Chapter 1

INTRODUCTION

Many interesting physical systems are successfully modeled with time-dependent conservation laws on smooth manifolds, $\mathscr{M}$. Examples of such systems include hydrodynamic flows in non-trivial geometry, important in aerodynamic modeling, for example. Though in many aerodynamic applications the manifold is simply Euclidean space, $\mathscr{M} = \mathbb{R}^3$, the curvilinear basis on which one computes is often non-orthogonal and quite complicated. Another example is geophysical flow, important for climate modeling and oceanography. In this case the manifold is the surface of the earth, which is approximated by a two-sphere, $\mathscr{M} = \mathcal{S}^2$. Researchers implement a wide range of choices for the bases that cover $\mathcal{S}^2$. One particular choice is shown in Figure 1.1, for which the sphere is covered by six non-orthogonal curvilinear sets of bases to avoid the polar singularity in standard coordinates. The algorithm presented in this thesis is being applied by Rossmanith and LeVeque [50] to various problems using such a coordinate setup.

The main example used in this thesis comes from modern numerical astrophysics. The hydrodynamic accretion of gas onto compact objects such as neutron stars and black holes is a current topic of research in astronomy and astrophysics. Figure 1.2 shows an example of gas spherically accreting onto a Schwarzschild black hole. The four dimensional curved manifold in this case is a solution to the Einstein equations of general relativity. Since we focus on relativistic applications in Chapter 8, a full description follows in Section 1.2.

Consider the flow of a substance whose state variables are ρ, $\vec{s}$, and e, the density, momentum, and energy, respectively. In the absence of any sources or sinks, the time rate of change of the integral of each state variable over the volume $\mathscr{V}$ is only dependent on the flux of that variable through the boundary, $\partial\mathscr{V}$. Mathematically, this is expressed with the following integral conservation law

$$\frac{d}{dt}\int_{\mathscr{V}} \boldsymbol{q}(t,\vec{x})\, d\mathscr{V} + \int_{\partial\mathscr{V}} \boldsymbol{f}(\boldsymbol{q}) \cdot \vec{\mathrm{d}\sigma} = 0, \tag{1.1}$$

where $\boldsymbol{q}$ is a vector of the state variables, $\vec{\mathrm{d}\sigma}$ is a surface normal, and $\boldsymbol{f}$ is the corresponding flux, possibly a nonlinear function of the components of $\boldsymbol{q}$. In the remainder of this thesis, bold-face quantities will denote vectors in state-space, for example, $\boldsymbol{q} = (\rho, s^i, e)^T$, where s^i represents the i^{th} component of the vector $\vec{s}$.

The differential form of (1.1) is written

$$\partial_t \boldsymbol{q}(t) + \vec{\nabla} \cdot \boldsymbol{f}(\boldsymbol{q}) = 0, \tag{1.2}$$

and is well known to a diverse community of computational fluid dynamicists. It is also well known that the integral conservation law may have discontinuous solutions that satisfy a weak form of (1.2) together with an entropy condition imposed by the underlying physics

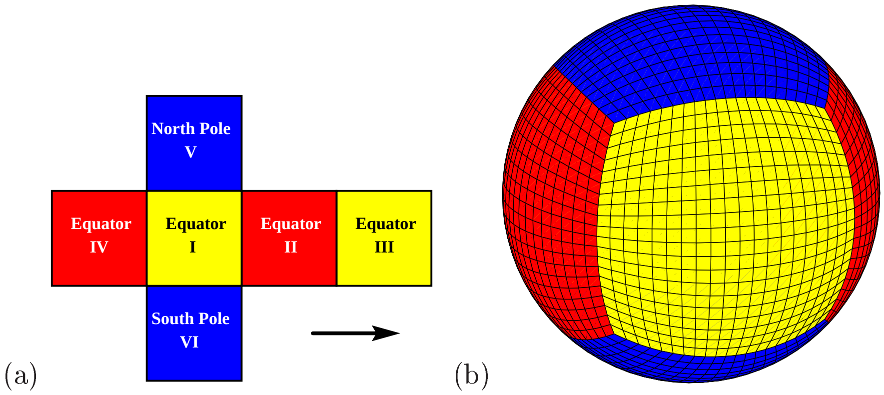

Figure 1.1: Six coordinate patches that cover the surface of a sphere, avoiding the polar singularity in standard coordinates.

