and temperature on the edge via

\[ \rho_0 = 10; \ \Psi_0[\phi] := 100 \sin(\phi) \]

The coefficients are

\[ \frac{1}{\pi} \int \cos(n \phi) \Psi_0[\phi], \{\phi, 0, 2\pi\} \]

\[ = -\frac{200 \sin(n \pi)^2}{(-1 + n^2) \pi} \]

i.e., really zero for all \( n \) and

\[ \frac{1}{\pi} \int \sin(n \phi) \Psi_0[\phi], \{\phi, 0, 2\pi\} \]

\[ = \frac{100 \sin(2n \pi)}{(-1 + n^2) \pi} \]
In[117]:= $b[1]$

\[ *\text{Power::infy} \rightarrow \text{Infinite expression } 0^\infty \text{ encountered.}\]

\[ *\text{Infinity::indet} \rightarrow \text{Indeterminate expression } 0 \text{ComplexInfinity encountered.}\]

Out[117]= Indeterminate

So all zero except \( n = 1 \) but must be careful

In[118]:= Limit[$b[n]$, \( n \to 1 \)]

Out[118]= 100

In[119]:= \[3\rho, \phi \] := 100 \( \frac{\rho}{\rho_0} \text{ Sin}[\phi] \text{ UnitStep}[\rho_0 - \rho] \]

In[120]:= ContourPlot[33\sqrt{x^2 + y^2}, \text{ArcTan}[x, y]], \{x, -\rho_0, \rho_0\}, \{y, -\rho_0, \rho_0\},

\text{Contours} \rightarrow \{\{80\}, \{60\}, \{40\}, \{20\}, \{0\}, \{-20\}, \{-40\}, \{-60\}, \{-80\}\},

\text{ContourLabels} \rightarrow \text{Automatic}, \text{FrameLabel} \rightarrow \{"x", "y"\}, \text{LabelStyle} \rightarrow \{\text{Large}\}]

Out[120]=

Finally consider a 3-D rectangular situation as in the Lecture in side a cube of side 10.

In[121]:= \text{Lx} = 10; \text{Ly} = 10; \text{Lz} = 10;

Next assume all sides are at zero temperature except the side at \( z = 0 \) where \( \Psi_2 = 100 \). The coefficients are
\[
\begin{align*}
\text{In[122]:=} & \quad \frac{100}{(\frac{Lx}{2})(\frac{Ly}{2})\sinh\left(\pi Lz \sqrt{\left(\frac{m}{Lx}\right)^2 + \left(\frac{n}{Ly}\right)^2}\right)} \\
\text{Integrate[} & \quad \sin\left(\frac{m\pi x}{Lx}\right) \sin\left(\frac{n\pi y}{Ly}\right), \{x, 0, Lx\}, \{y, 0, Ly\}\text{]} \\
\text{Out[122]:=} & \quad 1600 \csc\left[10 \sqrt{\frac{m^2}{100} + \frac{n^2}{100}} \pi\right] \sin\left[\frac{m\pi}{2}\right]^2 \sin\left[\frac{n\pi}{2}\right]^2 \\
\text{In[123]:=} & \quad \frac{1600 \csc\left[10 \sqrt{\frac{m^2}{100} + \frac{n^2}{100}} \pi\right] \sin\left[\frac{m\pi}{2}\right]^2 \sin\left[\frac{n\pi}{2}\right]^2}{m n \pi^2} \\
\text{In[124]:=} & \quad b[m\_, n\_] := \frac{1600 \csc\left[10 \sqrt{\frac{m^2}{100} + \frac{n^2}{100}} \pi\right] \sin\left[\frac{m\pi}{2}\right]^2 \sin\left[\frac{n\pi}{2}\right]^2}{m n \pi^2} \\
\text{In[125]:=} & \quad \mathbb{W}[x\_, y\_, z\_, M\_] := \text{Sum}\left[\right. \\
& \quad b[m, n] \sin\left(\frac{m\pi x}{Lx}\right) \sin\left(\frac{n\pi y}{Ly}\right) \sinh\left(\pi (Lz - z) \sqrt{\left(\frac{m}{Lx}\right)^2 + \left(\frac{n}{Ly}\right)^2}\right), \{m, 1, M\}, \{n, 1, M\}\right] \\
\text{In[126]:=} & \quad \mathbb{W}[x\_, y\_, z\_] := \mathbb{W}[x, y, z, 10] \\
\text{In[127]:=} & \quad \text{ContourPlot3D[}\mathbb{W}[x, y, z] == 50, \{x, 0, Lx\}, \\
& \quad \{y, 0, Ly\}, \{z, 0, Lz\}, \text{LabelStyle} \to \{\text{Large}\}\text{]} \\
\text{Out[127]:=} & \quad \text{ContourPlot3D[}\mathbb{W}[x, y, z], \{x, 0, Lx\}, \\
& \quad \{y, 0, Ly\}, \{z, 0, Lz\}, \text{Contours} \to 2, \text{LabelStyle} \to \{\text{Large}\}\text{]} \\
\$\text{Aborted}
\end{align*}
\]