Lecture 23 Appendix B (Solutions around regular singular points and Bessel)

Ex 12.11: 4

Let's use Mathematica to solve the equation

In[28]:= DSolve[x^2 y''[x] - 6 y[x] == 0, y[x], x]

Out[28]= 

\[
\begin{cases}
{\{ y[x] \to x^3 C[1] + \frac{C[2]}{x^2} \}} \\
{\{ y[x] \to x^3 C[1] + \frac{C[2]}{x^2} \}} \\
{\{ y[x] \to x^3 C[1] + \frac{C[2]}{x^2} \}} \\
{\{ y[x] \to x^3 C[1] + \frac{C[2]}{x^2} \}} \\
{\{ y[x] \to x^3 C[1] + \frac{C[2]}{x^2} \}}
\end{cases}
\]

As expected we obtain one solution regular at the origin and one that is singular (the origin is a regular singular point).

Ex 12.11: 8

Next solve

In[29]:= soln = DSolve[x^2 y''[x] + 2 x^2 y''[x] - 2 y[x] == 0, y[x], x]

Out[29]= 

\[
\begin{cases}
{\{ y[x] \to e^{-x} \sqrt{\frac{2}{\pi}} \sqrt{x} \left( -\frac{\text{Cosh}[x]}{x} + i \text{Sinh}[x] \right) \}} \\
{\{ y[x] \to e^{-x} \sqrt{\frac{2}{\pi}} \sqrt{x} \left( \text{Cosh}[x] - \frac{\text{Sinh}[x]}{x} \right) \}}
\end{cases}
\]

Out[29]= 

FullSimplify[soln[[1, 1, 2]]]

Out[30]= 

\[
\frac{e^{-x} \sqrt{\frac{2}{\pi}} \left( -\left( x C[1] + i C[2] \right) \text{Cosh}[x] + \left( C[1] + i x C[2] \right) \text{Sinh}[x] \right)}{\sqrt{-i x} \sqrt{x}}
\]


In[32]:= y1[x]

Out[32]= 

\[
(-1)^{3/4} e^{-x} \sqrt{x} \left( -\text{Cosh}[x] + \frac{\text{Sinh}[x]}{x} \right) \]

Out[32]= 

In[33]= \texttt{Simplify[\%]}

\texttt{Out[33]=}

\[\frac{(-1)^{1/4} e^{-x} \sqrt{x} \left(-x \cosh[x] + \sinh[x]\right)}{(-i x)^{3/2}}\]

In[34]= \texttt{y1[0]}

\texttt{Out[34]=}

\[\frac{(-1)^{3/4} e^{-x} \sqrt{x} \left(-\cosh[x] + \frac{\sinh[x]}{x}\right)}{\sqrt{-i x}}\]

In[35]= \texttt{N[\%]}

\texttt{Out[35]=}

\[-\left(0.707107 - 0.707107 i\right) 2.71828^{-1} x \sqrt{x} \left(-1. \cosh[x] + \frac{\sinh[x]}{x}\right)\]/\left(\sqrt{0. - 1. i \ x}\right)

In[36]= \texttt{FullSimplify[soln[[1, 1, 2]] /. C[2] \rightarrow 0]}

\texttt{Out[36]=}

\[e^{-x} \sqrt{\frac{2}{\pi}} C[1] \left(-x \cosh[x] + \sinh[x]\right)\]

\[\sqrt{-i x} \sqrt{x}\]

A substantially more complicated expression than above, and harder to compare to our analytic result. Let's try a power series and separate the nonsingular bit first.

