Noise-sensitivity in stochastic nonlinear models with delays and discontinuities

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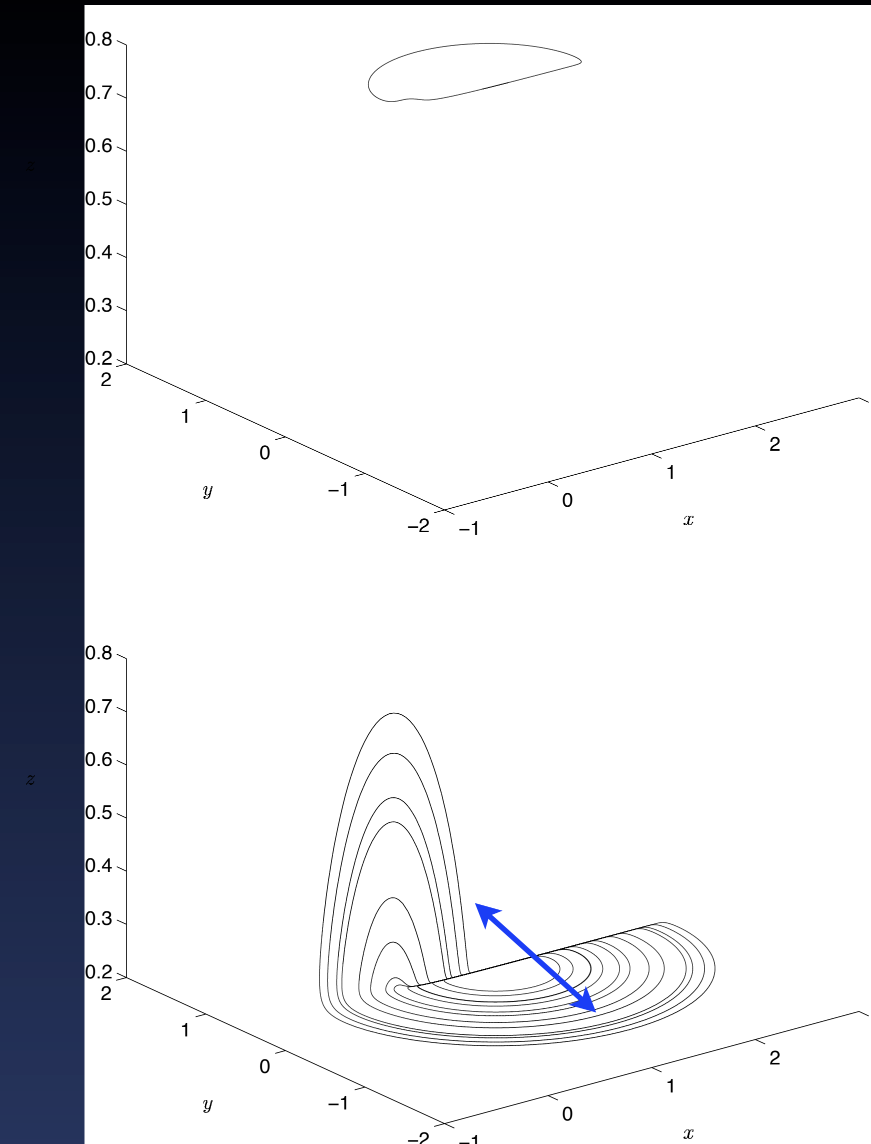
Noise reducing complexity:

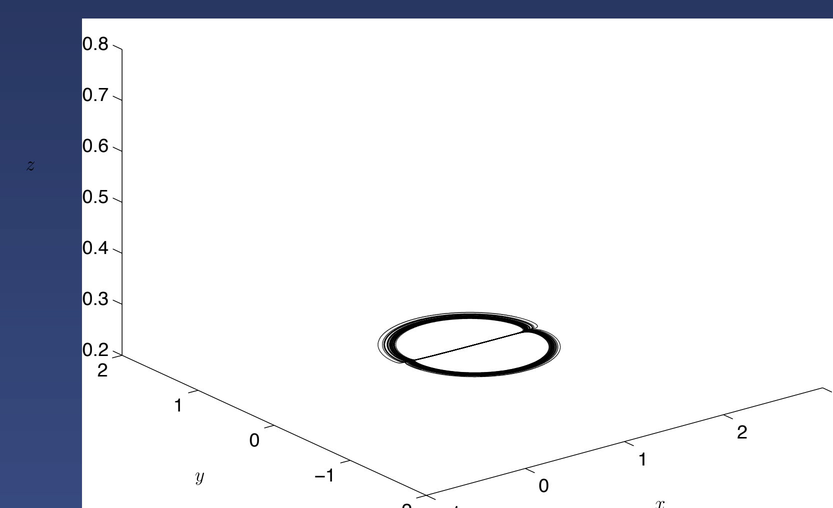
$$
\begin{aligned}
dx &= (\mu x - y^2 + 2z^2 - \delta z)dt \\
dy &= y(x-1)dt + \sqrt{2}\epsilon dW \\
dz &= (\mu z + \delta x - 2xz)dt,
\end{aligned}
$$

Hughes, Proctor 1990



Noise (dW = white noise) perturbs trajectories near slow manifold ($\mu \ll 1$)

Nearly periodic trajectory, well approximated by 2D probability density



Also, important in computations: identify potential computational error

K., Papanicolaou, 1998

Computational questions:

- Convergence of numerical methods: Strong (pathwise) convergence vs. Weak convergence
- Dynamics of numerical schemes
- Stochastic bifurcations: qualitative changes of dynamics at specific parameter values
- Stability of schemes
- Questions can vary with types of noise

(Concentrating on SDE's)

Examples:

- Convergence of numerical methods: Strong (pathwise) convergence vs. Weak (in distribution) convergence

Strong: $E|X - Y_h|$

Weak: $|E[X] - E[Y_h]|$ $\quad |f_X(x) - f_{Y_n}(y)|$

Forward, Backward Kolmogorov equations: PDE's

Karniadakis (2013), Schwab (2012) (DG, discontinuous dynamics)

Examples: $dY = a(Y)dt + b(Y)dW$

Strong:

$O(\sqrt{h})$ Euler-Maruyama $Y_{n+1} = Y_n + a(Y_n)h + b(Y_n)Z_n\sqrt{h}$

$O(h)$

θ - Maruyama method

Milstein

$$Y_{n+1} = Y_n + a(Y_n)h + b(Y_n)Z_n\sqrt{h} + b(Y_n)b'(Y_n)/2(Z_n^2 - 1)h$$

Weak:

$O(h)$ Euler-Maruyama

$O(h^2)$: Multi-step uses: $Y_n + ah + bZ_n\sqrt{h}$ $Y_n + ah \pm b\sqrt{h}$

Higher order: Additional r.v.'s, derivatives needed

Kloeden, Platen 1992

Recent examples:

- Nonlinearities:, tamed explicit methods
  Hutzenhaler, 2012

- Stability of schemes: Questions can vary with types of noise, or quantities of interest

$$dx = a(X)dt + b(X)dW$$

Multiplicative noise: could have X=0 as an equilibrium

$$dx = a(X)dt + bdW$$

Additive noise: X=0 is not an equilibrium

Buckwar, Riedler, Kloeden, 2011

- Dynamics of schemes
- Dynamical behavior

Stability for nonlinear SDE's: contractive conditions

Buckwar, Riedler, Kloeden, 2011

Non-normal drift - interaction of drift and diffusion in discretized system (stability) Buckwar, et al 20

Ito vs. Stratonovich interpretation of dW:
endpt vs. midpt evaluation of integrand, Ito friendlier for coding, Stratonovich usually used for parametric noise in applications

Dynamical Questions:

Noise driven order: "Stabilized" transients

Can't ignore: Transients from the deterministic dynamics
"Small" random perturbations drive qualitative changes

Stochastic facilitation: Constructive roles of biologically relevant noise in the nervous system

McDonnell, Ward Nature Neuroscience Reviews 2011

Various types of dynamics: bifurcations + delays

Discontinuous, Piecewise Smooth, dynamics: Sliding, grazing, impacts, virtual dynamics, control

Dynamical Questions:

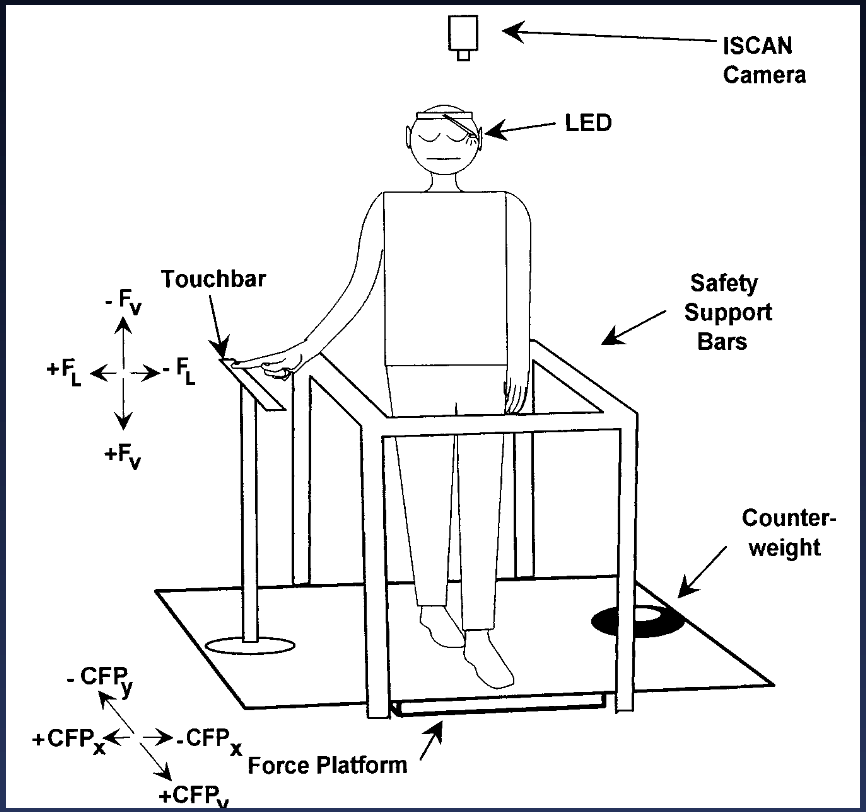
- Interplay of computational results and analysis
- Interplay of dynamics (and time scales) with stochastic perturbations - not necessarily separable
- Relatively fast, easy-to-code simulations to test and motivate "interesting" cases/parameter ranges
- Whose time is more valuable: researcher time or machine time? Significance of higher order methods

Delays: Models of Balance

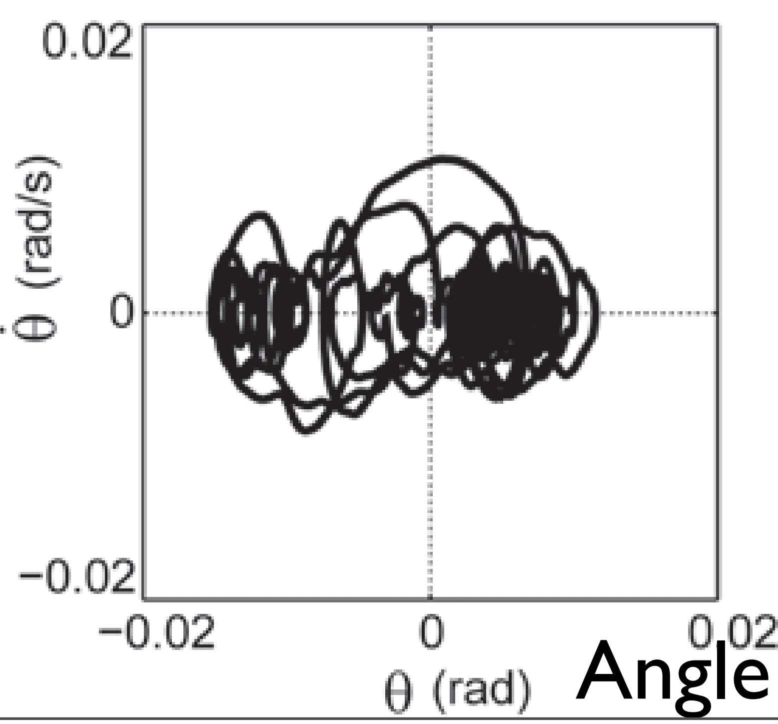
- Applications: Human Postural Sway, Stick Balancing, Robotics
- What are the contributing factors to stability, instability, balance, sway, other behaviors?

Transfer ideas between models in mechanics/optics and biological applications: transients sustained by stochastic effects