of the system at hand. Finite difference numerical methods applied to such discontinuous solutions often produce non-physical oscillations in the solution. High-resolution finite volume algorithms, based on Godunov's original method [27], make use of local Riemann problems and correction fluxes that are limited to produce non oscillatory solutions with high accuracy in smooth regions. Since we are often interested in capturing discontinuous solutions of (1.1), we focus on the development of high-resolution finite volume methods.

In order to solve (1.2) numerically, it is necessary to express the conservation law in some basis. Typically, equations (1.2) are written, and solved, in a Cartesian basis $\{\vec{e}_{(1)}, \vec{e}_{(2)}, \vec{e}_{(3)}\}$. In such a basis these equations have the form

$$\frac{\partial}{\partial t}\boldsymbol{q}(t,\vec{x}) + \sum_i \frac{\partial}{\partial x^i}\boldsymbol{f}^{(i)}(\boldsymbol{q}) = 0, \tag{1.3}$$

with $i \in \{1,2,3\}$ for three dimensions, and we have introduced the notation that brackets around a subscript or superscript denote the vector along that particular coordinate direction (i.e. $\boldsymbol{f}^{(1)}$ is the flux in the x^1-direction).

Many systems of interest, however, involve complex geometry requiring that (1.2) be solved on a smooth manifold, $\mathscr{M}$, covered by a set of non-Cartesian basis vectors. In general, on a coordinate basis equation (1.2) takes the form of a balance law

$$\frac{\partial}{\partial t}\boldsymbol{q}(t,\vec{x}) + \sum_i \frac{\partial}{\partial x^i}\boldsymbol{f}^{(i)}(\vec{x},\boldsymbol{q}) = \boldsymbol{\psi}(\vec{x},\boldsymbol{q}), \tag{1.4}$$

where the flux has gained an explicit dependence on the spatial coordinates, and a geometrically induced source term has appeared. Details of the derivation of (1.4) and the forms of both the spatially varying flux and induced source terms can be found in Section 5.8.

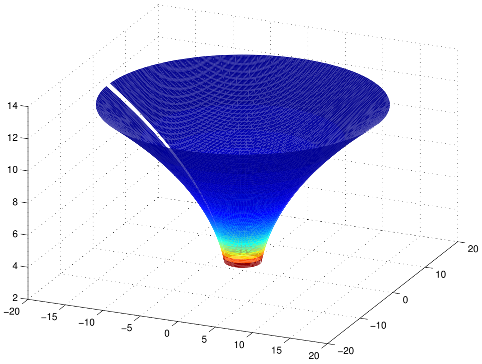

Figure 1.2: Steady-state accretion of a gas onto a Schwarzschild black hole. Plotted on an embedding diagram is the density at a late time of a typical simulation.

In this thesis I present a new unsplit high-resolution finite volume algorithm, an extension of the wave propagation method of LeVeque [42], to solve hyperbolic systems of conservation laws on general manifolds. The representation of such systems on $\mathscr{M}$, as discussed above, introduces a number of possible numerical complications due to geometric terms introduced by the representation of the divergence on a general basis. The algorithm presented here is based on both a *local* treatment of the divergence operator, and the solution of *Riemann problems* in locally orthonormal sets of bases, to avoid such numerical complications. We demonstrate that the method works well on both flat and curved manifolds by presenting examples of the algorithm for different systems of equations on various manifolds, focusing on results for a relativistic test fluid.