In[37]= \texttt{Series[soln[[1, 1, 2]] /. C[2] \rightarrow 0 /. C[1] \rightarrow \left\{\sqrt{\frac{\pi}{2}} \ \frac{-3}{\sqrt{-1}}\right\}, \{x, 0, 7\}]}

\texttt{Out[37]=}

\[\left\{x^2 - x^3 + \frac{3}{5} x^4 - \frac{4}{15} x^5 + \frac{2}{21} x^6 - \frac{x^7}{35} + O[x]^8\right\}\]

This is the series with coefficient b in the analytic solutions. The bit with a singular part is

In[38]= \texttt{Series[soln[[1, 1, 2]] /. C[1] \rightarrow 0, \{x, 0, 7\}]}

\texttt{Out[38]=}

\[-\frac{(-1)^{3/4} \sqrt{\frac{2}{\pi}} C[2]}{x} + (-1)^{3/4} \sqrt{\frac{2}{\pi}} C[2] - \frac{1}{3} \left(-1\right)^{3/4} \sqrt{\frac{2}{\pi}} C[2] \ x^2 + \]

\[\frac{1}{3} \ (-1)^{3/4} \sqrt{\frac{2}{\pi}} C[2] \ x^3 - \frac{1}{5} \ (-1)^{3/4} \sqrt{\frac{2}{\pi}} C[2] \ x^4 + \frac{4}{45} (-1)^{3/4} \sqrt{\frac{2}{\pi}} C[2] \ x^5 - \]

\[\frac{2}{63} \ (-1)^{3/4} \sqrt{\frac{2}{\pi}} C[2] \ x^6 + \frac{1}{105} (-1)^{3/4} \sqrt{\frac{2}{\pi}} C[2] \ x^7 + O[x]^{15/2}\]

With some care we recognize this part as the coefficient of a in our analytic solution PLUS the nonsingular series above. We can see this more clearly with the choice
Out[40]= Series[soln[[1, 1, 2]] /. C[1] → 0 /. C[2] → \(\sqrt{\frac{\pi}{2}}\ \frac{x}{\sqrt{-1}}\), \(x, 0, 7\)]

Out[40]= \(\frac{1}{x} - 1 + \frac{x^2}{3} - \frac{x^3}{5} + \frac{4 x^4}{45} + \frac{2 x^5}{63} - \frac{x^7}{105} + O[x]^{15/2}\)

So we can cancel the second bit with the choice

Out[41]= FullSimplify[soln[[1, 1, 2]] /. C[1] → \(\sqrt{\frac{\pi}{2}}\ \frac{1}{\sqrt{-1}}\) /. C[2] → \(\sqrt{\frac{\pi}{2}}\ \frac{x}{\sqrt{-1}}\)]

Out[41]= \(\left\{\left\{-\frac{1}{x^{3/2}}\right\}\right\}\)

Out[41]= \(\left\{-\frac{1}{x^{3/2}}\right\}\)

Some trouble simplifying but we do have

Out[42]= Series[soln[[1, 1, 2]] /. C[1] → \(\sqrt{\frac{\pi}{2}}\ \frac{1}{\sqrt{-1}}\) /. C[2] → \(\sqrt{\frac{\pi}{2}}\ \frac{x}{\sqrt{-1}}\), \(x, 0, 21\)]

Out[42]= \(\left\{\frac{1}{x} - 1 + O[x]^{43/2}\right\}\)

So it is just the first 2 terms. Note that for this equation it is not at all clear that using Mathematica makes the analysis easier.

**Ex 12.1:11**

Another equation:

Out[43]= soln = DSolve[36 \(x^2\) y''[x] + \((5 - 9 \ x^2)\) y[x] = 0, y[x], x]

Out[43]= \(\left\{\left\{y[x] \rightarrow \sqrt{x} \text{BesselJ}\left[\frac{1}{3}, -\frac{x}{2}\right] \text{C}[1] + \sqrt{x} \text{BesselY}\left[\frac{1}{3}, -\frac{x}{2}\right] \text{C}[2]\right\}\right\}\)

So Mathematica recognize these solutions as Bessel functions of fractional powers and imaginary arguments - cool! Let's looks at the series expansions

Out[44]= Series[\(\sqrt{x} \text{BesselJ}\left[\frac{1}{3}, -\frac{x}{2}\right]\), \(x, 0, 5\)]