- Applications: Human Postural Sway, Stick Balancing, Robotics



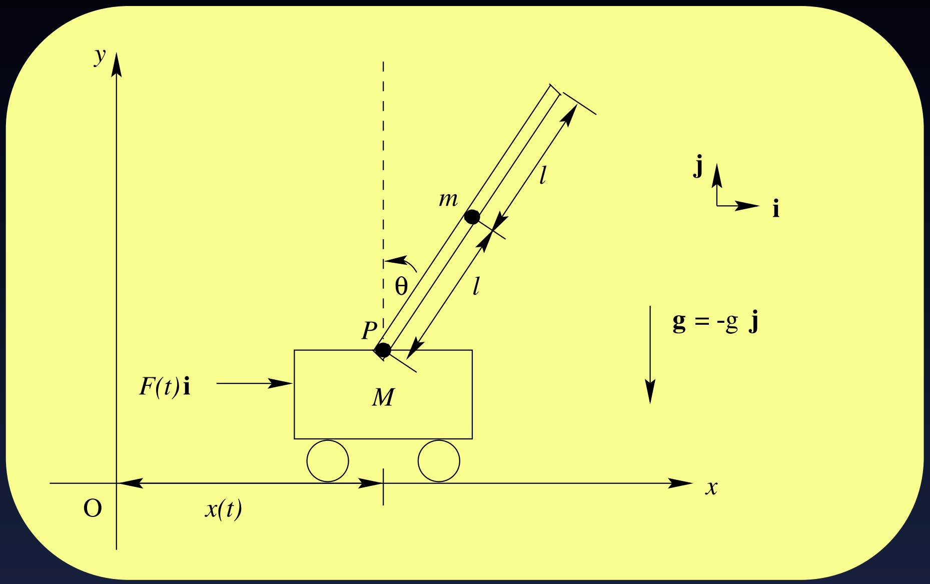
Derivative of the Angle from vertical



Asai, etal, 2009, PLOS1

Angle from vertical

Simple model: inverted pendulum

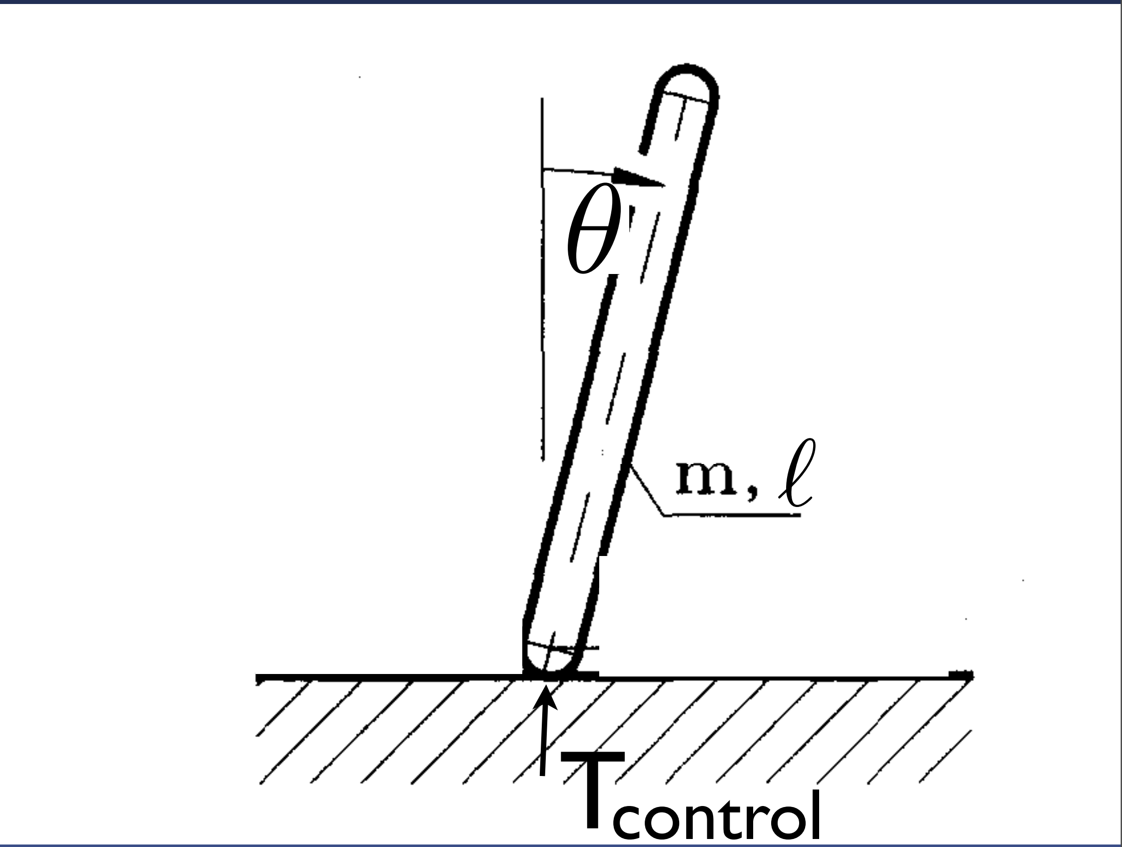


Stabilized on a cart

$$\left(1-\frac{3m}{4}\cos^2\theta\right)\ddot{\theta}+\frac{3m}{8}\dot{\theta}^2\sin(2\theta)-\frac{3}{2}\frac{g}{L}\sin\theta+\frac{3F}{2L(M_\mathrm{p}+M_\mathrm{c})}\cos\theta=0.$$

$$\frac{4}{3}m\ell^2\ddot{\theta}-mg\ell\,\sin(\theta)=T_{\mathrm{control}},$$

Even more simple model: inverted pendulum w/ pivot control (torque at the pivot)



Reduced to essentials

$$\dot{\theta} = \phi$$
$$\dot{\phi} = \sin\theta - F(\theta, \dot{\theta})\cos\theta$$

$$F = a\theta(t-\tau) + b\dot{\theta}(t-\tau)$$

(P) (D)

$$F = a\theta + b\dot{\theta}$$

Proportional Derivative

PD control

Biological considerations:

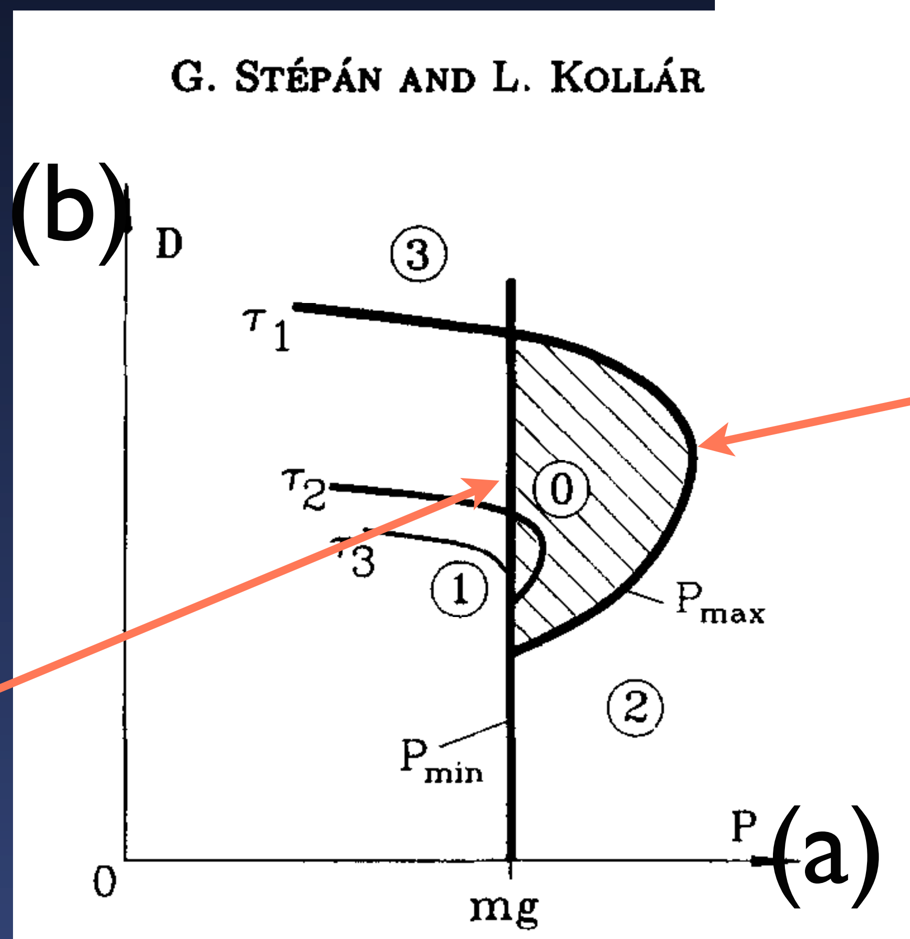
- Delay in the application of the control: neural transmission
- On-off control: not active control all the time
- Noise: Small random fluctuations can result in large changes in certain circumstances

Balance model:

$$\begin{aligned}\dot{\theta} &= \phi \\ \dot{\phi} &= \sin\theta - F(\theta,\dot{\theta})\cos\theta\end{aligned}$$

$$F = a\theta(t-\tau) + b\dot{\theta}(t-\tau)$$

Stability diagram for PD control (w/delay), control always on



Hopf bifurcation: oscillations for larger values of the control parameters a,b

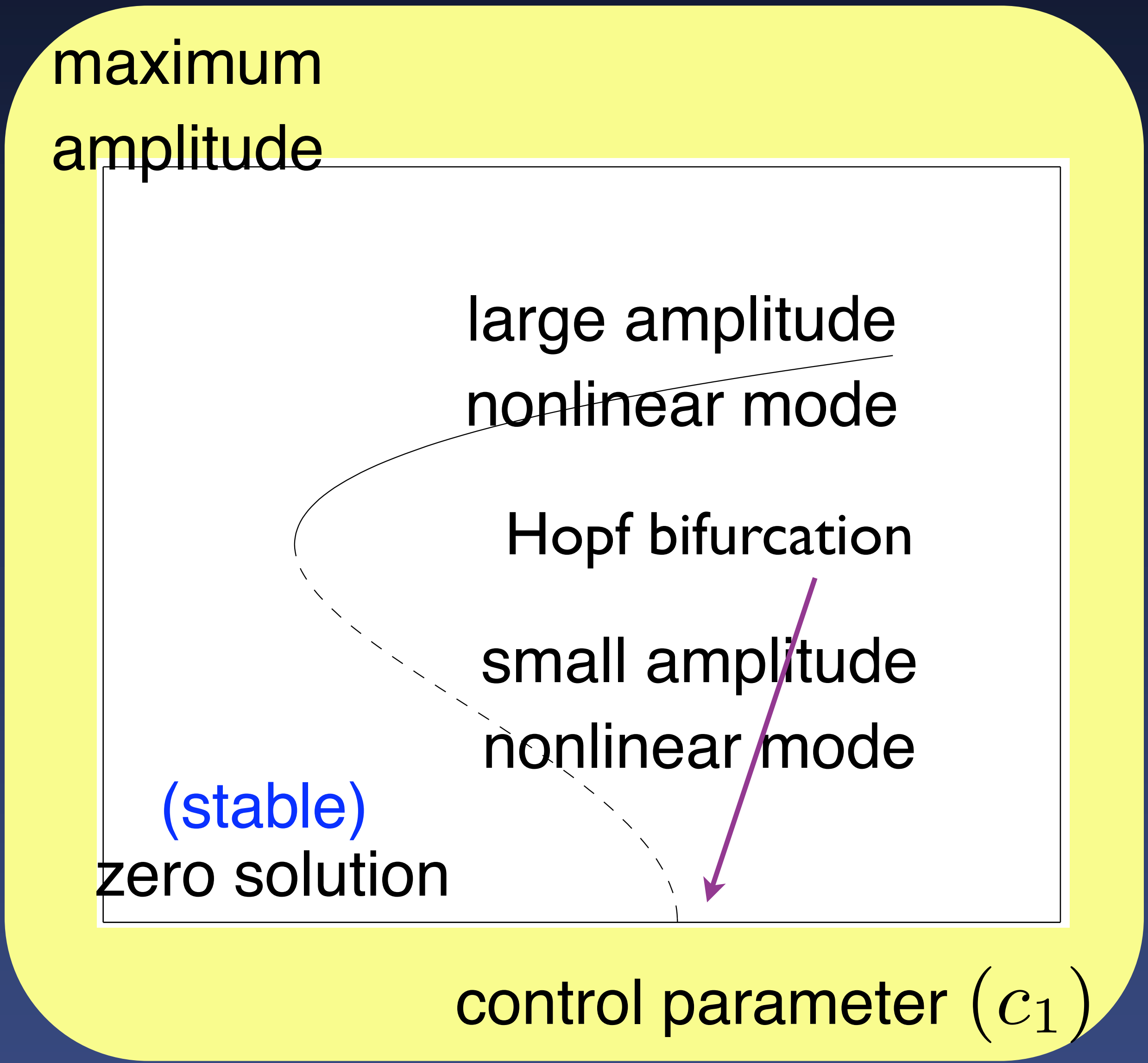
Pitchfork bifurcation

θ = 0 is stable in shaded area, D-shaped region

Pitchfork bifurcation: stability of non-zero fixed point

Hopf bifurcation, stability of oscillatory behavior

Fully nonlinear model: w/ Hopf bifurcation



Hopf bifurcation: oscillatory solutions for certain delay

Smaller nonlinear oscillatory solutions unstable

Larger nonlinear oscillatory solutions bi-stable with x=0 solution

Mathematical Challenges:

- Nonlinearity: non-uniqueness, complex dynamics
- Delays: Delay Differential Equations (DDEs)
- Noise: Stochastic DDEs (SDDEs), minimal theory, limited computational methods

Dynamical References: Campbell, Milton, Ohira, Sieber, Krauskopf, Stepan, K. others

SDDE's: $dx = f(x(t), x(t-\tau)dt + g(x(t), x(t-\tau)dW(t)$

- Dynamics: e.g. Mean Square Stability: Linear system w/ delays, Buckwar et al 2013

- Numerical methods

Milstein method ($O(h)$ strong convergence) for SDDE's:
Kloeden, Shardlow, 2012 (Taylor-like expansions)

Euler-Maruyama, ($O(h)$weak convergence) for SDDE's:

Distributed delays:
Buckwar, et al, 2005 (linear), Clement, et al 2006

Anticipating (Malliavin) calculus: tame Ito formula on segments of solution

$$\int_{t_1}^{t_2} f^{(i)}(Y(u-\tau), Y(u)) \dots dW(u-\tau)$$

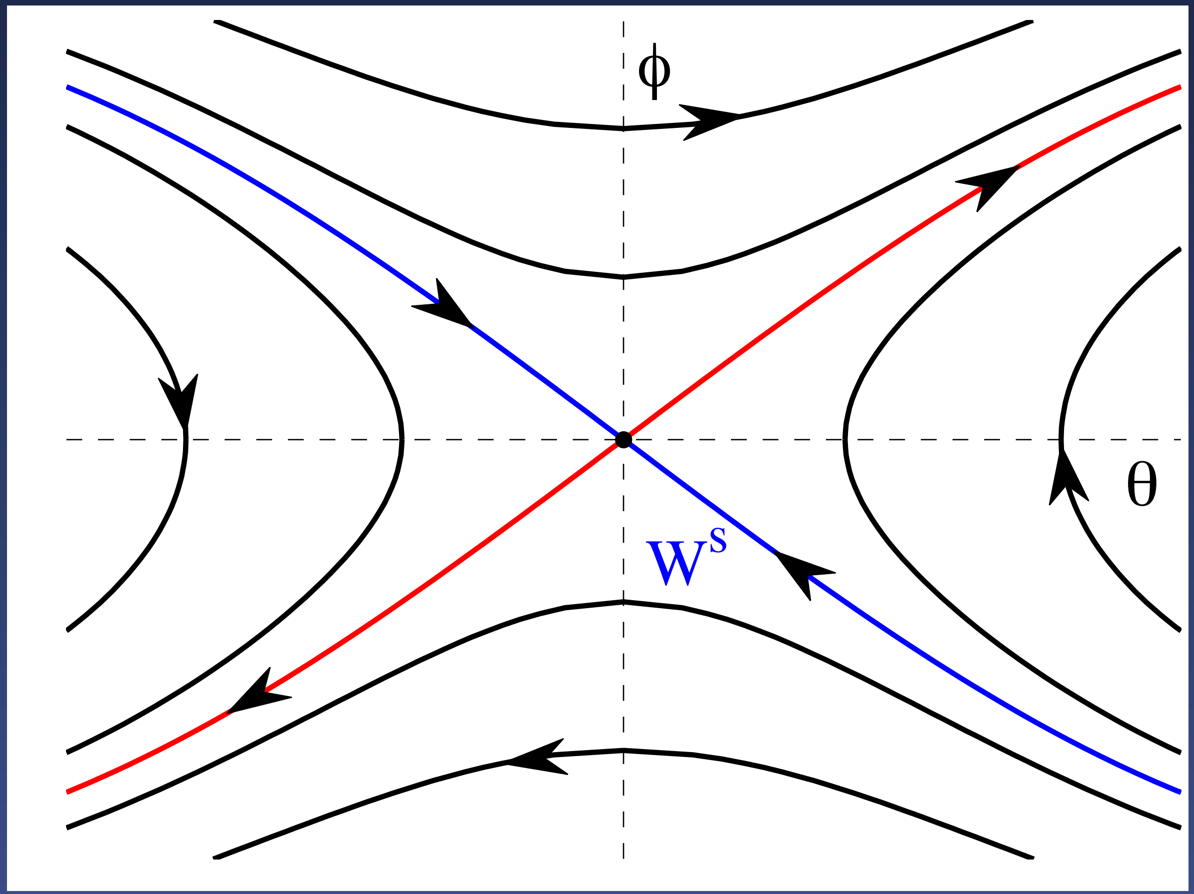
Buckwar, K. , Mohammed, Shardlow, 2008

State-dependent control in balance models:

