Chapter 8

CONCLUSIONS AND FUTURE WORK

In this chapter we summarize the work presented in this thesis and list some possible future research directions that are relevant to this work.

8.1 Conclusions

In this work, a wave propagation method in conjunction with a novel constrained transport technique was developed for approximately solving both the ideal and the shallow water magnetohydrodynamic equations. By using constrained transport, we are guaranteed that the divergence of the magnetic field is numerically conserved; and therefore, no unphysical magnetic monopoles are introduced into the numerical solution. This method does not require the solution of an elliptic equation as does the projection method [90], nor does it require solving a set of modified equations as in the 8-wave formulation [71] or the generalized divergence-cleaning method [21].

The constrained transport methods proposed by Balsara and Spicer [7] and Ryu et al. [78] were both found to be inconsistent and the Dai and Woodward [20] approach was found to introduce large unphysical oscillations near shocks. The methods proposed by De Sterck [84] and Londrillo and Del Zanna [63] were found to be incompatible with the wave propagation method.

Therefore, we introduced in this work a new constrained transport method that requires the solution of an advection equation for the magnetic potential. To accurately solve this equation without introducing spurious oscillations in the magnetic field, we introduced a modified composite scheme [61]. This method was shown to be consistent and significantly less oscillatory than the Dai and Woodward [20] approach.

In Chapter 5 the wave propagation method in conjunction with this new constrained transport scheme was tested on several problems in order to to verify its accuracy and robustness. We considered test cases for both the shallow water magnetohydrodynamic and the ideal magnetohydrodynamic equations. For the ideal MHD equations several test cases are available such as those presented in [76, 77, 90]. From these test cases we chose a coordinate aligned shock tube problem as well as some fully two-dimensional problems.

Solving the SMHD equations is a relatively new research area that began with the derivation of the equations in a paper by Gilman [30]. For shallow water magnetohydrodynamics there currently exist only two papers that have dealt with the numerical treatment of the equations. Numerical solutions to the one-dimensional SMHD equations are considered in [83] and a numerical method for the two-dimensional SMHD equations is presented in [84]. In order to demonstrate that our wave propagation method with constrained transport is also applicable to SMHD flows, we considered in Chapter 5 the same test problems that are presented in [83] and [84].

In Chapter 6 we considered the problem of modifying our scheme to generalized coordinates. In particular, we modified the curved manifold wave propagation method developed in Bale [3]. The curved manifold wave propagation method makes use of parallel transport and orthonormalization in order to simplify the solution procedure. Parallel transport turns out to be equivalent to the source term treatment we discussed in Section 2.7.2 and allows us to solve vector conservation laws on curved manifolds without the use of operator splitting techniques. Furthermore, orthonormalization allows us to update the solution by solving orthonormal (Cartesian) Riemann problems instead of the more complicated coordinate Riemann problems. In this same chapter we also extended the new constrained transport method to generalized coordinates.

Finally, in Chapter 7 we applied the curved manifold extension of the wave propagation method to flow on the surface of a sphere. We made use of a cubed sphere grid in order to avoid the grid singularities at the North and South Poles. On this grid we solved the linear advection equation, the shallow water equations, and the shallow water magnetohydrodynamic equations. For each system we considered test problems that demonstrated the validity of the numerical method.

8.2 Future work

In this work we have laid the groundwork for an accurate and robust numerical method for solving the equations of magnetohydrodynamics. In order to use this method to solve interesting physical problems several improvements and generalizations must be considered. Below we list three areas in which we would like to improve the proposed method.

8.2.1 Numerical analysis: improving the constrained transport scheme

The wave propagation method with constrained transport as it currently stands is only formally first order accurate. This is due to the fact that we make use of a composite scheme strategy that requires us to use Godunov's method during 30% of the time step when updating the magnetic potential advection equation (see Chapter 5). This situation could be remedied if a better wave limiter were developed that would guarantee a TVD property on the magnetic field and make the method fully second order method in smooth flow regimes.

8.2.2 Computer science: adaptive mesh refinement and parallelization

Recently, a new parallel adaptive mesh refinement version of the CLAWPACK software called BEARCLAW was developed [66]. This software has been shown to work very well on several test problems. In order to apply the constrained transport method to this framework, an accurate and conservative grid refinement strategy must be developed to handle the staggered grids present in constrained transport. An adaptive and parallel version of the constrained transport method is crucial for accurately solving two-dimensional and three-dimensional flows.

8.2.3 Applications: astrophysical flows

In addition to the numerical and algorithmic improvements mentioned above, we would also like to apply our method to problems relevant to astrophysical fluid dynamics. In particular, it would be interesting to apply the numerical method we developed for solving the SMHD equations on the sphere to the problem of simulating the solar tachocline. Such simulations could be compared to observations in order to determine how accurately the SMHD equations approximate real tachocline dynamics. Furthermore, it may be possible use such information to gain new insights into the formation of sunspots [30].

Other astrophysical problems for which an accurate numerical MHD solver is important include the formation of stars [15] and the prediction of space weather [16]. Application to these problems would require the generalization of the wave propagation method to function with more complicated physics taken into account. For example, one could consider the following generalizations:

- Special and general relativistic effects;
- More complicated thermodynamics;
- Multi-fluid magnetohydrodynamics.

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