Out[44]= \(\frac{\sqrt{x}}{2 \pi} \text{Floor}\left[\frac{2 - \arg(x)}{2 \pi}\right] \left\{-\frac{(-1)^{5/6} x^{5/6}}{2^{2/3} \text{Gamma}\left[\frac{4}{3}\right]} - \frac{3 (-1)^{5/6} x^{17/6}}{64 \left(2^{2/3} \text{Gamma}\left[\frac{4}{3}\right]\right)} - \frac{9 (-1)^{5/6} x^{39/6}}{14336 \left(2^{2/3} \text{Gamma}\left[\frac{4}{3}\right]\right)} + O[x]^{35/6}\right\}\)

So the overall constant is a mess, but the powers start with \(x^{5/6}\) and then go up by factor of \(x^2\), while the ratio of coefficients match our results, \(c2/c0 = 3/2^{6} = 3/64, c4/c0 = \)
The other solutions is

\[
\text{Series}\left[\sqrt{x} \ BesselY\left[\frac{1}{3}, -\frac{x}{2}\right], \{x, 0, 5\}\right]
\]

\[
\begin{align*}
&= e^{-\frac{1}{2} \text{Floor}\left[\frac{1}{2} \text{Arg}[x]\right]} \left(\text{Floor}\left[\frac{1}{2} \text{Arg}[x]\right] \right) \left(\frac{(1)^{5/6} \Gamma\left[-\frac{1}{3}\right] x^{5/6}}{2 \times 2^{2/3} \pi} + \frac{3 \left(1 \right)^{5/6} \Gamma\left[-\frac{1}{3}\right] x^{13/6}}{128 \times 2^{2/3} \pi} + \frac{9 \left(1 \right)^{5/6} \Gamma\left[-\frac{1}{3}\right] x^{29/6}}{28 \times 2^{2/3} \pi} + O(x^{35/6})\right) + \\
&+ \left(\frac{(1)^{1/6} \Gamma\left[-\frac{1}{3}\right] x^{11/6}}{\pi} - \frac{3 \left(1 \right)^{1/6} \Gamma\left[-\frac{1}{3}\right] x^{13/6}}{16 \left(2^{1/3} \pi\right)} - \frac{9 \left(1 \right)^{1/6} \Gamma\left[-\frac{1}{3}\right] x^{25/6}}{2560 \left(2^{1/3} \pi\right)} + O(x^{31/6})\right)
\end{align*}
\]

which includes the one above plus a series starting at \(x^{1/6}\) and going up by \(x^{2}\), while the ratio of coefficients match our results, \(c2/c0 = 3/2^5 = 3/32\), \(c4/c0 =

\[
\frac{3^2}{2^{10}/5} = \frac{9}{5120}
\]

**Exercise 12.19:**

Let's see what Mathematica says about the integral of 2 Bessel functions

\[
\text{FullSimplify}\left[\text{Integrate}\left[x \ BesselJ[p, BesselJZero[p, n] \ x] \ x^2, \{x, 0, 1\}\right], \quad \text{Assumptions} \rightarrow \{p \in \text{Reals}, p > -1\}\right]
\]

\[
\begin{align*}
&= \frac{1}{2} \left(\text{BesselJ}[-1 + p, \text{BesselJZero}[p, n]]^2 - \text{BesselJ}[p, \text{BesselJZero}[p, n]]^2\right) - \\
&- \frac{2 \ p \text{BesselJ}[-1 + p, \text{BesselJZero}[p, n]] \text{BesselJ}[p, \text{BesselJZero}[p, n]]}{\text{BesselJZero}[p, n]}
\end{align*}
\]

If we use the vanishing of the Bessel function at its own zero, we have

\[
\% /. \text{BesselJ}[p, \text{BesselJZero}[p, n]] \rightarrow 0
\]

\[
\frac{1}{2} \text{BesselJ}[-1 + p, \text{BesselJZero}[p, n]]^2
\]

which is one of the forms of the expressions we have in the analytic results. We, however, general use the version with \(-1+p \rightarrow 1+p\).