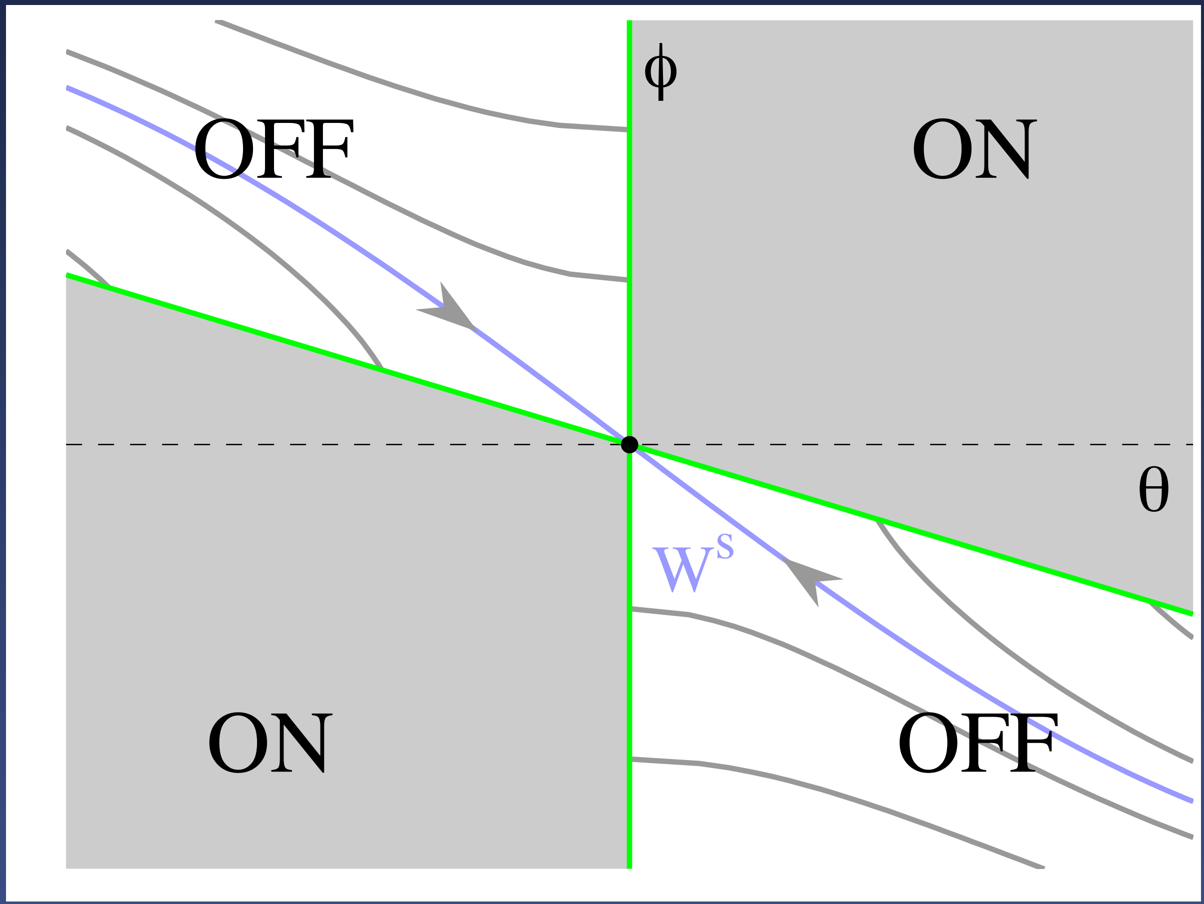
PD control with delay

$$\dot{\theta} = \phi$$
$$\dot{\phi} = \sin\theta - F(\theta, \dot{\theta})\cos\theta$$
$$F = a\theta(t-\tau) + b\dot{\theta}(t-\tau)$$

Phase plane for system with control off



State-dependent on/off control: control is on in state drifting away from origin



Asai, etal, 2009, PLOS1

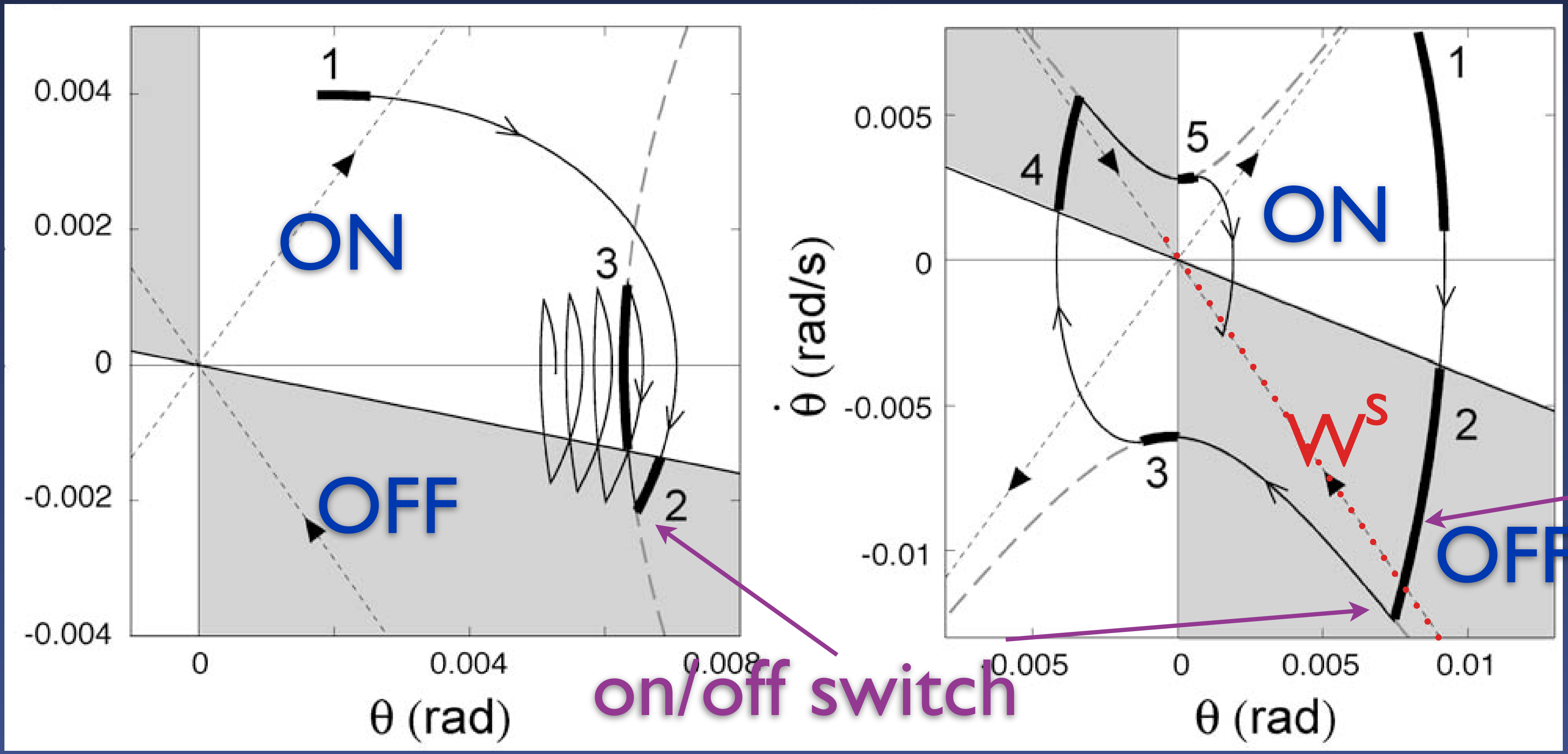
Zig-zags and spirals:

PD control with delay

$$\ddot{\theta} - \sin\theta + F(\theta, \dot{\theta})\cos\theta = 0$$
$$F = a\theta(t-\tau) + b\dot{\theta}(t-\tau)$$

On/Off Control : Note delay in switching from on/off, system enters off region before control is switched off

Weaker control: zig-zag behavior



Larger delay/ Stronger control: yields spiral

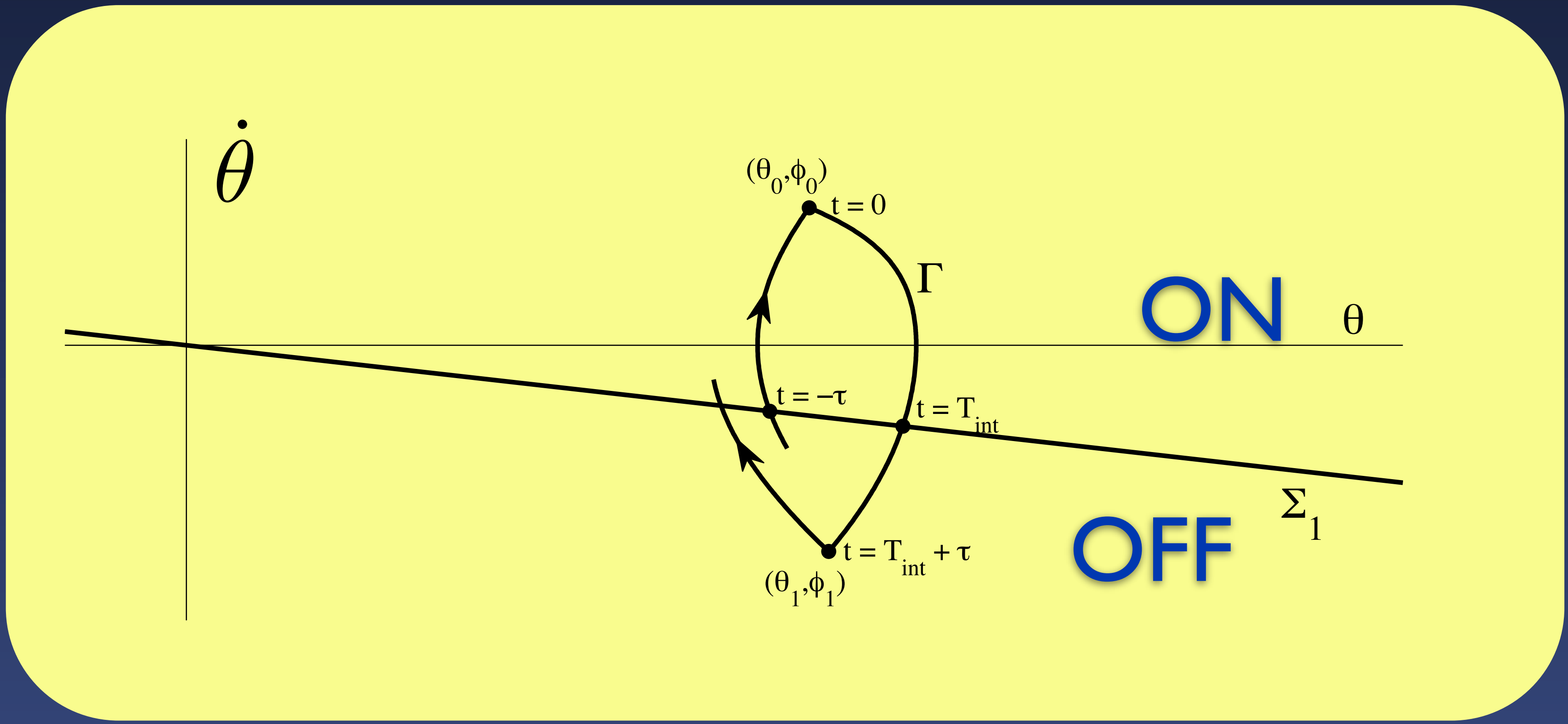
Asai, etal, 2009, PLOS1

Periodic behavior:

$$\ddot{\theta} - \sin\theta + F(\theta, \dot{\theta}) \cos\theta = 0$$
$$F = a\theta(t-\tau) + b\dot{\theta}(t-\tau)$$

PD control with delay

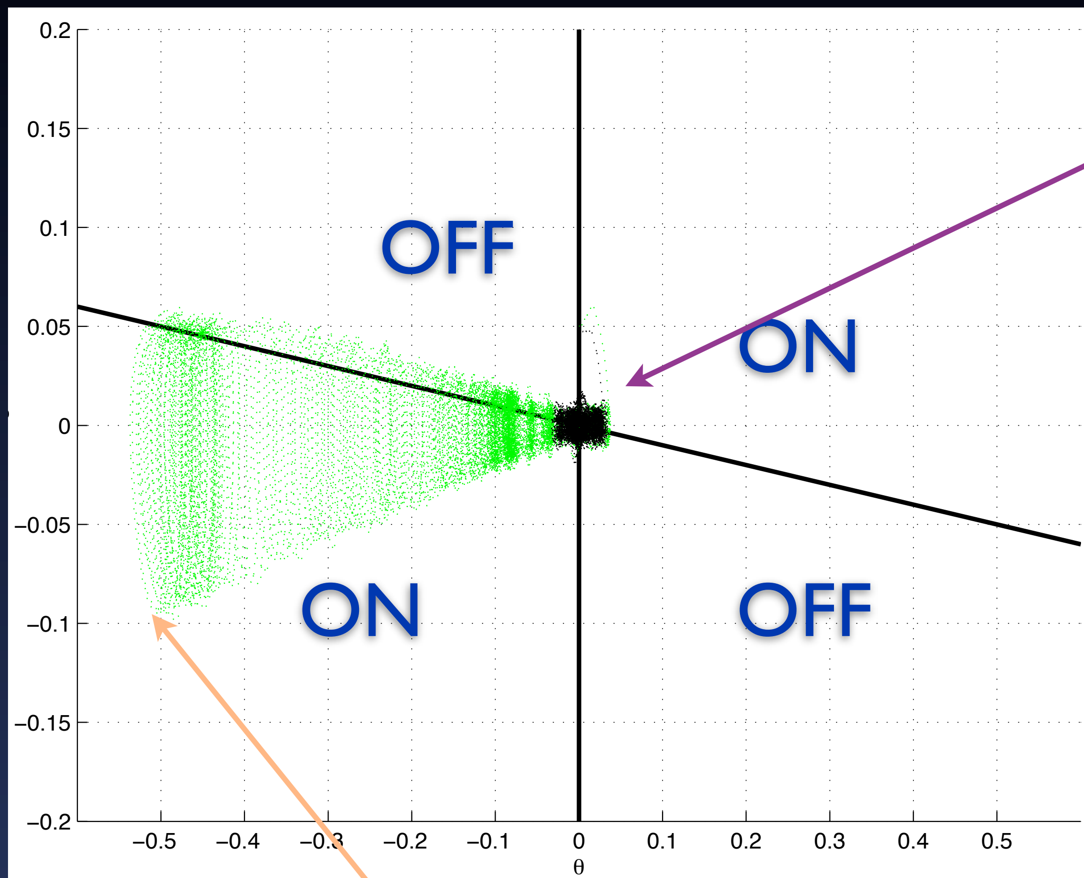
Zig-zag orbits - periodic solutions away from origin, moving back and forth from on/off