1.1 A class of modified wave propagation algorithms for conservation laws on curved manifolds

The representation of the divergence operator in (1.2) on a coordinate basis in $\mathscr{M}$ is responsible for both the explicit dependence of the flux on spatial coordinates, and the geometrically induced source term in (1.4). Such terms increase the computational complexity of numerical solutions and, in some cases, introduce numerical inaccuracies. The class of wave propagation algorithms developed in this thesis include the following features:

1. The algorithm Handles the spatially varying flux by defining f-waves (see Section 3.1), a set of waves that carry flux increments.

2. The method incorporates the effect of the geometrically induced source term into Riemann problems, thereby avoiding numerical complications with source terms.

3. The algorithm solves Riemann problems in sets of locally orthonormal basis vectors.

This has advantages over directly solving the Riemann problems in the spatially varying coordinate basis.

1.1.1 CLAWMAN *software package*

Successful algorithms have long been developed for hyperbolic systems in curvilinear coordinates [12], [78], and for specific systems of equations in more general manifolds [6], [20], but we are aware of no unsplit, high-resolution finite volume code able to handle different systems of equations on a variety of curved manifolds. The implementation of our modified algorithm is both easy to use, and versatile, making it useful to a broad class of users.

The original wave propagation algorithm [39], [44], [45] is implemented in the software package CLAWPACK (**C**onservation **LAW**s **PACK**age), a collection of freely available Fortran routines [41]. This software requires the user to provide, in addition to initial and boundary conditions, only the Riemann solver for the system of hyperbolic PDEs at hand. Consequently, CLAWPACK is easily applied to a variety of systems on standard Cartesian grids.

The algorithm presented in this thesis is implemented in the software package CLAWMAN (**C**onservation **LAW**s on **MAN**ifolds), a modification of CLAWPACK to handle the numerical techniques discussed throughout this dissertation. These routines are written in Fortran 90 and freely available [3]. In addition to the necessary routines for CLAWPACK, users need only provide a routine to define the metric tensor (discussed in Section 5.5), thereby defining the manifold's geometry. The package internally computes all necessary geometric information derived from the metric, such as derivatives of the metric and Christoffel symbols (i.e. see Section 5.6). Since Riemann problems are solved in sets of locally orthonormal basis vectors, approximate Riemann solvers need only be modified to handle spatially varying flux functions, as discussed in Section 3.3. That is, no geometric modifications are necessary within the Riemann solvers. We consider this to be an attractive feature of CLAWMAN. We note, however, that solving Riemann problems in locally orthonormal sets of basis vectors assumes that curvature in the manifold is well resolved on the grid being used.

1.2 Numerical relativistic astrophysics

The Einstein equations of General Relativity describe the geometric structure of spacetime. Solutions to these equations describe astrophysical phenomena such as black holes, neutron stars, supernovas, and the large-scale structure of the universe. Unfortunately, the Einstein equations are a set of very complex, highly coupled, nonlinear PDEs involving ten functions of one time-like variable and three space-like variables. Most of what is known about this set of equations comes from idealized solutions arrived at by linearizing the equations, or perturbing the equations about a solution with a high degree of symmetry. In order to investigate the full set of Einstein equations with no prescribed symmetry it is necessary to turn to numerical approximations. The field of study in which the numerical solutions to the Einstein equations are sought is called *numerical relativity*.

1.2.1 *Numerical study of relativistic problems*

In the mid 1960s, Hahn and Lindquist [29] made the first attempt to numerically evolve the collision of black holes, marking the birth of numerical relativity. Their pioneering work, however, was hindered by a poor theoretical understanding of black hole spacetimes and a bad choice of coordinates. At about the same time, May and White [59] successfully evolved the one dimensional collapse of a relativistic system with matter. Since these very early days, numerical relativity has been divided into two branches :

1. Numerical Relativistic Hydrodynamics

2. Numerical Relativistic Geometrodynamics.

The first branch, numerical relativistic hydrodynamics, is the numerical study of relativistic systems in the presence of matter (i.e. a hydrodynamic fluid). This branch will simply be called *relativistic hydrodynamics.* Numerical relativistic geometrodynamics is the numerical study of the dynamics of the gravitational field in vacuum. It is this branch that is typically referred to as *numerical relativity* and we follow this convention.

