Lecture 17

In[75] = << FourierSeries`

In this Lecture we discuss how to use Mathematica in our studies of analytic functions, especially in our efforts to perform contour integrals, e.g., in the context of an inverse Fourier transform. Consider first determining where a function is analytic, e.g., using Cauchy-Riemann. We can start by defining some functions, e.g., as in the Lecture

In[76] = f1[ z_] := z^2

Then the appropriate real and imaginary parts are given by (note the effort to tell Mathematica that x,y are real and separate off U and V

In[77] = U1[ x_, y_] := Refine[Re[ComplexExpand[f1[x + I y]]], {x, y} ϵ Reals];
V1[ x_, y_] := Refine[Im[ComplexExpand[f1[x + I y]]], {x, y} ϵ Reals]

In[78] = U1[x, y]
Out[78] = x^2 - y^2

In[79] = V1[x, y]
Out[79] = 2 x y

Then we check the C-R relations

In[80] = D[U1[x, y], x] - D[V1[x, y], y]
Out[80] = 0

In[81] = D[U1[x, y], y] + D[V1[x, y], x]
Out[81] = 0

Both differences vanish and the separate functions (and their derivatives) are finite in the entire complex plane (except ∞), so we conclude that this function is entire. Next try

In[82] = f2[ z_] := Conjugate[z]

In[83] = U2[ x_, y_] := Refine[Re[ComplexExpand[f2[x + I y]]], {x, y} ϵ Reals];
V2[ x_, y_] := Refine[Im[ComplexExpand[f2[x + I y]]], {x, y} ϵ Reals]

In[84] = U2[x, y]
Out[84] = x

In[85] = V2[x, y]
Out[85] = - y

So perfectly finite expressions (in the finite complex plane) but from C-R

In[86] = D[U2[x, y], x] - D[V2[x, y], y]
Out[86] = 2
In[87]:= D[U2[x, y], y] + D[V2[x, y], x]
Out[87]= 0

So the C-R conditions are NOT satisfied ANYWHERE, i.e., the nonzero result does not depend on x,y, and this function is nowhere analytic. Finally consider

In[88]:= f3[z_] := 1/z

which we know is divergent at the origin - e.g., try a 3D plot

In[88]= Plot3D[Abs[f3[x + I y]], {x, -100, 100}, {y, -100, 100}, PlotRange -> {0, 100}, AxesLabel -> {x, y, f}]

This clearly suggests the pole at the origin. We can also look at C-R after we find U & V. Note the interesting "feature" that, when we tell Mathematica that x & y are real, it knows that \(x^2 + y^2\) is real, but NOT \(1/(x^2 + y^2)\)

In[90]= ComplexExpand[f3[x + I y]]
Out[90]= \[\frac{x}{x^2 + y^2} - \frac{i y}{x^2 + y^2}\]

In[91]= Refine[Re[%), Element[x | y, Reals]]
Out[91]= y \text{Im}\left[\frac{1}{x^2 + y^2}\right] + x \text{Re}\left[\frac{1}{x^2 + y^2}\right]

?? So consider

In[92]= Refine[Re[I (x^2 + y^2)], Element[x | y, Reals]]
Out[92]= 0

Knows that \(x^2 + y^2\) is real, but
Refine[Re[i/(x^2 + y^2)], Element[x | y, Reals]]

\[- \text{Im} \left( \frac{1}{x^2 + y^2} \right)\]

So tell Mathematica directly

Refine[Re[i/(x^2 + y^2)], Element[x | y | 1/(x^2 + y^2), Reals]]

\[0\]

Refine[Re[1/(x^2 + y^2)], Element[x | y | 1/(x^2 + y^2), Reals]]

\[1\]

\[\frac{1}{x^2 + y^2}\]

Ok, so now we try

U3[x_, y_] := Refine[Re[ComplexExpand[f3[x + i y]]], Element[x | y | 1/(x^2 + y^2), Reals]]; V3[x_, y_] := Refine[Im[ComplexExpand[f3[x + i y]]], Element[x | y | 1/(x^2 + y^2), Reals]]

