Here we solve for the Green function either directly with DSolve or using Laplace transforms as in the analytic results. First proceed directly with DSolve first with the notation used in the Lecture

\[
\text{In}[42] = \text{DSolve}\left\{x''[t] + 2 x'[t] + x[t] = \text{DiracDelta}[t - \tau] \text{HeavisideTheta}[\tau],
\quad x[0] = 0, x'[0] = 0, x[t], t\right\}
\]

\[
\text{DSolve}\text{::deqn} : \text{Equation or list of equations expected instead of True in the first argument}
\]

\[
\text{Out}[42] = \text{DSolve}\left\{x[t] + 2 x'[t] + x''[t] = \text{DiracDelta}[t - \tau] \text{HeavisideTheta}[\tau], x[0] = 0, \text{True}, x[t], t\right\}
\]

We can proceed directly with the Green function with 2 arguments if we are careful about how we express the derivatives, especially at the boundary. We have

\[
\text{In}[43] = \text{DSolve}\left\{\text{D[G17}[t, \tau], \{t, 2\}] + 2 \text{D[G17}[t, \tau], t] + G17[t, \tau] = \text{DiracDelta}[t - \tau] \text{HeavisideTheta}[\tau],
\quad G17[0, \tau] = 0, \text{Derivative}[1, 0][G17][0, \tau] = 0, G17[t, \tau], t\right\}
\]

\[
\text{Out}[43] = \left\{\left\{G17[t, \tau] \to e^{\tau t} (t \tau \text{DiracDelta}[\tau] + t \text{HeavisideTheta}[t - \tau] - t \text{HeavisideTheta}[-\tau] + t \text{HeavisideTheta}[-\tau]) \text{HeavisideTheta}[\tau]\right\}\right\}
\]

\[
\text{In}[44] = \text{FullSimplify}[\%]
\]

\[
\text{Out}[44] = \left\{\left\{G17[t, \tau] \to -e^{\tau t} (t - \tau) (\text{DiscreteDelta}[\tau] - \text{HeavisideTheta}[t - \tau, \tau])\right\}\right\}
\]

\[
\text{In}[45] = G17[t, 0]
\]

\[
\text{Out}[45] = G17[t, 0]
\]

Except for the extra \text{DiscreteDelta} term, which we saw in the Lecture notebook does not matter, this result matches our analytic work. The actual function looks like
Also using Laplace we have -

\[ \text{In}[47]= \text{Solve}[\{\text{LaplaceTransform[D[G17[t, \tau], \{t, 2\}] + 2 D[G17[t, \tau], t] + G17[t, \tau], t, p] =} \\
\text{LaplaceTransform[DiracDelta[t - \tau] HeavisideTheta[\tau], t, p], G17[0, \tau] = 0,} \\
\text{Derivative[1, 0][G17][0, \tau] = 0\}, \text{LaplaceTransform[G17[t, \tau], t, p]}\] \]

\[ \text{Out}[47]= \{\{\text{LaplaceTransform[G17[t, \tau], t, p] \rightarrow e^{-p \tau} \text{HeavisideTheta}[\tau]^2 / (1 + p)^2} \}\} \]

This matches the transform found analytically except for the extra theta function - since the Laplace structure builds one in. Then we can invert the transform

\[ \text{In}[48]= \text{Simplify[InverseLaplaceTransform}\left[\frac{e^{-p \tau} \text{HeavisideTheta}[\tau]^2}{(1 + p)^2}\right], p, t] \]

\[ \text{Out}[48]= e^{-t+\tau} (t - \tau) \text{HeavisideTheta}[t - \tau] \text{HeavisideTheta}[\tau] \]

We can also proceed in 1 step

\[ \text{In}[49]= \text{Simplify[InverseLaplaceTransform[} \]
\[ \{\{\text{LaplaceTransform[D[G17[t, \tau], \{t, 2\}] + 2 D[G17[t, \tau], t] + G17[t, \tau], t, p] =} \\
\text{LaplaceTransform[DiracDelta[t - \tau] HeavisideTheta[\tau], t, p], G17[0, \tau] = 0,} \\
\text{Derivative[1, 0][G17][0, \tau] = 0\}, \text{LaplaceTransform[G17[t, \tau], t, p]}\}, p, t] \] \]

\[ \text{Out}[49]= \{\{G17[t, \tau] \rightarrow e^{-t+\tau} (t - \tau) \text{HeavisideTheta}[t - \tau] \text{HeavisideTheta}[\tau]\}\} \]

**Ex 8.11.13**

Not relevant to using Mathematica.