Test fluid approximation

Relativistic hydrodynamics continued to grow through the sixties with numerical calculations of supernova explosions by Wilson [81], and more investigations of relativistic collapse by May and White [60], and Matsuda and Sota [58]. Progress continued through the 1980s with calculations of nonspherical black hole accretion by Hawley and Smarr [33], [32], and relativistic stellar collapse by Nakamura and Sato [63], [64]. The nonspherical accretion problems of Hawley and Smarr were done using the *test fluid approximation.* All relativistic numerical calculations presented in this thesis have been done using this approximation.

The test fluid approximation refers to systems in which the flow of matter fields is significant, but the matter itself has little influence on the dynamics of the background gravitational fields. The gravitational field, therefore, is not changed by the flow of a test fluid. For example, a black hole accreting mass at the Eddington rate will take $\approx 10^8$ years to double its mass. The flow, in this case, has little influence on the gravitational dynamics on a relatively shorter time scale. Applications of the test fluid approximation include, but are by no means limited to, fluid accretion onto compact objects, the creation and subsequent propagation of extra-galactic jets, tidal disruption of stellar size objects in strong gravitational fields, shock acceleration of cosmic rays, and supernovas explosions. Though the test fluid approximation applies to any matter field with insufficient energy density to change the gravitational field, we concentrate on hydrodynamic matter fields in this thesis.

Today, increased computer speed and memory have allowed modern, high-resolution finite volume codes to accurately capture the flow of a test fluid for a number of astrophysical systems. The simplest test fluid is that of a *special relativistic* perfect fluid for which the background gravitational field is negligible. Riemann solvers have been developed for this set of equations by Martí and Müller [55], Balsara [5], Pons, Martí and Müller [68], and others

[17], [20], [76]. These Riemann solvers have been applied in finite volume methods for the special relativistic fluid equations by Font et al. [21], Martí et al. [54], [56], [57], Marquina et al. [52], and others. These numerical methods have been applied to extra-galactic radio jets [1], [53]. In Chapter 8, we present high-resolution numerical results for two standard test cases found in the literature. We demonstrate that the algorithm presented in this thesis performs well for a special relativistic test fluid.

In recent years, astronomers have been interested in the relativistic flow of matter in the presence of compact objects like neutron stars and black holes. In this case the background gravitational field is not negligible, but fixed. In such ultra-relativistic flows, gases emit high-energy radiation that provides astronomers with clues to the underlying physical processes taking place. A powerful way to improve the understanding of such complex flows is through accurate and stable numerical simulation. There are only a few numerical codes based on high-resolution finite volume updates that have been developed for a relativistic perfect fluid in a fixed background spacetime [6], [20], [38], [67]. The main goal of the research presented in this thesis is to develop a code for a relativistic perfect fluid in a fixed background. The novel feature of the algorithm presented is that it is unsplit and based on wave propagation. In Chapter 8, test cases from the literature are presented to demonstrate that this goal has been achieved.

1.2.2 Vacuum Spacetimes

It was not until the mid 1970s that the second branch, numerical relativity, became active again. In 1975 Eppley [19] successfully evolved the collision of two equal mass black holes in vacuum. The gravitational waves generated in systems like this should propagate to observers here on earth, but the large distance of propagation will attenuate the strength of the waves. Researchers are hard at work constructing ground based laser interferometers (*LIGO and GEO*)[77], to measure these gravitational waves. However, the very weak signal received will be noisy and it will be important to know what wave forms the Einstein equations predict to make the process of signal extraction easier. An important role of numerical relativity will be to catalog many such systems to aid in the data analysis from laser interferometry.

Progress in this branch has been slow due to the complexity of the vacuum Einstein equations. Significant challenges in numerical relativity that lead to open questions include:

- The Einstein equations can be cast in many different forms. It is not yet clear what analytical form of the equations is best suited for numerical implementation.