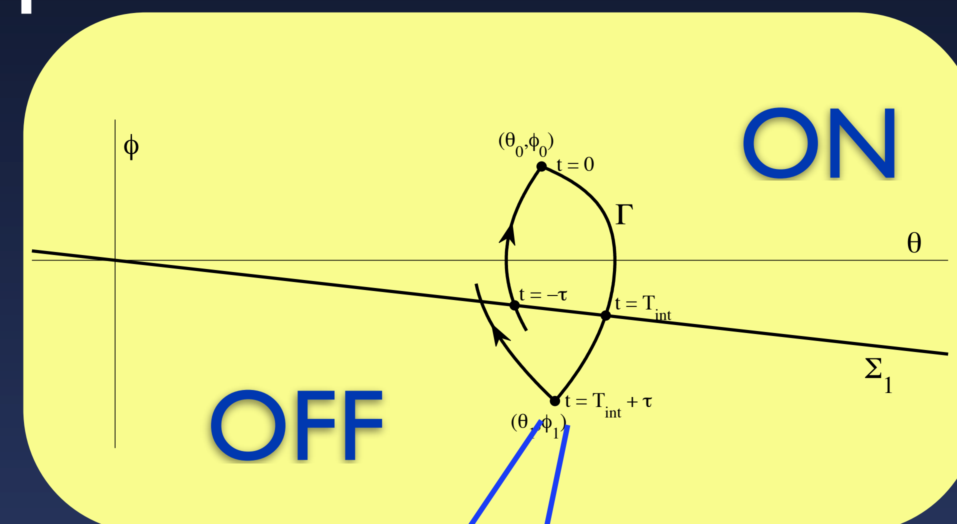


Note: w/o delay: PWS system is Filipov, solution slides along the switching manifold, can calculate analytically

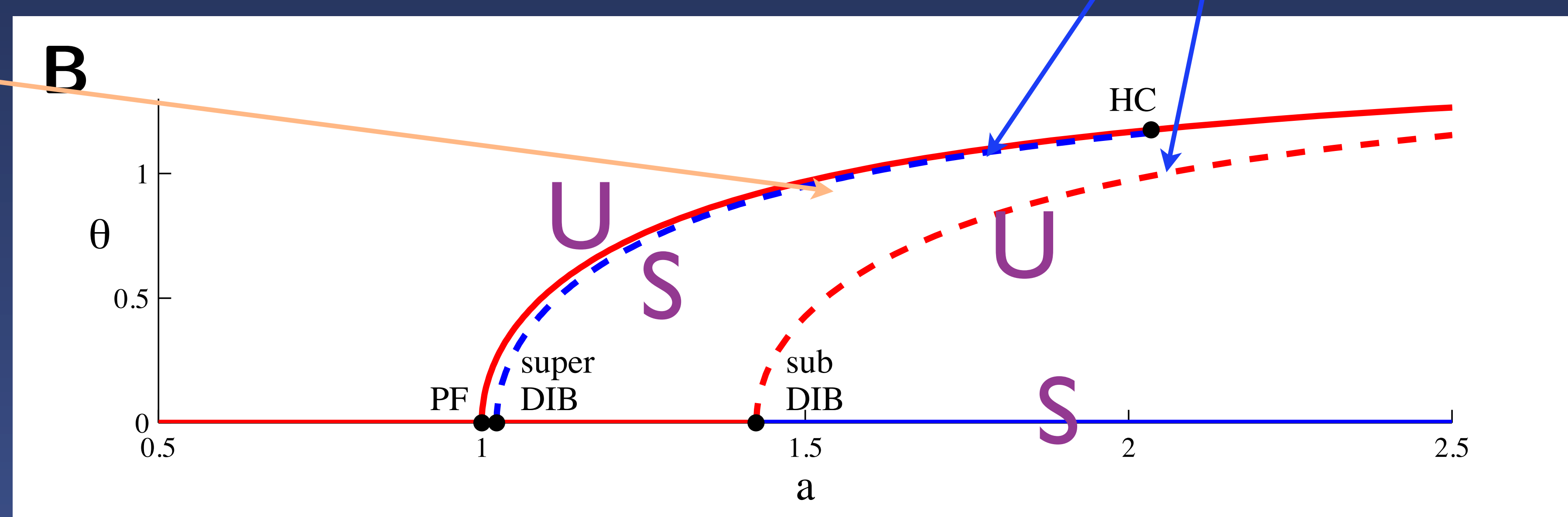
Balance model w/ noise: small random fluctuation

For larger control: oscillations near the origin: weak attraction to the origin





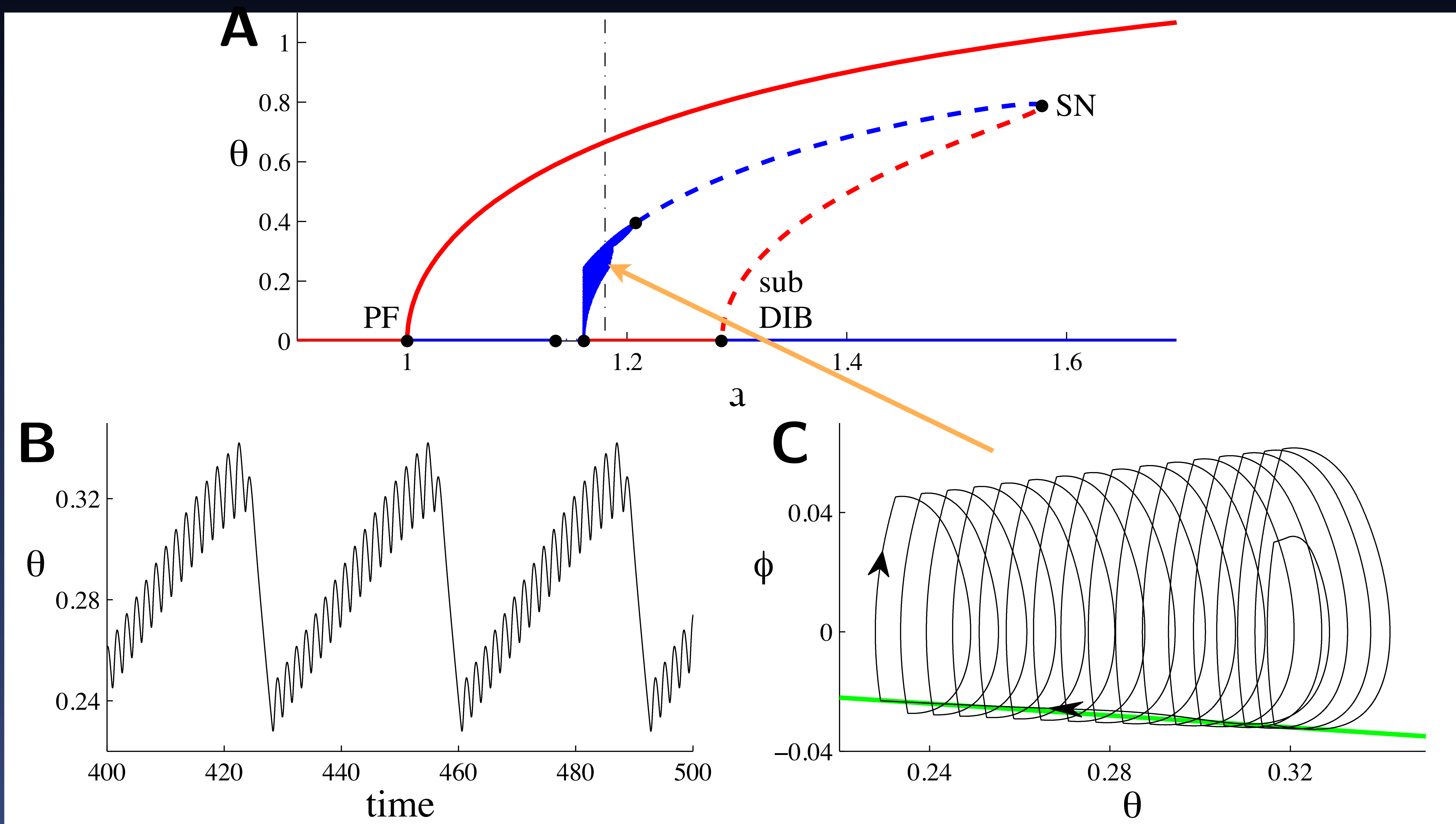
For smaller control, noise can cause transition to zig-zag orbit



Bifurcation diagram: control a vs. θ

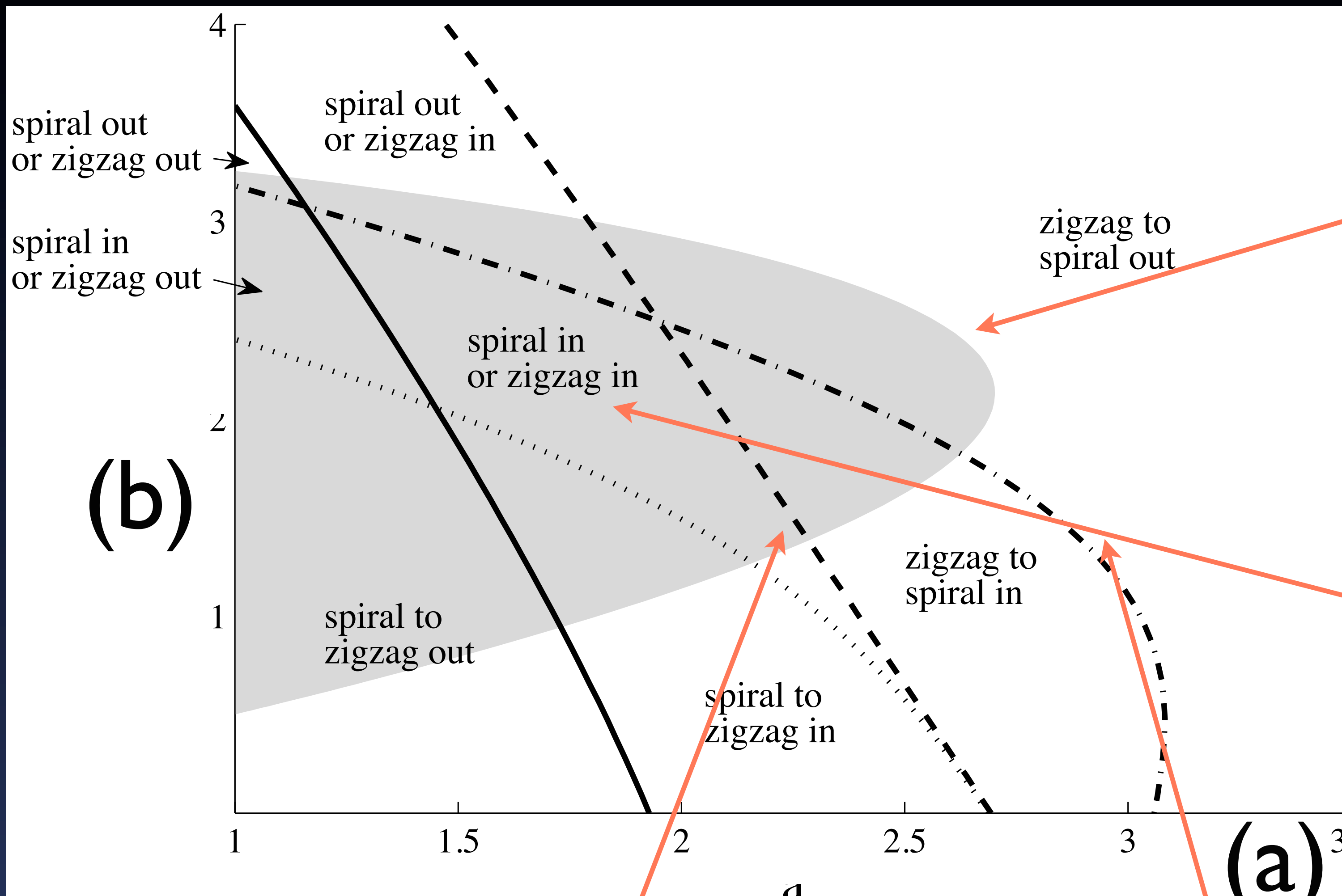
Bursting-like oscillations

Experimental observations



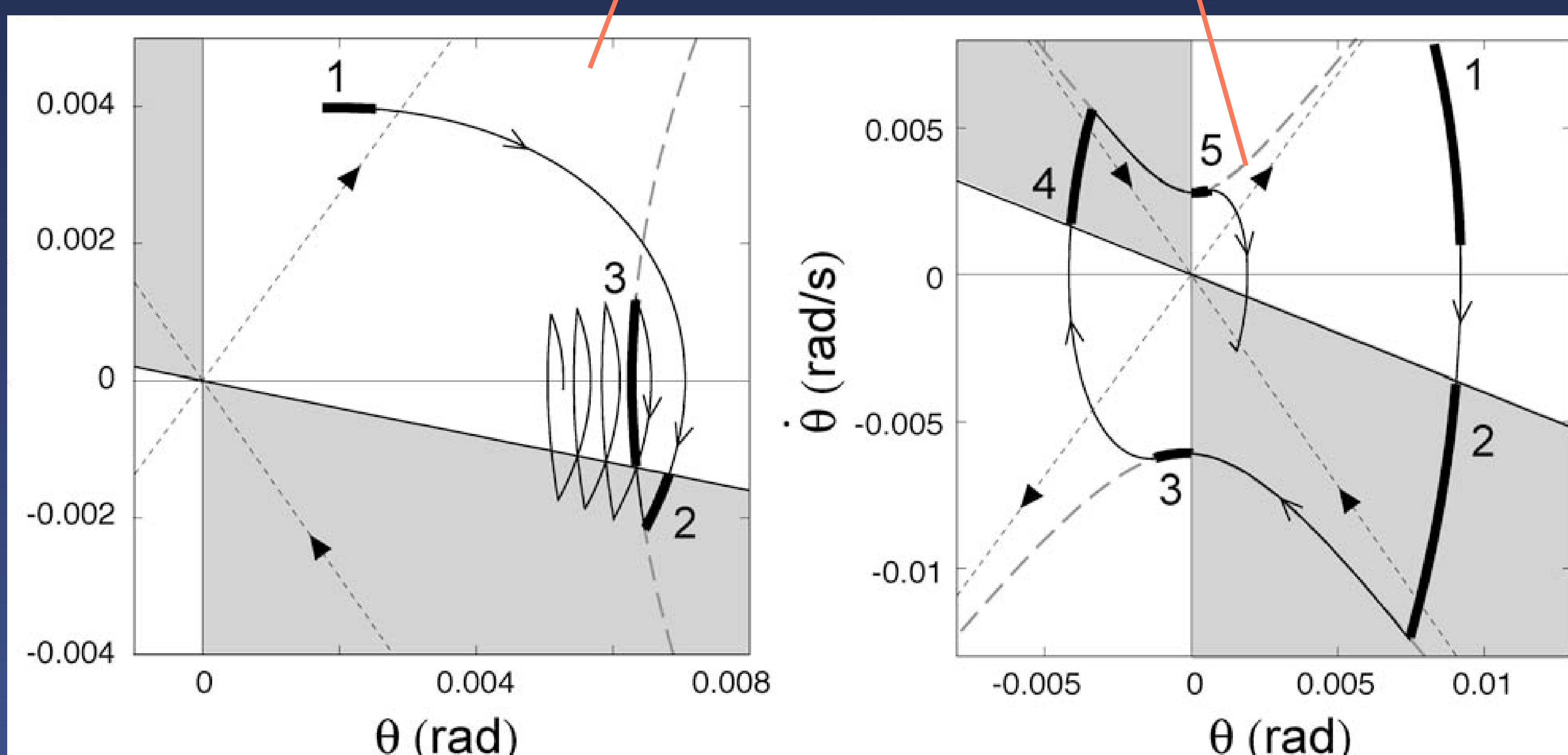
Also prominent in cart model with significant mass

Transitions from zig-zag to spirals?