**Ex 8.11.15**

We want to consider the various integrals using Mathematica

(a)
In[50]:= \text{Integrate}[\sin(x)\ \text{DiracDelta}[x - \frac{\pi}{2}], \{x, 0, \pi\}]

Out[50]= 1

(b)

In[51]:= \text{Integrate}[\sin(x)\ \text{DiracDelta}[x + \frac{\pi}{2}], \{x, 0, \pi\}]

Out[51]= 0

(c)

In[52]:= \text{Integrate}[\exp(3\ x)\ \text{D}[\text{DiracDelta}[x], x], \{x, -1, 1\}]

Out[52]= -3

(d)

In[53]:= \text{Integrate}[\cosh(x)\ \text{D}[\text{DiracDelta}[x - 1], \{x, 2\}], \{x, 0, \pi\}]

Out[53]= \cosh(1)

All results agree with the analytic results.

**Ex 8.11:21**

This is similar to the previous exercise practicing performing integrals with delta functions in Mathematica.

(a)

In[54]:= \text{Integrate}[(5\ x - 2)\ \text{DiracDelta}[2 - x], \{x, 0, 3\}]

Out[54]= 8

(b)

In[55]:= \text{Integrate}[\phi(x)\ \text{DiracDelta}[x^2 - a^2], \{x, 0, \infty\}]

Out[55]= \text{ConditionalExpression}\left[\frac{\text{Boole}[a < 0] \ \phi[-a] + \text{Boole}[a > 0] \ \phi[a]}{2 \ \text{Abs}[a]}, a \neq 0\right]

Interesting, this result suggests that Mathematica includes a contribution outside of the original interval, i.e., with a > 0 meaning x = -a < 0. Looks like a bug!??

(c)

In[56]:= \text{Integrate}[\cos(x)\ \text{DiracDelta}[-2\ x], \{x, -1, 1\}]

Out[56]= \frac{1}{2}

(d)

In[57]:= \text{Integrate}[\cos(x)\ \text{DiracDelta}[\sin(x)], \{x, -\pi/2, \pi/2\}]

Out[57]= 1
Ex 8.12: 2

First we can re-derive the harmonic Green function that we already found in the Lecture. We have

\[ G_2(t, \tau) = \frac{\text{HeavisideTheta}(t - \tau) \sin((t - \tau) \omega)}{\omega} \]

So the solution to the differential equation is

\[
G_2(t, \tau, \omega) = \frac{\text{HeavisideTheta}(t - \tau) \sin((t - \tau) \omega)}{\omega}
\]

Or, due to the explicit theta functions, we can actually integrate "everywhere"

\[
G_2(t, \tau, \omega) = \frac{\text{HeavisideTheta}(t - \tau) \sin((t - \tau) \omega)}{\omega}
\]

This agrees with our analytic result which we can also obtain directly using Mathematica

\[
\text{DSolve}\left[\left\{y''[t] + \omega^2 y[t] = \sin(\omega t) \text{ HeavisideTheta}[t], y[0] = 0, y'[0] = 0, y[t], t\right\}\right]
\]

\[
\left\{\left\{y[t] \to -\frac{t \omega \cos[t \omega]}{4 \omega^2} \sin[t \omega] + 2 \cos[t \omega] \text{ UnitStep}[t] \right\}, \left\{y[t] \to -\frac{t \omega \cos[t \omega]}{2 \omega^2} \sin[t \omega] + 2 \cos[t \omega] \text{ UnitStep}[t] \right\}\right\}
\]

which looks like
The solution is (no surprise) a growing oscillation.

