Lecture 18

In this Lecture we discuss how to use Mathematica in our studies of the expansion (to use in the solution of differential equations) of non-integrable functions. The tool to be used here is the Laplace transform. This is effectively a Fourier transform (and inverse transform) continued into the complex plane, i.e., the path of the underlying integral is parallel to the imaginary axis instead of the real axis. Recall that we apply this transform to functions that are zero for arguments < 0, by assumption. We can start by considering some examples (See Eq. (18.4))

\[ \text{In[6]} = f_1[t_] := \text{UnitStep}[t] \]
\[ \text{In[7]} = \text{LaplaceTransform}[f_1[t], t, p] \]
\[ \text{Out[7]} = \frac{1}{p} \]

Note that Mathematica does not remind us of the constraints on the value of \( p \), e.g., here that \( \text{Re}[p] > 0 \). Next consider powers

\[ \text{In[8]} = f_2[t_] := t^n \text{UnitStep}[t] \]
\[ \text{In[9]} = \text{LaplaceTransform}[f_2[t], t, p] \]
\[ \text{Out[9]} = p^{-1-n} \text{Gamma}[1+n] \]

where again we must really require that \( \text{Re}[p] > 0 \). Now exponentials

\[ \text{In[10]} = f_3[t_] := \text{Exp}[-a t] \text{UnitStep}[t] \]
\[ \text{In[11]} = \text{LaplaceTransform}[f_3[t], t, p] \]
\[ \text{Out[11]} = \frac{1}{a + p} \]

where \( \text{Re}[p+a] > 0 \) is required to make sense.

\[ \text{In[12]} = f_4[t_] := \text{Sin}[\beta t] \text{UnitStep}[t] \]
\[ \text{In[13]} = \text{LaplaceTransform}[f_4[t], t, p] \]
\[ \text{Out[13]} = \frac{\beta}{p^2 + \beta^2} \]

\[ \text{In[14]} = f_5[t_] := \text{Cos}[\beta t] \text{UnitStep}[t] \]
\[ \text{In[15]} = \text{LaplaceTransform}[f_5[t], t, p] \]
\[ \text{Out[15]} = \frac{p}{p^2 + \beta^2} \]

\[ \text{In[16]} = f_6[t_] := \text{Sinh}[\gamma t] \text{UnitStep}[t] \]
\textbf{In[17]} := \texttt{LaplaceTransform[f6[t], t, p]} \\
\textbf{Out[17]} := \frac{\gamma}{p^2 - \gamma^2} \\
\textbf{In[18]} := f7[t_] := \text{Cosh}[\gamma t] \text{UnitStep}[t] \\
\textbf{In[19]} := \text{LaplaceTransform}[f7[t], t, p] \\
\textbf{Out[19]} := \frac{p}{p^2 - \gamma^2} \\
Then we have the inverse transform (a defined function to use instead of the contour integrals discussed in the Lecture)

\textbf{In[20]} := \text{InverseLaplaceTransform}\left[\frac{1}{p}, p, t\right] \\
\textbf{Out[20]} := 1 \\
\textbf{In[21]} := \text{InverseLaplaceTransform}\left[p^{-1-n} \text{Gamma}[1 + n], p, t\right] \\
\textbf{Out[21]} := t^n \\
Note that Mathematica does not explicitly return the implied theta function, UnitStep[t], which is explicit in the analytic approach.

\textbf{In[22]} := \text{InverseLaplaceTransform}\left[\frac{1}{a + p}, p, t\right] \\
\textbf{Out[22]} := e^{-a t} \\
\textbf{In[23]} := \text{InverseLaplaceTransform}\left[\frac{\beta}{p^2 + \beta^2}, p, t\right] \\
\textbf{Out[23]} := \text{Sin}[t \beta] \\
\textbf{In[24]} := \text{InverseLaplaceTransform}\left[\frac{p}{p^2 + \beta^2}, p, t\right] \\
\textbf{Out[24]} := \text{Cos}[t \beta] \\
\textbf{In[25]} := \text{InverseLaplaceTransform}\left[\frac{\gamma}{p^2 - \gamma^2}, p, t\right] \\
\textbf{Out[25]} := \frac{1}{2} e^{-t \gamma} (\text{1} + e^{2 t \gamma}) \\
\text{Also known as Sinh}[\gamma t] \\
\textbf{In[26]} := \text{InverseLaplaceTransform}\left[\frac{p}{p^2 - \gamma^2}, p, t\right] \\
\textbf{Out[26]} := \frac{1}{2} e^{-t \gamma} (\text{1} + e^{2 t \gamma}) \\
Or Cosh[\gamma t].
For fun we can also try the inverse transform as the explicit (Bromwich) integral

\[
\text{In[27]} = \frac{1}{2 \pi i} \int \frac{p}{p^2 - \gamma^2} \exp(p t), \{p, \gamma + 1 - i \omega, \gamma + 1 + i \omega\}
\]

\[
\text{Out[27]} = \frac{1}{2 \pi} \int_{\gamma-i \infty}^{\gamma+i \infty} \frac{e^{p \tau}}{p^2 - \gamma^2} \, dp
\]