D-shaped region: stable vertical position with cts control

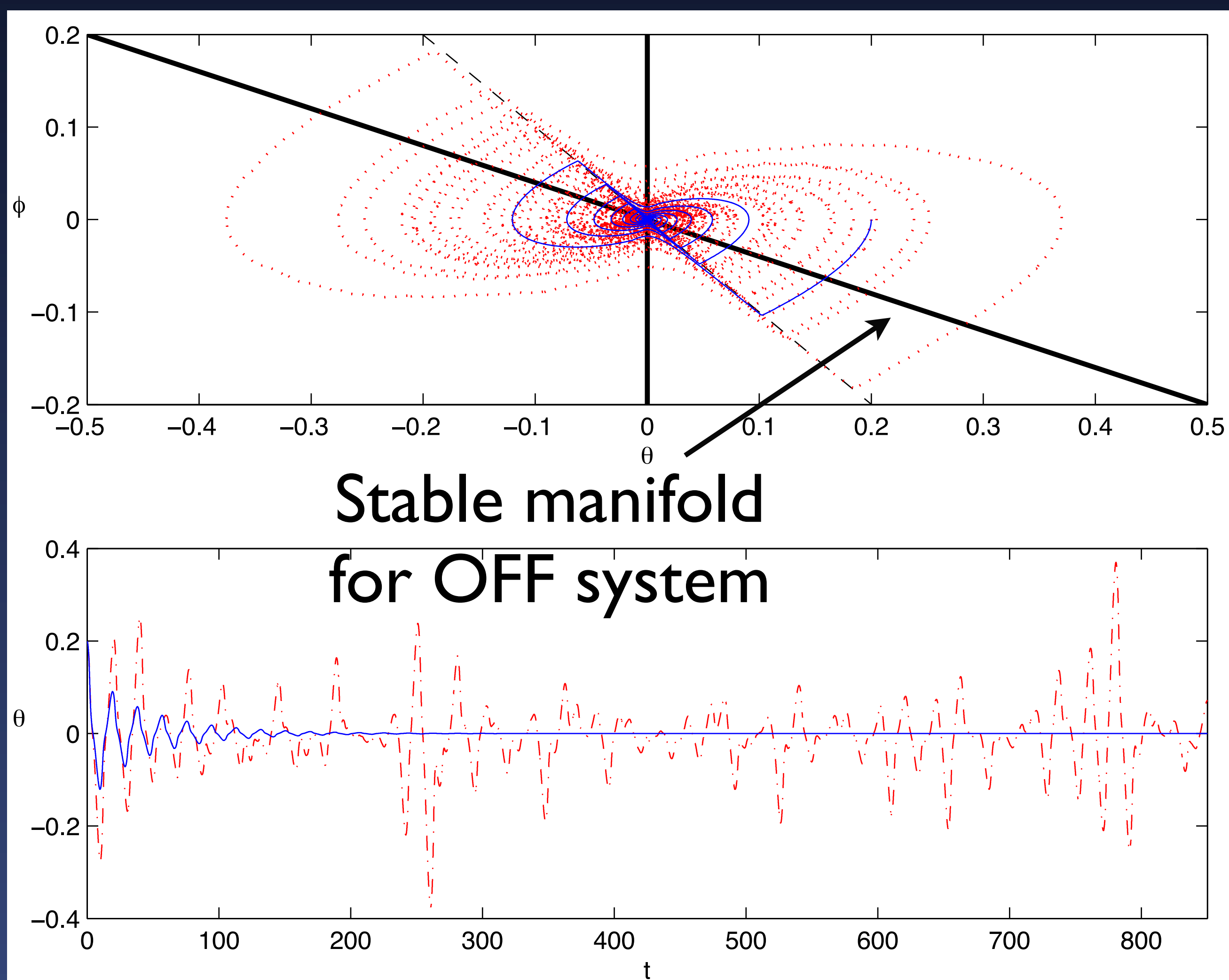
Regions: analysis of linearized model

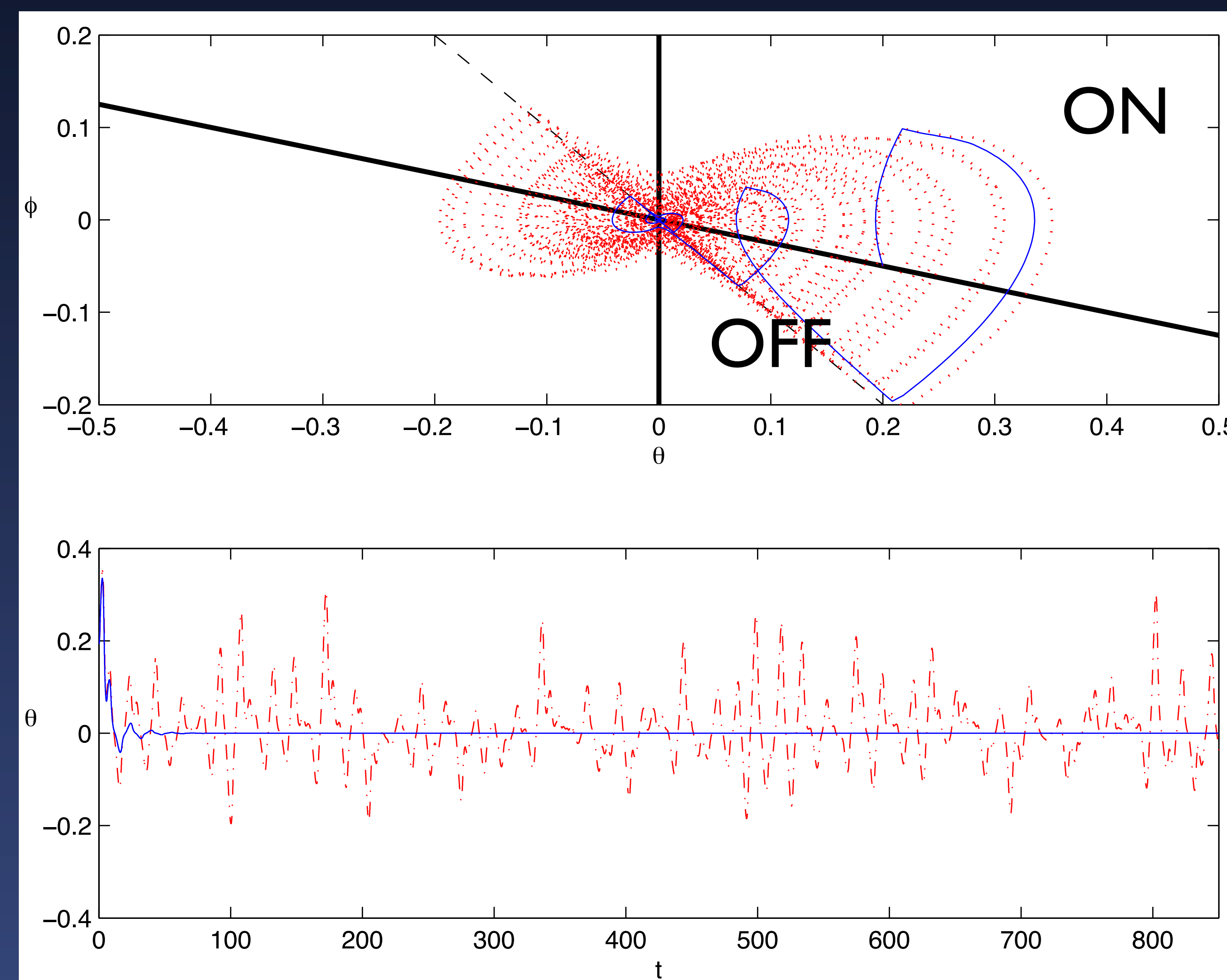


(unstable) zigzag and spirals w/ discts control

Transitions: sensitivity to noise

Noise driven/amplified spirals, stabilized transients: complex dynamics expected for larger systems





unstable spiral, sustained by noise
- similar to perturbed spiral

unstable ZZ sustained as
spiral via noise

Results for stochastic + discontinuous dynamics

Discontinous noise sources : Jump-diffusion: R-K methods: Mean square (strong) convergence and Lipschitz-type conditions for increment functions
Buckwar, et al 2011

Stochastic Flows for SDE's with singular coefficients: Many results carry-over for non-smooth, measurable drifts, as long as noise is "nice" (e.g. Brownian) Mohammed et al 2013

This result is counter-intuitive since the dominant `culture' in stochastic (and deterministic) dynamical systems is that the flow `inherits' its spatial regularity from the driving vector fields

Results for stochastic + discontinuous dynamics

Smoothed systems (approximating discts system):
Elemgard et al 2013, Simonsen et al 2013

Deterministic:

Acary, Brogliato, Numerical Methods for Nonsmooth, DS, 2008

Dieci, Lopez, Survey for IVPs, 2012

Weak existence and uniqueness vs. strong uniqueness, discontinuous drift bounded away from zero Pascu, 2013

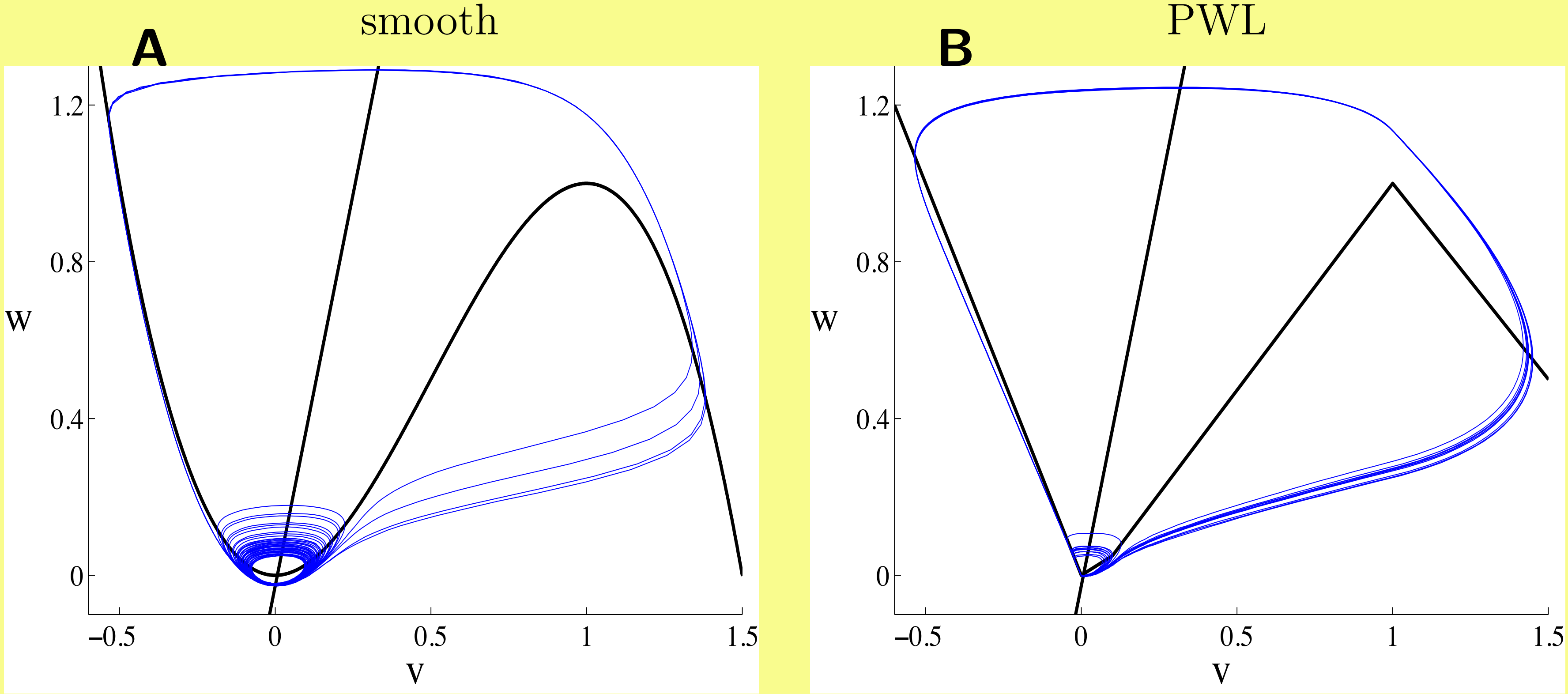
Smooth vs. Piecewise Linear (PWL) models: Complex dynamics in higher dimensions or noise driven?

Neuro-dynamics: Typical structure

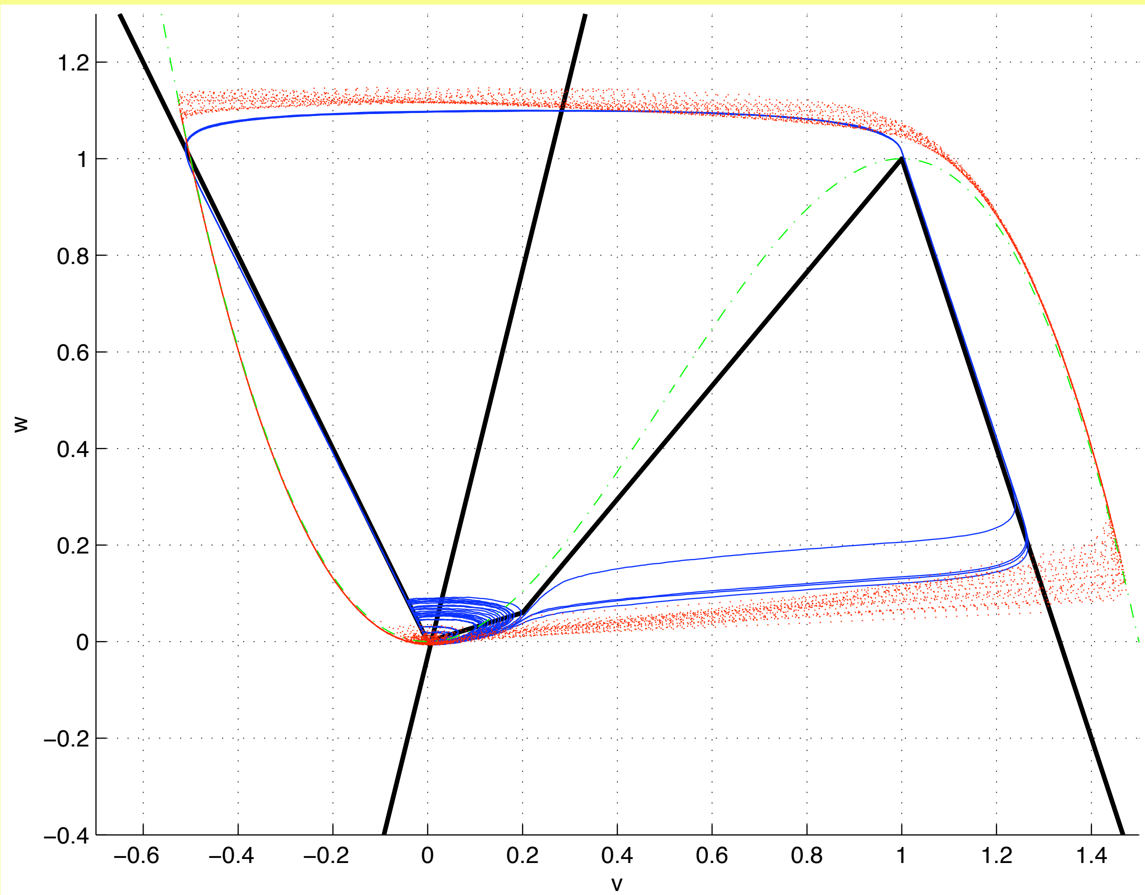
$$dv = (f(v) - w)\ dt\ ,$$
$$dw = \varepsilon(\alpha v - \sigma w - \lambda)\ dt + D\ dW$$