U3[x, y]

\[\frac{x}{x^2 + y^2}\]

V3[x, y]

\[- \frac{y}{x^2 + y^2}\]

D[U3[x, y], x] - D[V3[x, y], y]

\[- \frac{2 x^2}{(x^2 + y^2)^2} - \frac{2 y^2}{(x^2 + y^2)^2} + \frac{2}{x^2 + y^2}\]

Oops

Simplify[%]

\[0\]

OK

D[U3[x, y], y] + D[V3[x, y], x]

\[0\]

So satisfy C-R everywhere that the function is finite, i.e., not at the origin. We can see the presence of the pole using Cauchy's Integral formula, or more directly Eq. (17.12). First consider a contour integral on a circle of radius 1 about the origin, \(z = e^{i \theta}\) (note that \(dz \rightarrow i e^{i \theta} \, d\theta\))

\[\frac{1}{2 \pi i} \text{Integrate}[f3[e^{i \theta}] \cdot e^{i \theta}, \{\theta, 0, 2 \pi\}]\]

\[1\]

So there is a pole of residue 1, which we could find via
\textbf{In[103]} := \texttt{Residue[f3[z], \{z, 0\}]}

\textbf{Out[103]} = 1

But now integrate on a circle that does NOT enclose the origin \((z = 1.1 + e^{i\theta}, dz = i e^{i\theta} d\theta)\)

\textbf{In[104]} := \texttt{\frac{1}{2 \pi i} Integrate[f3[1.1 + e^{i\theta}] e^{i\theta}, \{\theta, 0, 2 \pi\}]}

\textbf{Out[104]} = 0

These 2 paths (and the pole) look like

\textbf{In[105]} := \texttt{Graphics[
\{Thick, Circle[{0, 0}, 1], Thick, Dashed, Circle[{1.1, 0}, 1], Disk[{0, 0}, .05],
\ Arrow[{{2.1, -.1}, {2.1, .1}}], Arrow[{{1, -.1}, {1, .1}}], Axes \to True\}]

Now the integral (correctly) vanishes. Another useful tool in \textit{Mathematica} (for contour integrals) is numerical integration around a square (or polygon) path with the vertices specified. First consider a square not enclosing the origin

\textbf{In[106]} := \texttt{\frac{1}{2 \pi i} NIntegrate[f3[z], \{z, 1, 2, 2 + i, 1 + i, 1\}]

\texttt{NIntegrate::ncvb : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in z near [z] = \{1.00000001793518215981501903631603912611441632374439464621563466 +
1.00000000000000000000000000000000000000000000000000000000000000000\ i. NIntegrate
obtained 6.245 \times 10^{-17} + 1.99493 \times 10^{-17} \ i and 1.353129043060503 \times 10^{-13} for the integral and error estimates. \Rightarrow}

\textbf{Out[106]} = 3.17503 \times 10^{-18} - 9.93923 \times 10^{-18} \ i

\textbf{In[107]} := \texttt{Chop[%]}

\textbf{Out[107]} = 0

\texttt{i.e., zero within the given numerical accuracy. If we enclose the origin, the integral is not zero}

\textbf{In[108]} := \texttt{\frac{1}{2 \pi i} NIntegrate[f3[z], \{z, 1 - i, 1 + i, -1 + i, -1 - i, 1 - i\}]

\textbf{Out[108]} = 1. + 3.53395 \times 10^{-17} \ i

Here the paths look like
Now consider the integral in Eq. (17.15). Mathematica is in fact happy to evaluate this one directly

\[ f_4(z) = \frac{1}{1 + z^2} \]

But recognizing there are (simple) poles at \(+i, -i\) (by looking at a plot of the absolute value)
First check that we can add the arc at infinity by checking that it has vanishing magnitude (the 1/2 is because we doubled the integration range to cover the entire real axis)

\[
\text{In[113]=} \quad \text{Integrate} \left[ \frac{1}{2} f_4[R e^{i\theta}] i R e^{i\theta}, \{\theta, 0, \pi\} \right]
\]

\[
\text{Out[113]=} \quad \frac{i e^{i\theta} \pi R}{2 \left(1 + e^{2 i\theta} R^2\right)}
\]