**Ex 8.12: 7**

Here we consider solving a driven second order equation (plus initial conditions) for a hyperbolic equation. As we discovered before the Green function looks like

\[
\text{In}[66]:= \text{FullSimplify}\left[ \text{DSolve}\left[ \begin{array}{c}
\text{D}(G7[t, \tau], \{t, 2\}) - a^2 G7[t, \tau] = \text{DiracDelta}[t - \tau] \text{HeavisideTheta}[\tau], \\
G7[0, \tau] = 0, \text{Derivative}[1, 0][G7][0, \tau] = 0, G7[t, \tau] \end{array} \right] \right] \\
\text{Out}[66]= \left\{ \begin{array}{l}
G7[t_\text{\_\text{\_}}, \tau_\text{\_\text{\_}}, a_\text{\_\text{\_}}] := \frac{\text{HeavisideTheta}[t - \tau, \tau] \text{Sinh}[(t - \tau) a]}{a}
\end{array} \right\}
\]

So the solution to the differential equation is

\[
\text{In}[67]:= \text{Integrate}[G7[t, \tau, a] \text{Exp}[\tau], \{\tau, -\infty, \infty\}, \text{Assumptions} \rightarrow t \in \text{Reals}] \\
\text{Out}[67]= \frac{1}{a (-1 + a^2)} \text{HeavisideTheta}[t] \left( a (-\text{Cosh}[t] + \text{Cosh}[a t] + \text{Sinh}[t]) - \text{Sinh}[a t] \right)
\]

which matches the analytic result (\text{Sinh}[t]-\text{cosh}[t] = -\text{Exp}[-t]). Proceeding directly

\[
\text{In}[68]:= \text{DSolve}\left[ \begin{array}{l}
y''[t] - a^2 y[t] = \text{Exp}[-t] \text{HeavisideTheta}[t], y[0] = 0, y'[0] = 0, y[t], t
\end{array} \right] \\
\text{Out}[68]= \left\{ \begin{array}{l}
y[t] \rightarrow \left( \text{e}^{-a t} (1 - a t) \left( \text{e}^{2 a t} - a \text{e}^{a t} + \text{e}^{(1 + a) t} + a \text{e}^{(1 + a) t} - a \text{e}^{(-1 - a) t} + a \text{e}^{(-1 - a) t} - a \text{e}^{(1 - a) t} + a \text{e}^{(1 - a) t} \right) + a \text{e}^{2 a t} \text{UnitStep}[t] \right) / (2 (-1 + a) a (1 + a))
\end{array} \right\}
\]
In[69] := FullSimplify[%]

Out[69] := \[\{\{y[t] \to \begin{cases} \frac{a \cosh[t] - a (\cosh[\alpha t] + \sinh[t]) \cosh[\alpha t]}{a - a^3} & t \geq 0 \\ 0 & \text{True} \end{cases} \}\}

which looks like

In[70] := Plot[\[\frac{1}{2 a (-1 + a^2)} e^{-(1 + a^2) t} (-2 a e^{2 a t} + (1 + a) e^{(1 + a) t} + (-1 + a) e^{t^3 a t}) \] \text{UnitStep}[t] / . \{a \to 2, t, -1, 10\}, AxesLabel \to \{"t", "y[t]"\}, LabelStyle \to \{\text{Large}\}, PlotStyle \to \{\text{Thick}\}, \text{Exclusions} \to \text{None}\]

Or from above

In[71] := Plot[\[\frac{1}{a (-1 + a^2)} \text{HeavisideTheta}[t] (a (-\cosh[t] + \cosh[\alpha t] + \sinh[t]) - \sinh[\alpha t]) \] / . \{a \to 2, t, -1, 10\}, AxesLabel \to \{"t", "y[t]"\}, LabelStyle \to \{\text{Large}\}, PlotStyle \to \{\text{Thick}\}, \text{Exclusions} \to \text{None}\]

The solution is (no surprise) a growing exponential as found both ways.
**Ex 8.12:11**

Here we want to consider the case of periodic boundary conditions, where generally we are only interested in the result for a single interval.

\[ \text{In[72]} = \text{FullSimplify}\left[\text{DSolve}\left[\left\{\text{D[GL1[x, \chi], \{x, 2\}} + \text{GL1[x, \chi]} = \text{DiracDelta[x - \chi]}, \text{GL1[0, \chi]} = 0, \text{GL1}\left[\frac{\pi}{2}, \chi\right] = 0\right\}, \text{GL1[x, \chi]}, x\right]\right] \]

\[ \text{Out[72]} = \{\{\text{GL1[x, \chi]} \rightarrow -\cos[\chi] \text{HeavisideTheta}[\pi - 2 \chi] \sin[x] + \text{HeavisideTheta}[x - \chi] \sin[x - \chi] + \cos[x] \text{HeavisideTheta}[-\chi] \sin[\chi]\}\} \]