$$f(v) = \begin{cases} \eta_L v\ , & v \le 0 \\ \eta_1 v\ , & 0 < v \le v_1 \\ \eta_2(v - v_1) + w_1\ , & v_1 < v \le 1 \\ \eta_R(v-1) + 1\ , & v > 1 \end{cases}$$

$$f(v) = 3v^2 - 2v^3$$

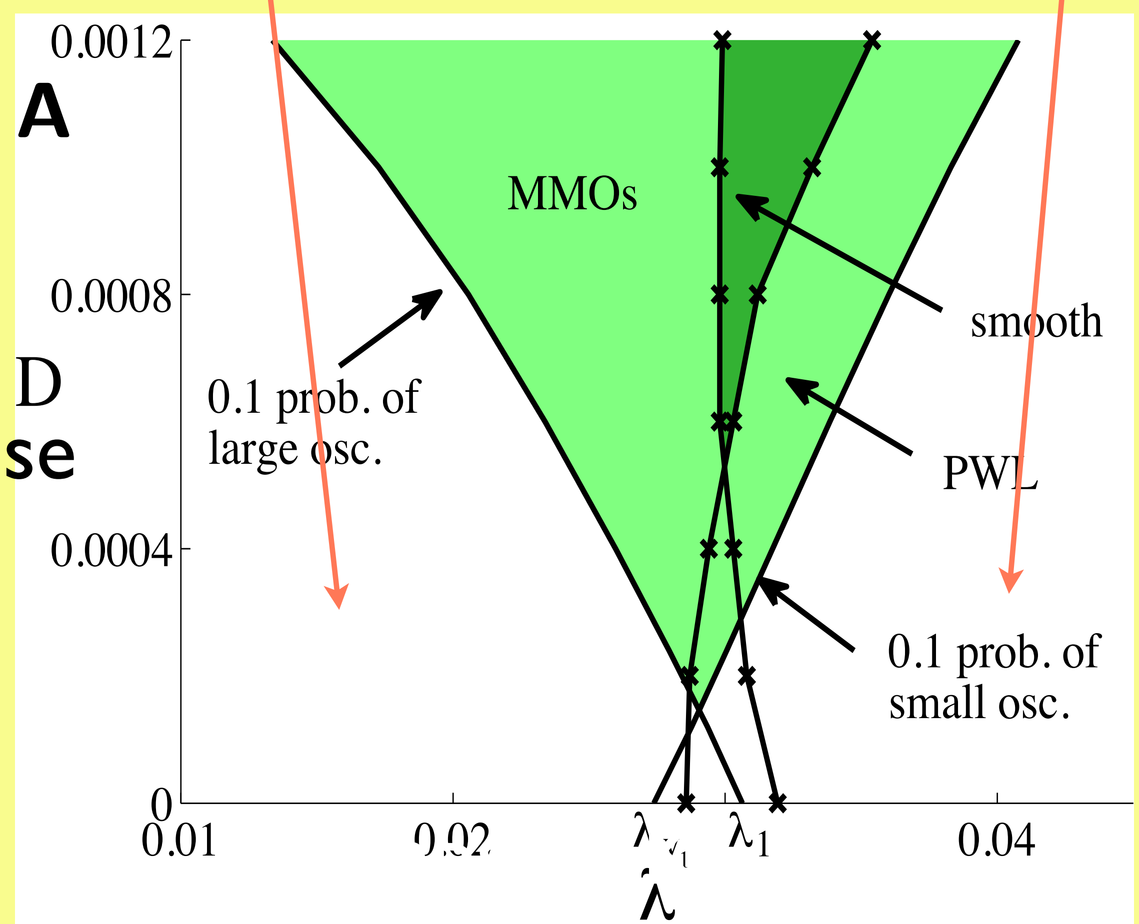


MMO's in PWL models w/ canard structure



primarily small oscillations

primarily large limit cycles



noise

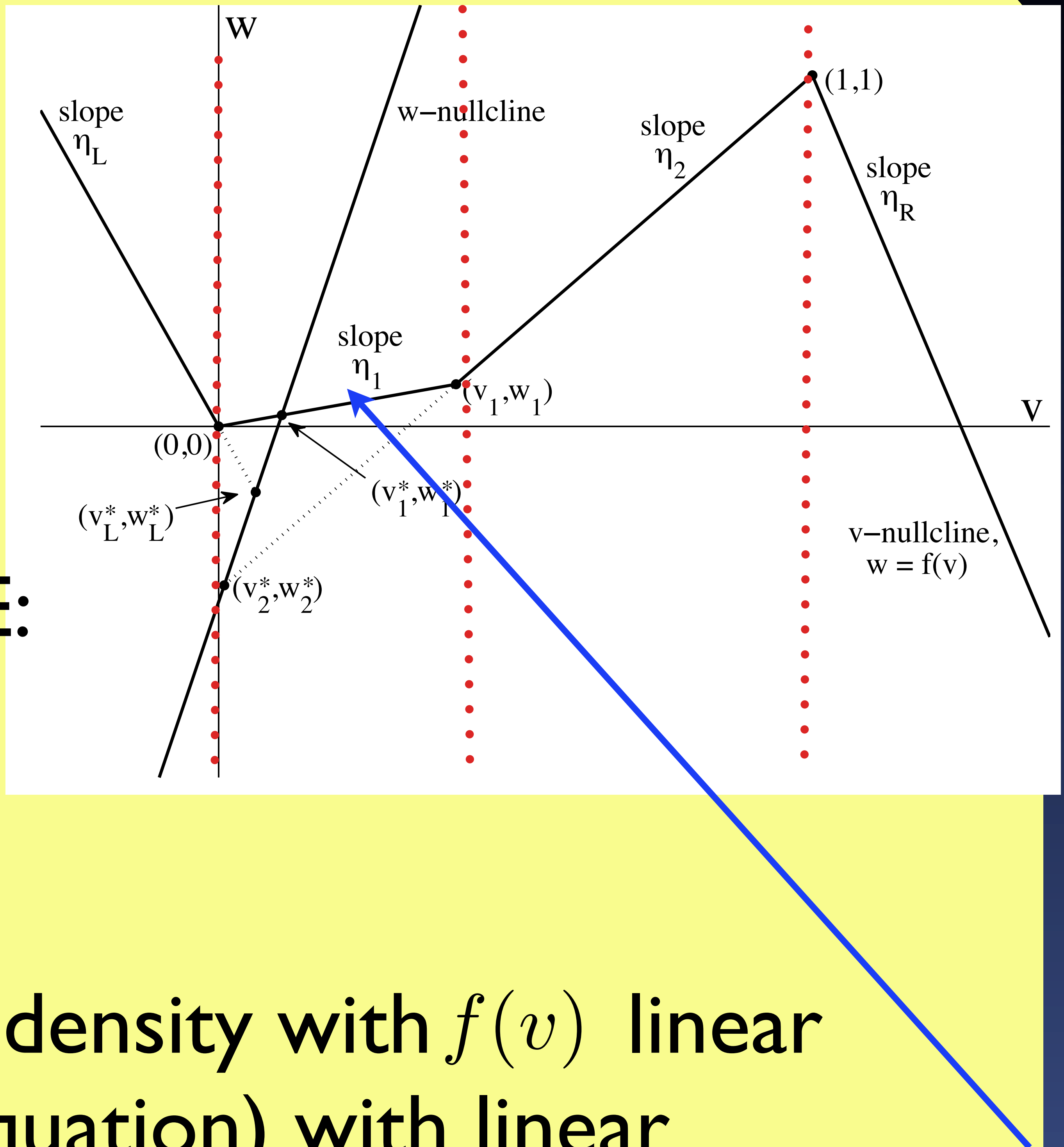
control parameter

MMO's robust over a larger range of control parameter for PWL

Simpson, K. Physica D, 2011

Analysis of PWL FHN type models, w/ noise

- Linear analysis on subregions
- Time dependent probabilities



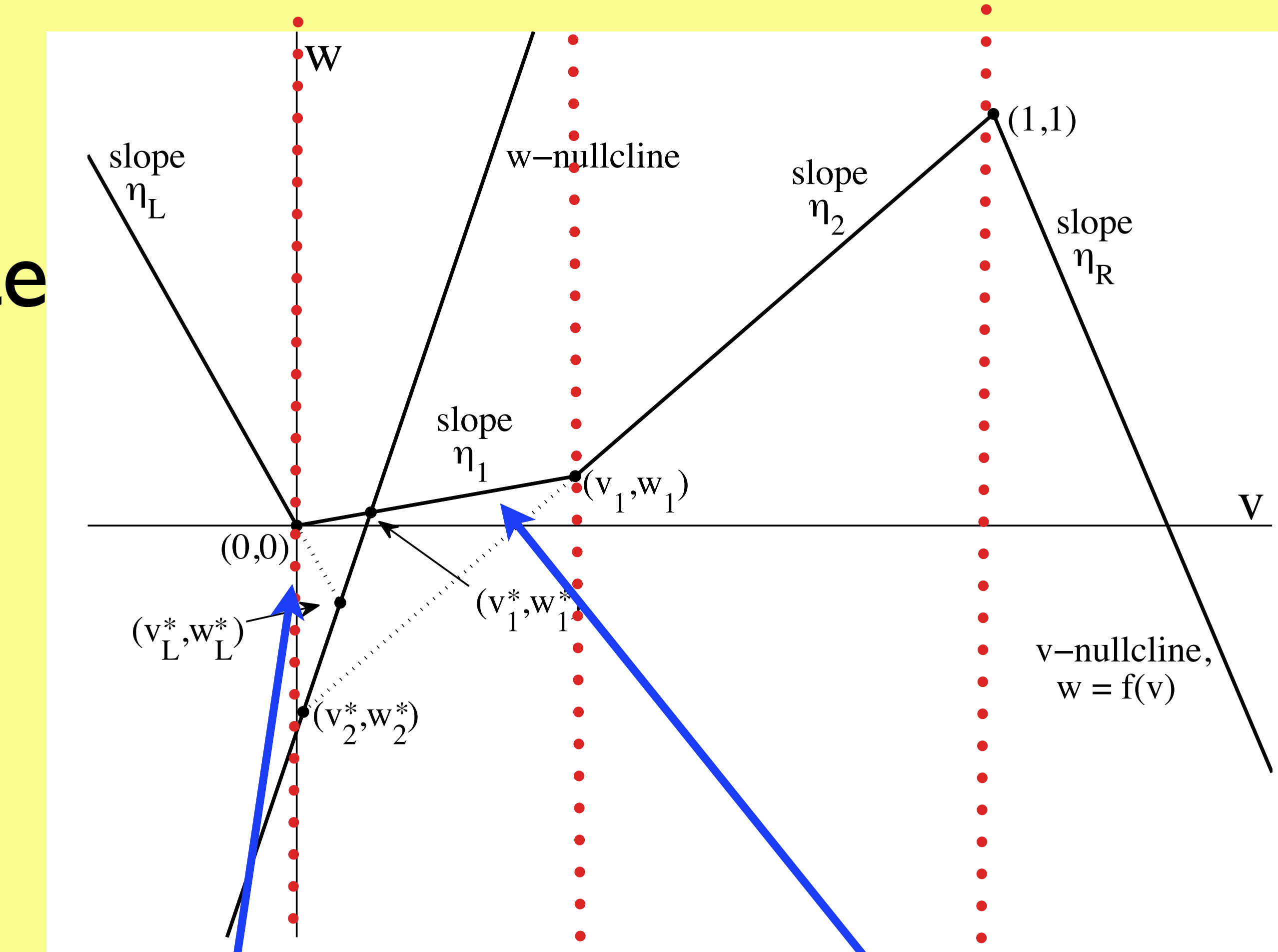
In each for four regions, SDE:

$$dv = (f(v) - w)\ dt\ ,$$
$$dw = \varepsilon(\alpha v - \sigma w - \lambda)\ dt + D\ dW$$

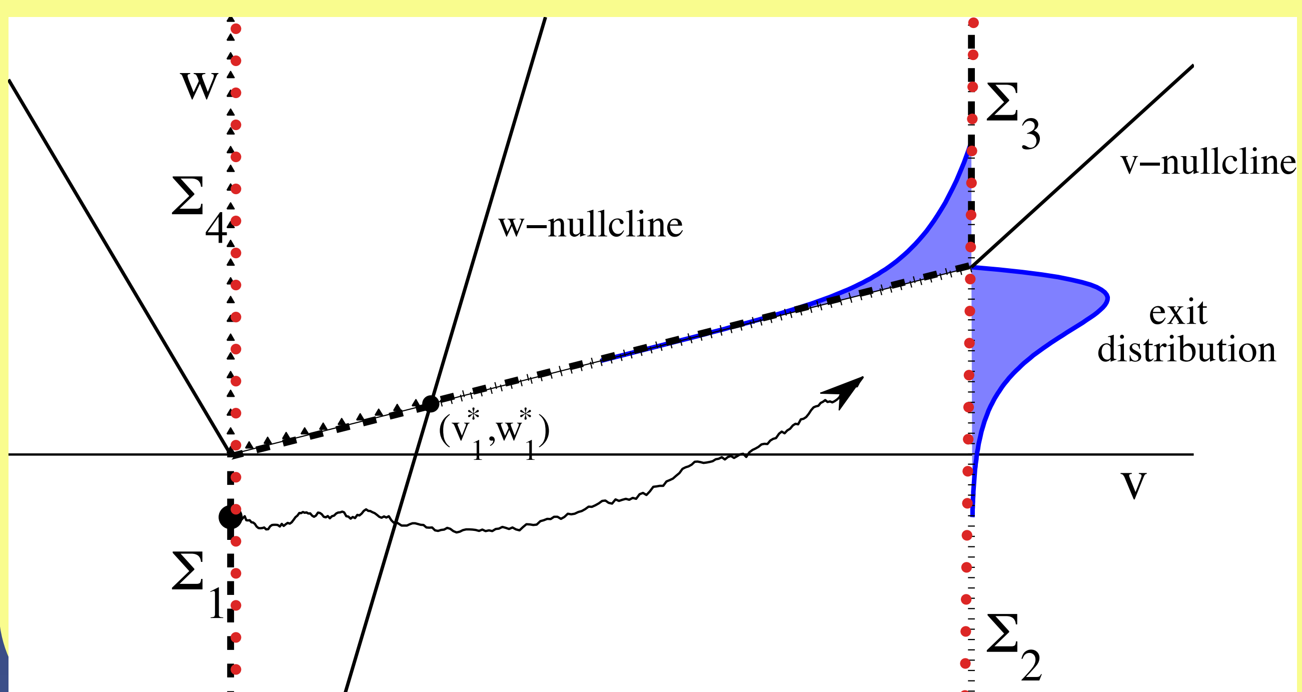
Time dependent probability density with $f(v)$ linear
Solve PDE (Fokker Planck equation) with linear coefficients: Gaussian probabilities
Time dependent Ornstein-Uhlenbeck processes