Take \( R \) to \( \infty \)

\[
\text{In[114]=} \quad \text{Limit}[%, \text{R}\rightarrow \infty]
\]

\[
\text{Out[114]=} \quad 0
\]

Now we have a (closed) contour integral and we use Cauchy to tell us that we want the residue at +i if we closed above

\[
\text{In[115]=} \quad \frac{2 \pi i}{2} \text{Residue}[f_4[z], \{z, i\}]
\]

\[
\text{Out[115]=} \quad \frac{\pi}{2}
\]

or if we close below so now contour is CW (with -1 factor in Cauchy)

\[
\text{In[116]=} \quad -\frac{2 \pi i}{2} \text{Residue}[f_4[z], \{z, -i\}]
\]

\[
\text{Out[116]=} \quad \frac{\pi}{2}
\]

(Must) Get the same answer (see Eq. (17.18)).

Now return to solving the differential equation with RHS a delta function

\[
\text{In[117]=} \quad f_5[t_] := C\text{DiracDelta}[t]
\]

\[
\text{In[118]=} \quad G_5[\omega_] := \text{FourierTransform}[f_5[t], t, \omega, \text{FourierParameters} \rightarrow \{0, -1\}]
\]

\[
\text{In[119]=} \quad G_5[\omega]
\]

\[
\text{Out[119]=} \quad C \sqrt{2 \pi}
\]

\[
\text{In[120]=} \quad \text{InverseFourierTransform}[G_5[\omega], \omega, t, \text{FourierParameters} \rightarrow \{0, -1\}]
\]

\[
\text{Out[120]=} \quad C\text{DiracDelta}[t]
\]

With the complementary solution defined by the parameters

\[
\alpha_1 = -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \quad ; \quad \alpha_2 = -\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}
\]

we have the Fourier transform of a particular solution as

\[
\text{In[121]=} \quad x_{p[t]}[\omega] := \frac{G_5[\omega]}{a (i \omega - \alpha_1) (i \omega - \alpha_2)}
\]
InverseFourierTransform[xpt[\omega], \omega, t, FourierParameters \rightarrow \{0, -1\}]

\begin{align*}
\text{Out[122]} &= -\frac{1}{2 a (a_1 - a_2)} C \left\{ \text{Sign} [t] \left( -e^{t a_1} + e^{t a_2} + e^{t a_1} \text{Sign} [\text{Re} [a_1]] - e^{t a_2} \text{Sign} [\text{Re} [a_2]] \right) + \\
&\quad 2 e^{t a_1} \text{HeavisideTheta} [-t \text{Sign} [\text{Re} [a_1]]] \text{Sign} [\text{Re} [a_1]] - \\
&\quad 2 e^{t a_2} \text{HeavisideTheta} [-t \text{Sign} [\text{Re} [a_2]]] \text{Sign} [\text{Re} [a_2]] \right\} + \\
&\quad \frac{1}{2 a \omega_0} \text{HeavisideTheta}[t]
\end{align*}

We recognize this as the result in Eq. (17.20). We could also use

Integrate[xpt[\omega] \text{Exp}[i \omega t], \{\omega, -\infty, \infty\}]

\begin{align*}
\text{Out[125]} &= \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \frac{C e^{i \omega t}}{a \sqrt{2 \pi} (y + i \omega - i \omega_0) (y + i \omega + i \omega_0)}
\end{align*}

Which doesn't work so well. Finally we can try to solve directly

Clear[b]

DSolve[a \ x''[t] + b \ x'[t] + c \ x[t] \rightarrow C \text{DiracDelta}[t], x[t], t]

\begin{align*}
\text{Out[127]} &= \left\{ \left\{ x[t] \rightarrow \left( \begin{array}{c}
\text{HeavisideTheta}[t] \\
\text{HeavisideTheta}[t] \\
\text{HeavisideTheta}[t]
\end{array} \right) \right\} \right\}
\end{align*}

We recognize the third term as the particular solution as above (with b/2 a = \gamma and \sqrt{b^2 - 4 a c} = 2 a \omega_0)

If we use the phase choice from class, we get the same answer since the function is even.