As usual (since Mathematica doesn't know exactly what we want) we want to carefully interpret this result for the interval \(0 < x, \chi < \frac{\pi}{2}\). Thus we can ignore the third term. If we write the second term, which contributes only for \(x > \chi\), as \(\sin[x - \chi] = \sin[x] \cos[\chi] - \cos[x] \sin[\chi]\), we see that added to the first term it serves to switch the role of \(x\) and \(\chi\). So on the interval we have

\[ \text{In[73]} = \text{GL1[x, _]} := -\cos[x] \text{HeavisideTheta}[\pi - 2 \chi] \sin[x] + \text{HeavisideTheta}[x - \chi] \sin[x - \chi] \]

which matches the analytic result in the interval of interest.

So the solution to the differential equation is

\[ \text{In[74]} = \text{Integrate}\left[\text{GL1[x, \chi]} \sin[2 \chi], \{\chi, 0, \frac{\pi}{2}\} \right] \text{, Assumptions} \rightarrow x \in \text{Reals} \]

\[ \text{Out[74]} = \frac{1}{3} \left(2 \cos[x] \text{HeavisideTheta}\left[-\frac{\pi}{2} + x\right] (-1 + \sin[x]) - 2 (1 + (-1 + \cos[x]) \text{HeavisideTheta}[x]) \sin[x]\right) \]

Thus for the interval \(0 < x < \pi/2\) we have just the second term

\[ \text{In[75]} = \text{FullSimplify}\left[\frac{1}{3} (-2 (1 + (-1 + \cos[x]) ) \sin[x])\right] \]

\[ \text{Out[75]} = -\frac{2}{3} \cos[x] \sin[x] \]

which matches the analytic result, \(2 \sin[x] \cos[x] = \sin[2x]\). Proceeding directly

\[ \text{In[76]} = \text{DSolve}\left[\left\{y''[x] + y[x] = \sin[2x], y[0] = 0, y\left[\frac{\pi}{2}\right] = 0\right\}, y[x], x\right] \]

\[ \text{Out[76]} = \{\{y[x] \rightarrow \frac{1}{6} (-3 \cos[x] \sin[x] - \cos[3 x] \sin[x] - 4 \cos[x] \sin[x]^3)\}\}\]

\[ \text{In[77]} = \text{FullSimplify}[\%] \]

\[ \text{Out[77]} = \{\{y[x] \rightarrow \frac{2}{3} \cos [x] \sin[x]\}\}\]

which looks like
In[78]= Plot[-\[1/3] Sin[2 x], {x, 0, \[Pi]/2}, AxesLabel -> {"x", "y[x]"}, Ticks -> {{0, \[Pi]/4, \[Pi]/2}}, LabelStyle -> {Large}, PlotStyle -> {Thick}, Exclusions -> None]

\[y[x] \quad \frac{\pi}{4} \quad \frac{\pi}{2} \]

\[\begin{align*}
\text{Out[78]=} & \quad -0.05 \\
& \quad -0.10 \\
& \quad -0.15 \\
& \quad -0.20 \\
& \quad -0.25 \\
& \quad -0.30 \\
\end{align*}\]

Now consider some general problems from the end of the chapter.

**Ex 8.13: 3**

Directly we try

In[79]= DSolve[y''''[x] + 2 y''[x] + 2 y'[x] == 0, y[x], x]

\[\text{Out[79]=} \quad \left\{ \left\{ y[x] \rightarrow C[3] + \frac{1}{2} e^{-x} (-C[1] + C[2]) \cos[x] + \left(-C[1] + C[2]\right) \sin[x] \right\} \right\} \]

This is what we obtained (with more work) analytically with the identifications \(C[3] = E, -\left(C[1] + C[2]\right)/2 = C, \left(-C[1] + C[2]\right)/2 = D\).