- There exists an intrinsic gauge freedom in general relativity resulting in a freedom in the choice of coordinates. A bad choice of coordinates will result in *coordinate shocks* that introduce numerical inaccuracies and perhaps instabilities.

- Once an analytic form of the Einstein equations has been chosen, a numerical scheme must be chosen to solve them. It is an open question as to what numerical properties are beneficial.

- Boundary conditions pose a numerical challenge both at interior boundaries near the event horizon of a black hole, and in the radiation field.

One approach in recent years has been to formulate the Einstein equations as a hyperbolic set of balance laws [9], [10], [11]. Once the equations have been cast as a hyperbolic set of balance laws, a large body of work for high-resolution finite volume methods is available. The hope is that the large gradients that develop near boundaries or due to a poor choice of coordinates will be handled in a stable way by such numerical schemes. Bardeen and Buchman are applying high-resolution wave propagation algorithms to the vacuum Einstein equations written as several different hyperbolic systems [7].

1.2.3 The full problem

We have discussed two active areas of research in relativistic astrophysics: numerical relativistic hydrodynamics (test fluids) and numerical relativity. As I mentioned earlier, these are approximations to the full problem in which gravitational fields evolve and push matter around, but the matter in motion affects the evolution of the gravitational fields. This is considerably more challenging than either problem alone.

One benefit to the hyperbolic formulation of the vacuum equations and their numerical solution is that this system can be combined with the hyperbolic equations for a test fluid. The entire system can then be evolved as one hyperbolic system of balance laws. The possible advantages of this include:

- The high-resolution capabilities of finite volume schemes will capture sharp hydrodynamic shock waves while providing additional robustness for coordinate shocks and ill-behaved boundary conditions.

- The gauge freedom in general relativity makes it difficult to impose physical boundary conditions on the correct fields. Since high-resolution finite volume schemes are based on local characteristic decompositions, there is a chance of handling such boundary conditions in a stable way.

The class of wave propagation algorithms developed in this thesis have been applied to problems with stationary metrics. These methods can be modified to handle a metric that evolves in time. Such modifications could then be combined with a hyperbolic code for the vacuum equations to solve the full problem.

1.3 Outline of this thesis

Since the methods developed here are based on the standard wave propagation algorithm, we present the standard algorithm in Chapter 2.

Chapter 3 is based on a recently completed paper [4], in which we develop high-resolution finite volume methods for hyperbolic conservation laws with spatially-varying flux functions. The calculation of f-waves is discussed, and examples are presented to demonstrate that the method works well.

Chapter 4 discusses various problems that can arise for conservation laws with source terms. We present two un-split ways to handle source terms. One, the quasi-steady method, assumes that flux gradients are being balanced by source functions. In such a situation, the solution should be near steady state. This method, however, is not very applicable to problems away from steady state. A second method that we discuss assumes nothing about such a balance between flux gradients and source terms.

We use concepts and notation from both differential geometry and general relativity in the remainder of the thesis. There exists no single standard for notation throughout these fields. Chapter 5 introduces some basic concepts we use throughout the thesis and sets the notational convention. We introduce the standard wave propagation method in Chapter 2.

In Chapter 6 we use ideas developed in Chapters 3 and 4 to develop the main algorithm presented in this dissertation. We present, in detail, the modifications to the standard wave propagation algorithm that are necessary to solve conservation laws on smooth manifolds. We present examples of systems of various manifolds to demonstrate that the method works well.

The algorithm developed in this dissertation is capable of handling many different systems of hyperbolic conservation laws on general manifolds. An implementation of the algorithm that takes advantage of this generality will be useful to a wide range of researchers. In Chapter 7 we present an implementation of the algorithm in the software package CLAWMAN.

Finally, in Chapter 8, we apply the new algorithm to the relativistic hydrodynamic equations. We develop the equations as a set of conservation laws, discuss some of the difficulties when trying to numerically solve them, and finally present examples to show that the algorithm performs well on relativistic problems.