Or for a specific case \( \gamma = 1 \)

\[
\text{In[28]} = \frac{1}{2 \pi i} \int \frac{p}{p^2 - 1} \exp(p t), \{p, 2 - i \omega, 2 + i \omega\}
\]

\[
\text{Out[28]} = \frac{1}{2 \pi} \int_{2-i \infty}^{2+i \infty} \frac{e^{p \tau}}{p^2 - 1} \, dp
\]

So that does not work so well, but the InverseLaplaceTransform function does work.

As with the Fourier transform as applied to differential equations, the Laplace transform serves to turn the differential equation (plus initial conditions at \( t = 0 \)) into an algebraic equation. Solving for the Laplace transform of the (full) solution, we can then invert the transform to find the solution. For example, from Eq. (18.19) we have

\[
\text{In[29]} = \text{InverseLaplaceTransform}\left[\frac{1}{p^2 (p+1)^2}, p, t\right]
\]

\[
\text{Out[29]} = -2 + 2 e^{-t} + t + e^{-t} t
\]

This is to be compared to Eq.(18.22), which agrees except for the understood theta function of \( t \). Of course, we also use

\[
\text{In[30]} = \text{DSolve}\left[\{x'[t] + 2 x'[t] + x[t] = t, x'[0] == 0, x[0] == 0\}, x[t], t\right]
\]

\[
\text{DSolve}:\text{deqn} : \text{Equation or list of equations expected instead of True in the first argument} \{x[t] + 2 x'[t] + x''[t] = t, True, True\}. \gg
\]

\[
\text{Out[30]} = \text{DSolve}\left[\{x[t] + 2 x'[t] + x''[t] = t, True, True\}, x[t], t\right]
\]

which agrees. We can also use Mathematica to solve this equation via the Laplace Transform but using Mathematica to do the algebra.

First consider the transform of the LHS

\[
\text{In[31]} = \text{LaplaceTransform}\left[\{x'[t] + 2 x'[t] + x[t], t, p\}\right]
\]

\[
\text{Out[31]} = \text{LaplaceTransform}\left[x[t], t, p\right] + 2 p \text{LaplaceTransform}\left[x[t], t, p\right] + p^2 \text{LaplaceTransform}\left[x[t], t, p\right]
\]

This is what we want except it asks for the initial conditions. So we provide them

\[
\text{In[32]} = x[0] = 0
\]

\[
\text{Out[32]} = 0
\]

\[
\text{In[33]} = x'[0] = 0
\]

\[
\text{Out[33]} = 0
\]

Now we have
In[34]= LaplaceTransform[x''[t] + 2 x'[t] + x[t], t, p] + 2 p LaplaceTransform[x[t], t, p] + p^2 LaplaceTransform[x[t], t, p]

Out[34]= LaplaceTransform[x'[t], t, p] + 2 p LaplaceTransform[x[t], t, p] + p^2 LaplaceTransform[x[t], t, p]

Then we can transform the differential equation and solve for the transform of the solution, which we identify in the function call

In[35]= Solve[LaplaceTransform[x''[t] + 2 x'[t] + x[t], t, p] == LaplaceTransform[t, t, p], LaplaceTransform[x[t], t, p]]

Out[35]= {{LaplaceTransform[x[t], t, p] -> 1/p^2 (1 + p)^2}}

Thus we are back to where we were above and the transform back

In[36]= InverseLaplaceTransform[%, p, t]

Out[36]= {x[t] -> -2 + 2 e^-t + t e^-t}

Alternatively we could include the initial conditions inside Solve

In[37]= Clear[x]

In[38]= Solve[{LaplaceTransform[x''[t] + 2 x'[t] + x[t], t, p] == LaplaceTransform[t, t, p], x[0] == 0, x'[0] == 0}, LaplaceTransform[x[t], t, p]]

Out[38]= {}

Or even force Mathematica to solve the equation via Laplace transforms in 1 step (maybe it does that with DSolve?).

In[39]= InverseLaplaceTransform[Solve[{LaplaceTransform[x''[t] + 2 x'[t] + x[t], t, p] == LaplaceTransform[t, t, p], x[0] == 0, x'[0] == 0}, LaplaceTransform[x[t], t, p]], p, t]

Out[39]= {}

To summarize, with a driving force of t[UnitStep[t], the response of the system looks like
Oops really zero for $t < 0$

So there is a transition region from $x = 0$ to $x$ responding linearly to the linear force, but for $t > 4$ (when the exponentials have damped out) $x$ is clearly linear (but displaced by 2, $x \sim t - 2$).