Lecture 12

In this Lecture we discuss more about the issue of partial derivatives and especially the gradient operator. As we saw in the previous lecture the gradient operation is given in terms of the standard Derivative command by (note the double curly brackets)

\[\nIn[187]:= \text{D}[f[x, y, z], \{\{x, y, z\}\}]\\
\text{Out}[187]= \{f^{(1,0,0)}[x, y, z], f^{(0,1,0)}[x, y, z], f^{(0,0,1)}[x, y, z]\}\\
\]

For example with

\[\nIn[188]:= f[x_, y_, z_] := xyz\\
In[189]:= \text{D}[f[x, y, z], \{\{x, y, z\}\}]\\
\text{Out}[189]= \{yz, xz, xy\}\\
\]

where we see that the result is a 3-vector.

Mathematica also offers a power Vector Analysis package with all of the familiar commands, grad, curl, divergence and Laplacian and the ability to use these operators in a variety of useful coordinate systems. First we must load the vector analysis package

\[\nIn[190]:= \text{\textless\textless VectorAnalysis}\text{\textgreater\textgreater}\\
\text{General::obspkg : VectorAnalysis} is now obsolete. The legacy version being loaded may conflict with current Mathematica functionality. See the Compatibility Guide for updating information.\\
\]

We can check the default coordinate system with

\[\nIn[191]:= \text{CoordinateSystem}\\
Out[191]= \text{Cartesian}\\
\]

where the default definition of the coordinates is

\[\nIn[192]:= \text{Coordinates[Cartesian]}\\
Out[192]= \{Xx, Yy, Zz\}\\
\]

Thus the analog of the operations above (in the default coordinates) is

\[\nIn[193]:= \text{Grad[Xx Yy Zz]}\\
Out[193]= \{Yy Zz, Xx Zz, Xx Yy\}\\
\]

As a vector derived from the gradient, it has no curl.

\[\nIn[194]:= \text{Curl[\%]}\\
Out[194]= \{0, 0, 0\}\\
\]

We can reset the names of the variables to more familiar choices with

\[\nIn[195]:= \text{Grad[x y z, Cartesian[x, y, z]}]\\
Out[195]= \{y z, x z, x y\}\\
\]
Now consider a vector that does have a curl (in the default coordinates)

```
In[196]:= Curl[{-Yy^2, Xx^2, Zz^2 + Xx^2 + Yy^2}]
Out[196]= {2 Yy, -2 Xx, 2 Xx + 2 Yy}
```

The divergence of a curl vanishes

```
In[197]:= Div[Curl[{-Yy^2, Xx^2, Zz^2 + Xx^2 + Yy^2}]]
Out[197]= 0
```

But for a non-curl vector

```
In[198]:= Div[{-Yy^2, Xx^2, Zz^2 + Xx^2 + Yy^2}]
Out[198]= 2 Zz
```

In the usual variables

```
In[199]:= Curl[{-y^2, x^2, z^2 + x^2 + y^2}, Cartesian[x, y, z]]
Out[199]= {2 y, -2 x, 2 x + 2 y}

In[200]:= Div[Curl[{-y^2, x^2, z^2 + x^2 + y^2}, Cartesian[x, y, z]], Cartesian[x, y, z]]
Out[200]= 0

In[201]:= Div[{-y^2, x^2, z^2 + x^2 + y^2}, Cartesian[x, y, z]]
Out[201]= 2 z
```

Or we can just reset the labels

```
In[202]:= SetCoordinates[Cartesian[x, y, z]]
Out[202]= Cartesian[x, y, z]

In[203]:= Coordinates[Cartesian]
Out[203]= {x, y, z}

In[204]:= Curl[{-y^2, x^2, z^2 + x^2 + y^2}]
Out[204]= {2 y, -2 x, 2 x + 2 y}

In[205]:= Div[Curl[{-y^2, x^2, z^2 + x^2 + y^2}]]
Out[205]= 0

In[206]:= Div[{-y^2, x^2, z^2 + x^2 + y^2}]
Out[206]= 2 z
```

Now consider switching the coordinate system to cylindrical

```
In[207]:= Coordinates[Cylindrical]
Out[207]= {Rr, Ttheta, Zz}

In[208]:= Grad[f[Rr, Ttheta, Zz], Cylindrical]
Out[208]= {Ttheta Zz, Zz, Rr Ttheta}
```
Notice the extra factor in the derivative with respect to the angle