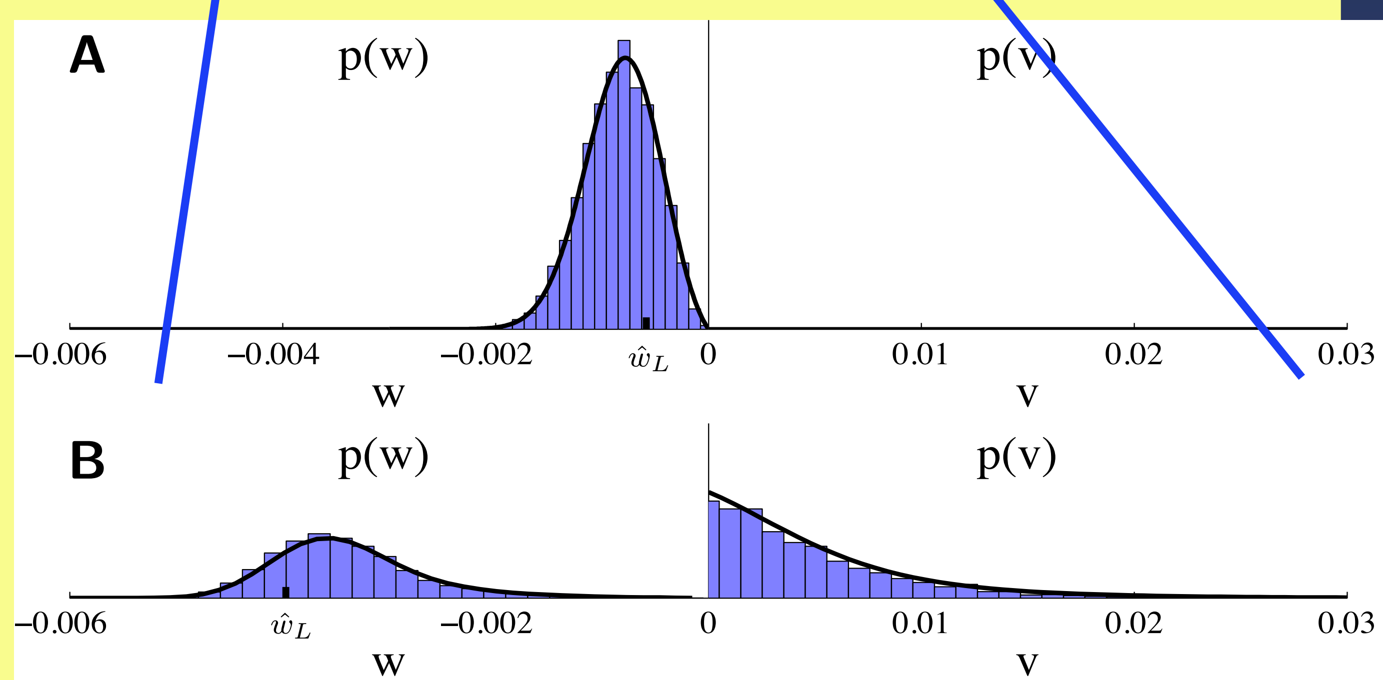
Time dependent densities: transition distributions

- Time dependent probabilities
- Semi-analytical approach: iterate on transitions between regions
- Additional local analysis at crossings: first vs. last crossing time
- Parametric dependence



Variability with λ and critical values v_1, w_1





Sliding dynamics: Relay control

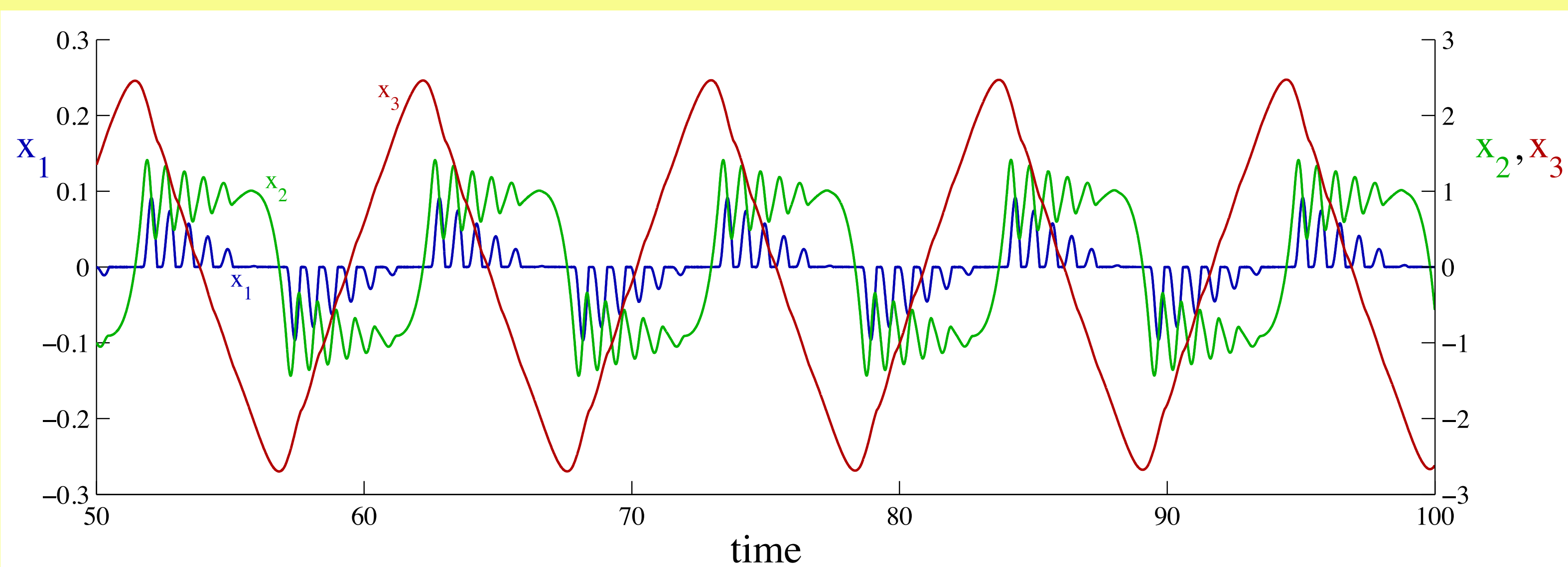
$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

$$\varphi = C^{\mathsf{T}}\mathbf{x}\ ,$$

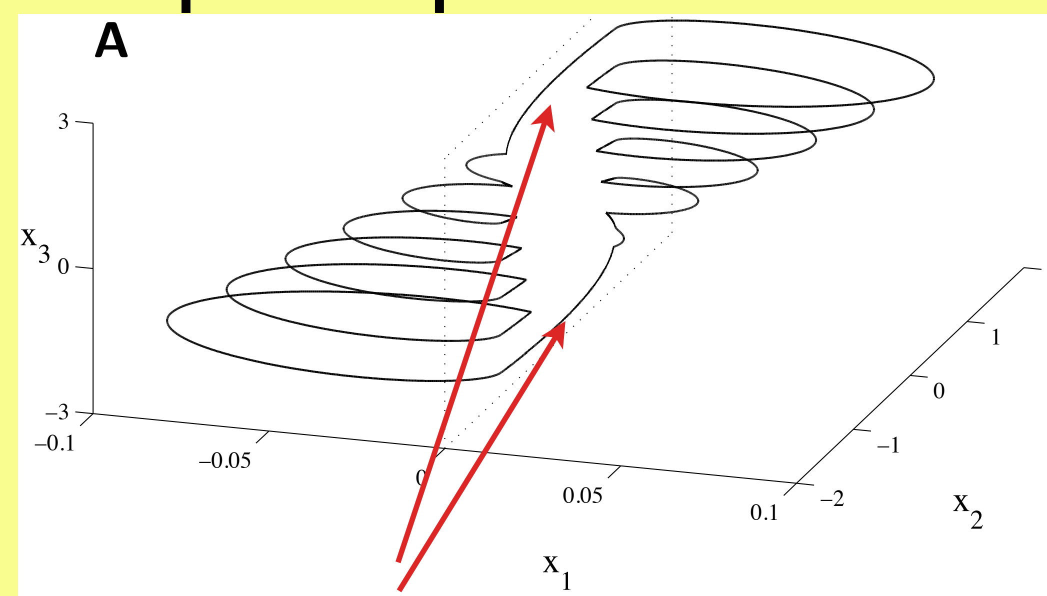
$$u = -\mathrm{sgn}(\varphi)$$

Control (u) depends on state x

no noise



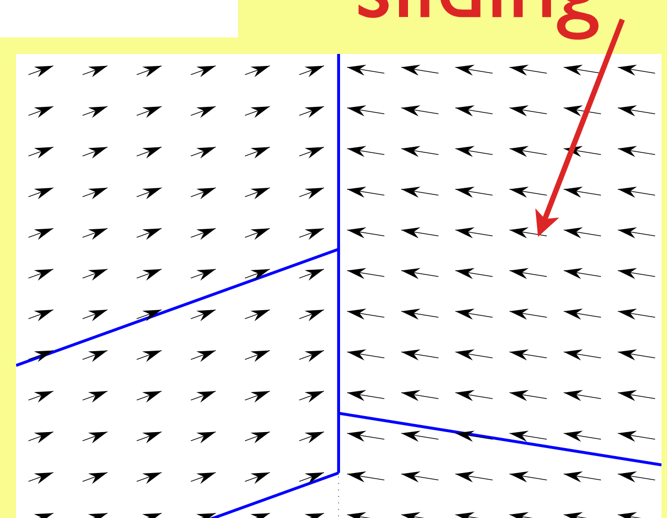
phase plane

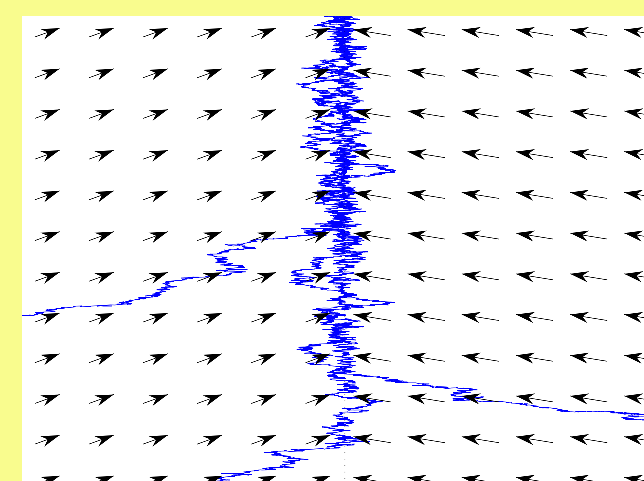


sliding

w/noise

vector field is discontinuous along the switching (sliding) manifold





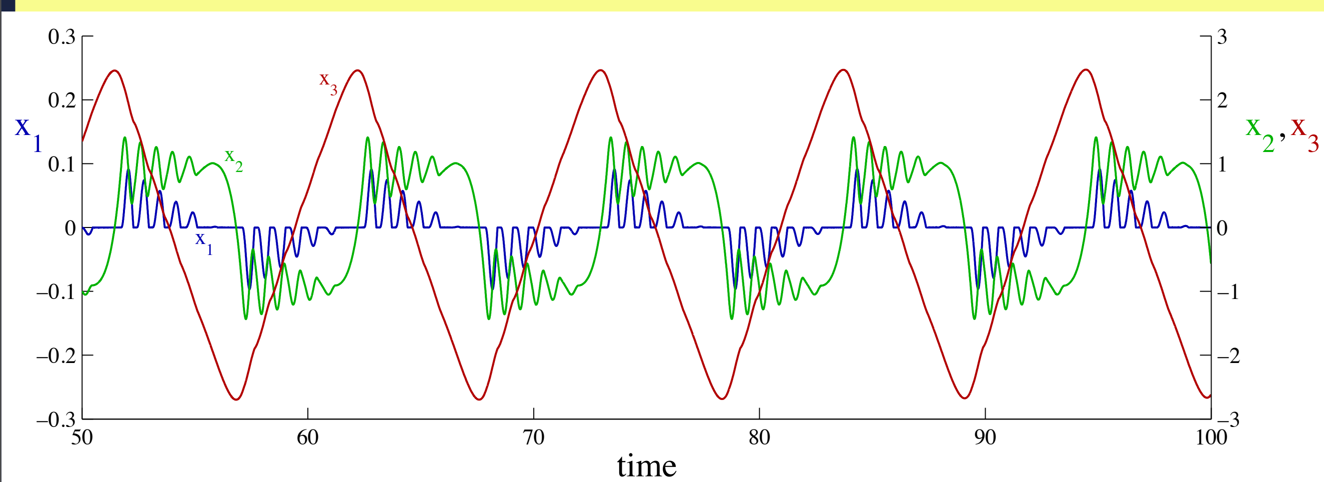
Potential contributions to O(1) change to average period

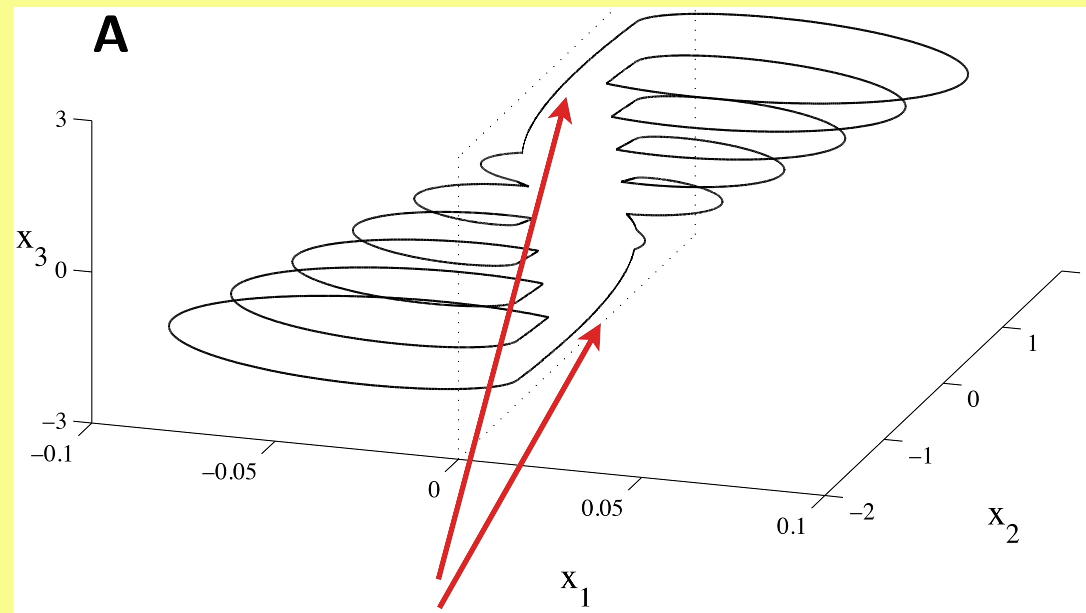
$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$
$$\varphi = C^{\mathsf{T}}\mathbf{x}\ ,$$
$$u = -\mathrm{sgn}(\varphi)$$