FourierTransform[f[t], t, \omega, FourierParameters \rightarrow \{0, -1\}]

\begin{align*}
\text{Out[128]} &= \text{FourierTransform}[f[t], t, \omega, \text{FourierParameters} \rightarrow \{0, -1\}]
\end{align*}

\begin{align*}
\text{Out[129]} &= \text{InverseFourierTransform}\left[ \sqrt{2 \pi} \text{DiracDelta} [\omega], \omega, t, \text{FourierParameters} \rightarrow \{0, -1\} \right]
\end{align*}

This calculation confirms the desired factors of \sqrt{2 \pi} but not yet the signs in the phases. So let's
consider something with time dependence as in Eq. (16.11).

\[ f_5[t] := \text{UnitStep}[1 - t] \ast \text{UnitStep}[1 + t] \]

\[ \text{Plot}[f_5[t], \{t, -3, 3\}, \text{AxesLabel} \to \{t, F\}, \]
\[ \text{LabelStyle} \to \{\text{Large}\}, \text{PlotStyle} \to \{\text{Thick}\}, \text{Exclusions} \to \text{None}] \]

The Fourier transform is (see Eq. (16.12) - again the function is even (and the transform is real)

\[ F(\omega) = \sqrt{\frac{2}{\pi}} \sin(\omega) \]

\[ \text{Plot}[% \to \{\omega, -20, 20\}, \text{PlotRange} \to \{-.2, 1.0\}, \text{AxesLabel} \to \{\omega, G\}, \]
\[ \text{LabelStyle} \to \{\text{Large}\}, \text{PlotStyle} \to \{\text{Thick}\}, \text{Exclusions} \to \text{None}] \]
In[134]= \[\text{InverseFourierTransform}\left[ \frac{\sqrt{\frac{2}{\pi}} \sin[\omega]}{\omega}, \omega, t, \text{FourierParameters} \to (0, -1) \right] \]

Out[134]= \[\frac{1}{2} (\text{Sign}[1 - t] + \text{Sign}[1 + t]) \]

which is the original function.

In[135]= \[\text{Plot}\left[ \frac{1}{2} (\text{Sign}[1 - t] + \text{Sign}[1 + t]), \{t, -3, 3\}, \text{AxesLabel} \to \{t, F\}, \right. \]

LabelStyle \to \{\text{Large}\}, \text{PlotStyle} \to \{\text{Thick}\}, \text{Exclusions} \to \text{None} \]

Out[135]=

\begin{figure}
\begin{center}
\includegraphics[width=0.5\textwidth]{plot.png}
\end{center}
\end{figure}

We could also obtain these results in Mathematica by explicitly performing the integral transforms

In[136]= \[\left( \frac{1}{\sqrt{2\pi}} \right) \text{Integrate}\left[ f5[t] \exp[-i \omega t], \{t, -\infty, \infty\} \right] \]

Out[136]= \[\frac{\sqrt{\frac{2}{\pi}} \sin[\omega]}{\omega} \]

In[137]= \[\left( \frac{1}{\sqrt{2\pi}} \right) \text{Integrate}\left[ \sqrt{\frac{2}{\pi}} \frac{\sin[\omega]}{\omega} \exp[i \omega t], \{\omega, -\infty, \infty\} \right] \]

Out[137]= \[\text{ConditionalExpression}[0, \text{Im}[t] = 0 \&\& (\text{Re}[t] < -1 \| \text{Re}[t] > 1)] \]

However, note that Mathematica has difficulty obtaining the correct result for \(|t| < 0\), but it does know that it vanishes for \(|t| > 1\).