Or with my choices for the coordinate labels

\[ \text{In[209]} = \text{SetCoordinates[Cylindrical[\(\rho, \phi, z\)]]} \]
\[ \text{Out[209]} = \text{Cylindrical[\(\rho, \phi, z\)]} \]

\[ \text{In[210]} = \text{Coordinates[Cylindrical]} \]
\[ \text{Out[210]} = \{\(\rho, \phi, z\)\} \]

\[ \text{In[211]} = \text{Grad}[f[\(\rho, \phi, z\)], Cylindrical] \]
\[ \text{Out[211]} = \{z \phi, z, \rho \phi\} \]

Now consider the same starting function - xyz in cartesian coordinates

\[ \text{In[212]} = \text{Simplify[Grad[\(\rho^2 \sin[\phi] \cos[\phi] z\), Cylindrical]]} \]
\[ \text{Out[212]} = \{z \rho \sin[2 \phi], z \rho \cos[2 \phi], \rho^2 \cos[\phi] \sin[\phi]\} \]

Note that all terms have dimensions of \((\text{length})^2\) - now going to spherical coordinates

\[ \text{In[213]} = \text{Coordinates[Spherical]} \]
\[ \text{Out[213]} = \{Rr, Ttheta, Pphi\} \]

\[ \text{In[214]} = \text{Grad}[f[Rr, Ttheta, Pphi], Spherical] \]
\[ \text{Out[214]} = \{Pphi Ttheta, Pphi, Ttheta \csc[Ttheta]\} \]

Now with my choices of labels

\[ \text{In[215]} = \text{SetCoordinates[Spherical[\(r, \theta, \phi\)]]} \]
\[ \text{Out[215]} = \text{Coordinates::invalid : Spherical[\(x, y, z, \theta, \phi\)] is not a valid coordinate system specification.} \]

\[ \text{In[216]} = \text{Coordinates[Spherical]} \]
\[ \text{Out[216]} = \{Rr, Ttheta, Pphi\} \]

Again for the function xyz

\[ \text{In[217]} = \text{Simplify[Grad[\(r^3 \sin[\phi] \cos[\phi] \sin[\theta] \cos[\theta] \sin[\theta]^2 \cos[\theta]\), Spherical]]} \]
\[ \text{Grad::nocoord : Spherical is not a non-empty list of valid variables.} \]
\[ \text{Out[217]} = \{x^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], y^3 \cos[\theta] \cos[\phi] \sin[\phi] \sin[\theta]^2 \sin[\phi], z^3 \cos[\theta] \cos[\phi] \sin[\theta] \cos[\theta] \sin[\theta]^2 \sin[\phi]\} \]

Having set the coordinates we no longer need to specify them - we get the same result with

\[ \text{In[218]} = \text{Simplify[Grad[\(r^3 \sin[\phi] \cos[\phi] \sin[\theta] \cos[\theta] \sin[\theta]^2 \cos[\theta]\)]} \]
\[ \text{Grad::argtu : Grad called with 1 argument; 2 or 3 arguments are expected.} \]
\[ \text{Out[218]} = \text{Grad[\{x^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], y^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], z^3 \cos[\theta] \cos[\phi] \sin[\theta] \cos[\theta] \sin[\theta]^2 \sin[\phi]\}]} \]
\textbf{In[219]}: \texttt{Curl[]} \\
\texttt{Curl::argtu : Curl called with 1 argument; 2 or 3 arguments are expected. \gg} \texttt{Curl[Grad[[x^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], \\
y^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], z^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi]]]]} \\
\texttt{Out[219]}: \texttt{Curl[Grad[[x^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], \\
y^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], z^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi]]]]} \\
\textbf{In[220]}: \texttt{Simplify[]} \\
\texttt{Out[220]}: \texttt{Curl[Grad[[x^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], \\
y^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi], z^3 \cos[\theta] \cos[\phi] \sin[\theta]^2 \sin[\phi]]]]} \\
\textbf{We also have} \\
\textbf{In[221]}: \texttt{SetCoordinates[Cartesian[x, y, z]]} \\
\texttt{Out[221]}: \texttt{Cartesian[x, y, z]} \\
\textbf{In[222]}: \texttt{Grad[x y z]} \\
\texttt{Out[222]}: \texttt{\{yz, xz, xy\}} \\
\textbf{In[223]}: \texttt{Div[Grad[x y z]]} \\
\texttt{Out[223]}: 0 \\
\textbf{Since there is no quadratic behavior, it must vanish as does the same quantity}