**Ex 8.13:15**

Another equation to solve directly

In[80]= DSolve[y''[x] - 4 y'[x] + 4 y[x] == 6 Exp[2 x], y[x], x]

\[\text{Out[80]=} \quad \left\{ \left\{ y[x] \rightarrow 3 e^{2 x} x^2 + e^{2 x} C[1] + e^{2 x} x C[2] \right\} \right\} \]

This matches our analytic result and we can clearly see the particular solution and the complementary solution

In[81]= DSolve[y c''[x] - 4 y c'[x] + 4 y c[x] == 0, y c[x], x]

\[\text{Out[81]=} \quad \left\{ \left\{ y c[x] \rightarrow e^{2 x} C[1] + e^{2 x} x C[2] \right\} \right\} \]

**Ex 8.13:30**

Here the equation has non-constant coefficients and initial conditions (which we must work hard at analytically. With Mathematica we have
In[82] = DSolve[{my ''[t] + \(\frac{2 m}{1 + t}\) y'[t] == m g, y[0] == 0, y'[0] == 0)}, y[t], t]

Out[82] = \[
\begin{align*}
[y[t] & \rightarrow \frac{3 g t^2 + g t^3}{6 (1 + t)}] \\
\end{align*}
\]

In[83] = Expand[Out[82]]

Out[83] = \[
\begin{align*}
[y[t] & \rightarrow \frac{g t^2}{2 (1 + t)} + \frac{g t^3}{6 (1 + t)}] \\
\end{align*}
\]

In[84] = Apart[Out[83]]

Out[84] = \[
\begin{align*}
[y[t] & \rightarrow -\frac{1}{3} + \frac{g t}{3} + \frac{g t^2}{6} + \frac{g}{3 (1 + t)}] \\
\end{align*}
\]

which in this final form we see in the analytic result.

**Ex 8.13:47**

Finally consider the equation

In[85] = DSolve[y''[t] + y[t] = Sec[t]^2, y[t], t]

Out[85] = \[
\begin{align*}
[y[t] & \rightarrow -1 + C[1] \cos[t] + 2 \text{ArcTanh}\left[\tan\left(\frac{t}{2}\right)\right] \sin[t] + C[2] \sin[t]] \\
\end{align*}
\]

The third term is not obviously the same as the analytic result so compare

In[86] = Series[2 ArcTanh[Tan[t/2]], (t, 0, 10)]

Out[86] = \[
\begin{align*}
t + \frac{t^3}{6} + \frac{61 t^7}{5040} + \frac{277 t^9}{72576} + O[t^{11}] \\
\end{align*}
\]

to

In[87] = Series[Log[Abs[Tan[t] + Sec[t]]], (t, 0, 10)]

Out[87] = Log[Abs[Sec[t] + Tan[t]]]

Better in the form

In[88] = Series[Log[1 + Sin[t]]/\sqrt{Cos[t]^2}, (t, 0, 10)]

Out[88] = \[
\begin{align*}
t + \frac{t^3}{6} + \frac{61 t^7}{5040} + \frac{277 t^9}{72576} + O[t^{11}] \\
\end{align*}
\]

In[89] = \[
\begin{align*}
2 \text{ArcTanh}\left[\tan\left(\frac{t}{2}\right)\right] / \sqrt{\frac{1 + \sin[t]}{\cos[t]^2}} \\
\end{align*}
\]

Out[89] = \[
\begin{align*}
\frac{2 \text{ArcTanh}\left[\tan\left(\frac{t}{2}\right)\right]}{\sqrt{\frac{1 + \sin[t]}{\cos[t]^2}}} \\
\end{align*}
\]
\begin{align*}
\text{In[90]} &= \text{Series}\left[\frac{2 \text{ArcTanh}[\tan\left(\frac{t}{2}\right)]}{\log\left[\frac{1+\sin[t]}{\sqrt{\cos[t]^2}}\right]}, \{t, 0, 100\}\right] \\
\text{Out[90]} &= 1 + O[t]^{101}
\end{align*}

So they really look to be the same, but due to the singularity structure the plots don’t come out the same!!

\begin{align*}
\text{In[91]} &= \text{Plot}\left[2 \text{ArcTanh}[\tan\left(\frac{t}{2}\right)], \{t, -10, 10\}\right] \\
\text{Out[91]} &= \\
\text{In[92]} &= \text{Plot}\left[\log\left[\frac{1+\sin[t]}{\sqrt{\cos[t]^2}}\right], \{t, -10, 10\}\right] \\
\text{Out[92]} &= 
\end{align*}

Life can be fun with Mathematica!!