Finally we want to consider performing the inverse transform by closing the contour and using Cauchy. The first point that we notice is that, in order to add the arc at infinity we must split the sine function into the 2 (complex) exponentials. We start with the Fourier transform
\[ G5_\omega := \frac{\sqrt{\frac{2}{\pi}} \sin(\omega)}{\omega} \]

and consider its behavior on the upper arc.

\[ \text{Integrate}[G5[R e^{i \theta}] \exp[i R e^{i \theta} t], \{\theta, 0, \pi\}] \]

\[ \text{Limit}[\text{Integrate}[G5[R e^{i \theta}] \exp[i R e^{i \theta} t], \{\theta, 0, \pi\}], R \to \infty] \]

Mathematica can't seem to handle the integral. Let's just consider the behavior along the positive imaginary axis, \( \theta = \pi/2 \)

\[ \text{Limit}[G5[R e^{i \theta}] \exp[i R e^{i \theta} t], \{\theta, \pi/2\}, R \to \infty] \]

This will diverge for \( t < 1 \), while on the negative imaginary axis

\[ \text{Limit}[G5[R e^{i \theta}] \exp[i R e^{i \theta} t], \{\theta, -\pi/2\}, R \to \infty] \]

we will get divergence for \( t > -1 \). So we are encouraged to split the function

\[ G5A_\omega := \frac{\sqrt{\frac{2}{\pi}} \exp[i \omega]}{2 i \omega} \]

\[ G5B_\omega := -\frac{\sqrt{\frac{2}{\pi}} \exp[-i \omega]}{2 i \omega} \]
which is well behaved for \( t > -1 \) and

\[
\text{In[146]} = \text{Limit}[GSA\{Re^\theta\} \text{Exp}[i \text{Re}^\theta t] i \text{Re}^\theta / \theta \to -\pi/2, R \to \infty]
\]

\[
\text{Out[146]} = \text{Limit}\left[\frac{e^{R-R t}}{\sqrt{2 \pi}}, R \to \infty\right]
\]

which well behaved for \( t < -1 \) - i.e., one of the arcs always works. Then for the B bits

\[
\text{In[147]} = \text{Limit}[GSB\{Re^\theta\} \text{Exp}[i \text{Re}^\theta t] i \text{Re}^\theta / \theta \to \pi/2, R \to \infty]
\]

\[
\text{Out[147]} = \text{Limit}\left[\frac{-e^{R-R t}}{\sqrt{2 \pi}}, R \to \infty\right]
\]

where we want \( t > 1 \) and

\[
\text{In[148]} = \text{Limit}[GB\{Re^\theta\} \text{Exp}[i \text{Re}^\theta t] i \text{Re}^\theta / \theta \to -\pi/2, R \to \infty]
\]

\[
\text{Out[148]} = \text{Limit}\left[\frac{-e^{R+R t}}{\sqrt{2 \pi}}, R \to \infty\right]
\]

where we want \( t < 1 \) - one arc works for all \( t \). Now the only problem, as discussed in the Lecture is that we are forced to integrate through the pole at the origin for each separate case. In the analytic discussion in the Lecture we study this situation using a small semi-circle at the pole. \textit{Mathematica} can also make sense of this situation by setting the PrincipalValue parameter to true - i.e., define the integral as the principal value, which is essentially the average of having the pole in the upper half plane and in the lower. Such a situation is best studied analytically and not using \textit{Mathematica}.

\[
\text{In[149]} = \text{Integrate}\left[\frac{\text{Exp}[i x]}{x}, \{x, -\infty, \infty\}\right]
\]

\text{Integrate::idiv} : Integral of \( \frac{\text{Exp}[i x]}{x} \) does not converge on \( [-\infty, \infty] \).

\[
\text{Out[149]} = \int_{-\infty}^{\infty} \frac{\text{Exp}[i x]}{x} \, dx
\]

\[
\text{In[150]} = \text{Integrate}\left[\frac{\text{Exp}[i x]}{x}, \{x, -\infty, \infty\}, \text{PrincipalValue} \to \text{True}\right]
\]

\[
\text{Out[150]} = i \pi
\]