Control (u) depends on state x

phase plane

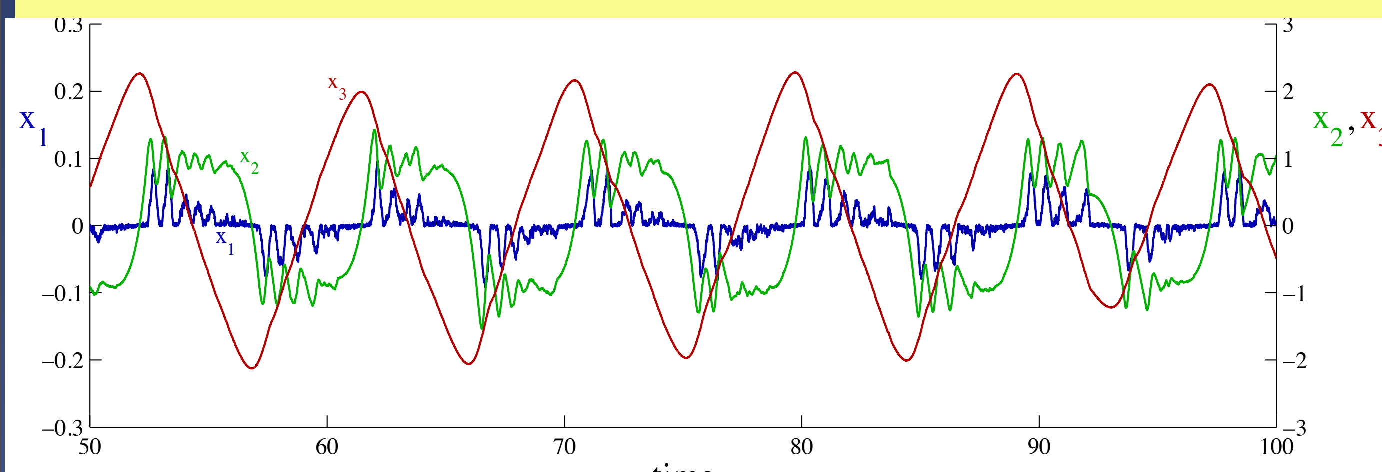
no noise

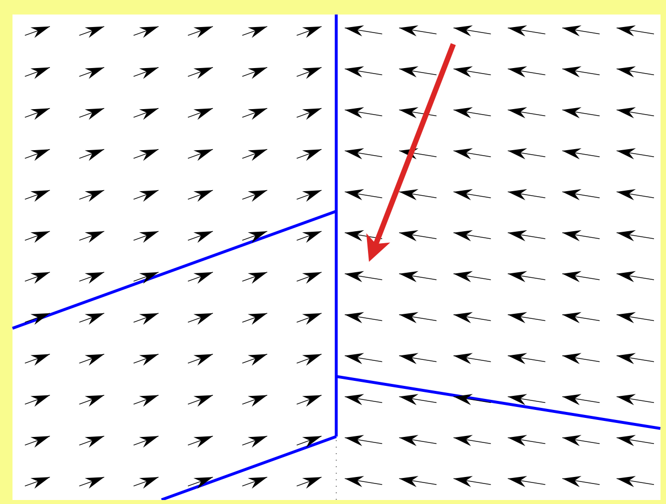




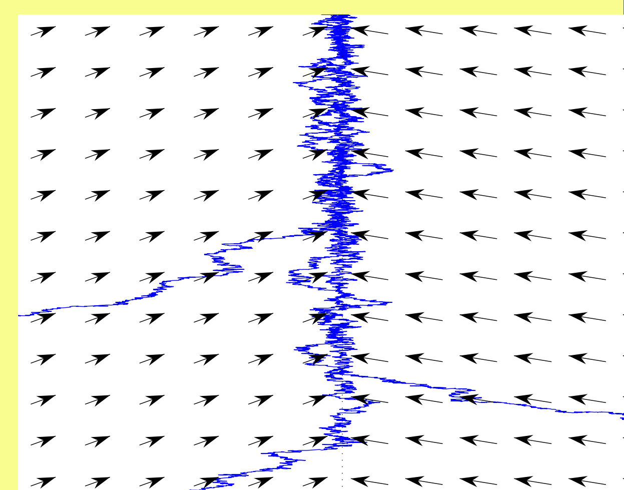
sliding

w/ noise



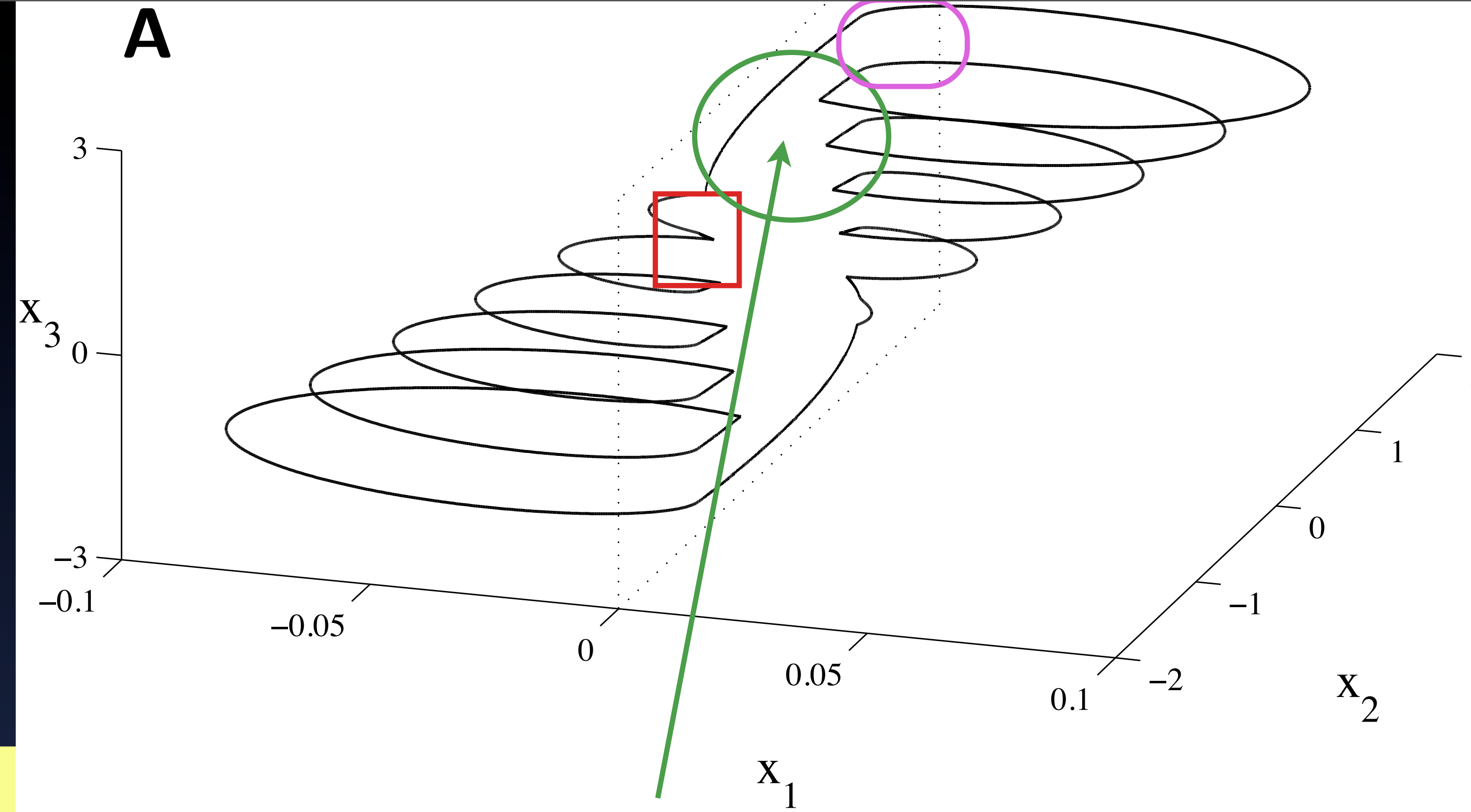


w/noise



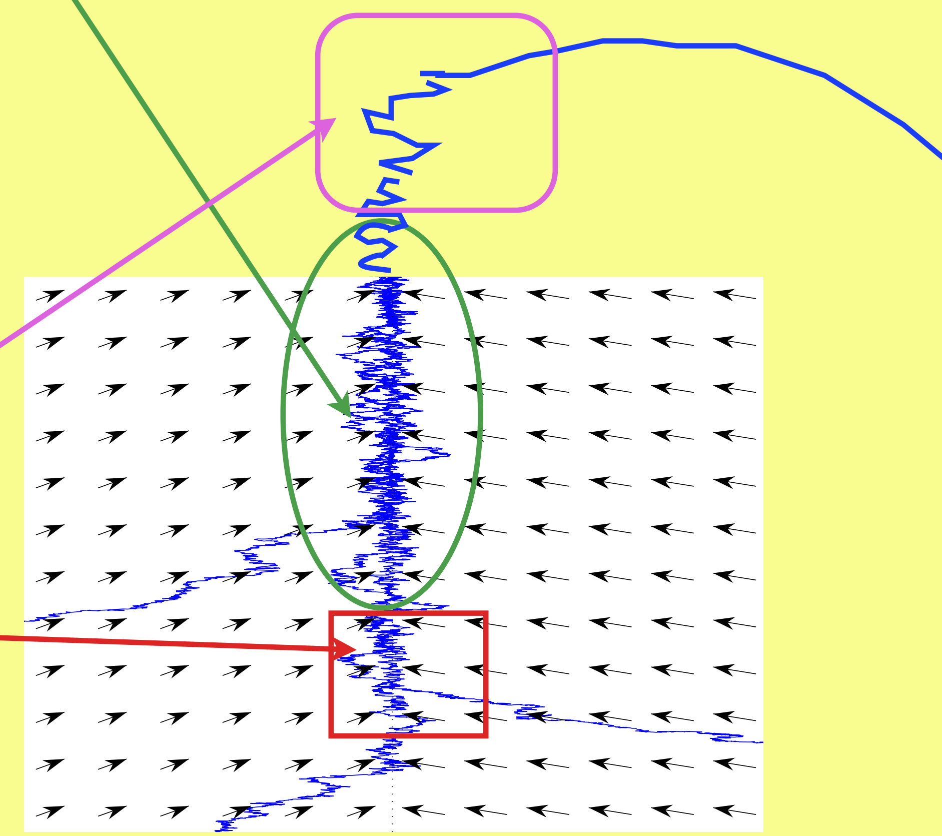
3D model

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$
$$\varphi = C^{\mathsf{T}}\mathbf{x}\ ,$$
$$u = -\mathrm{sgn}(\varphi)$$



sliding

Results: influence of noise in deviation from sliding

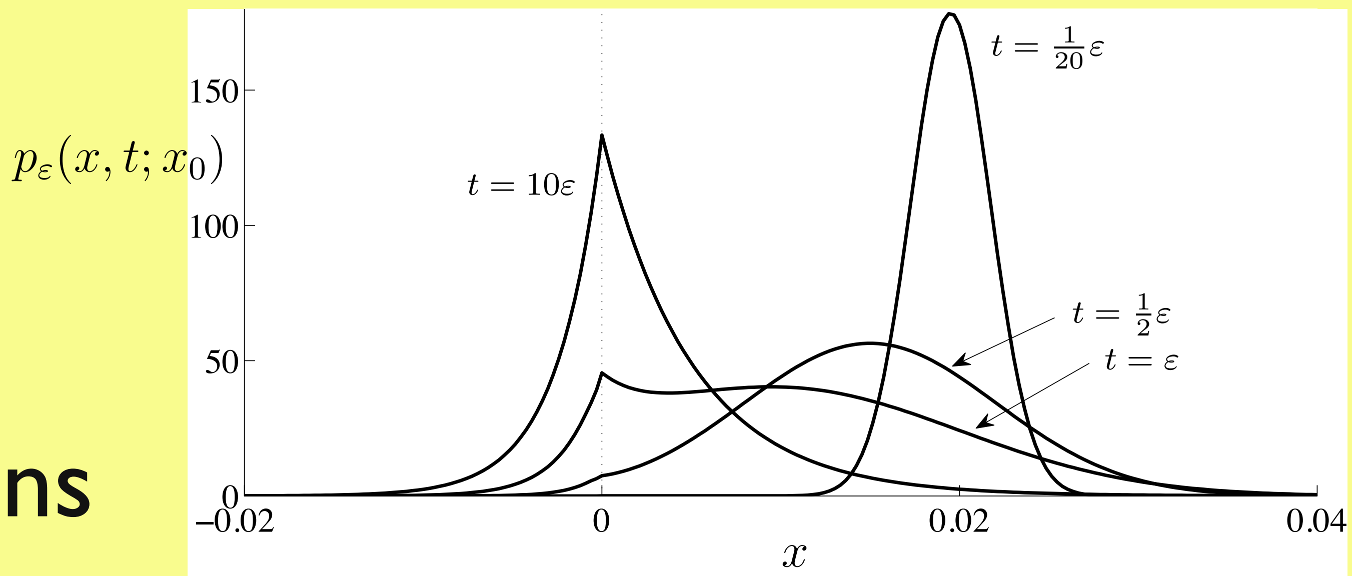


Other regions: connection of sliding with other dynamics (exit/ entrance), use of stochastic averaging, asymptotic analysis of FPE

Distributions, moments

Constant drift case:

$$Var(y(t)) = \varepsilon\kappa t + \frac{(b_L - b_R)^2}{(a_L + a_R)^2}\varepsilon t + O(\varepsilon^2)$$



Time dependent density:

Needed to compute correlations

(Karatzas, Shreve)

p(x,t|x_0) =

$$\begin{cases} \frac{2}{\varepsilon}e^{\frac{2a_L x}{\varepsilon}} \int_0^\infty h_\varepsilon(t, b, a_R) * h_\varepsilon(t, b - x - x_0, a_L)\, db + G_{\mathrm{absorb},\varepsilon}(x, t, a_L | x_0)\ , & x_0 \le 0,\ x \le 0 \\ \frac{2}{\varepsilon}e^{\frac{-2a_R x}{\varepsilon}} \int_0^\infty h_\varepsilon(t, b + x, a_R) * h_\varepsilon(t, b - x_0, a_L)\, db\ , & x_0 \le 0,\ x \ge 0 \\ \frac{2}{\varepsilon}e^{\frac{2a_L x}{\varepsilon}} \int_0^\infty h_\varepsilon(t, b + x_0, a_R) * h_\varepsilon(t, b - x, a_L)\, db\ , & x_0 \ge 0,\ x \le 0 \\ \frac{2}{\varepsilon}e^{\frac{-2a_R x}{\varepsilon}} \int_0^\infty h_\varepsilon(t, b + x + x_0, a_R) * h_\varepsilon(t, b, a_L)\, db + G_{\mathrm{absorb},\varepsilon}(x, t, -a_R | x_0)\ , & x_0 \ge 0,\ x \ge 0 \end{cases}$$

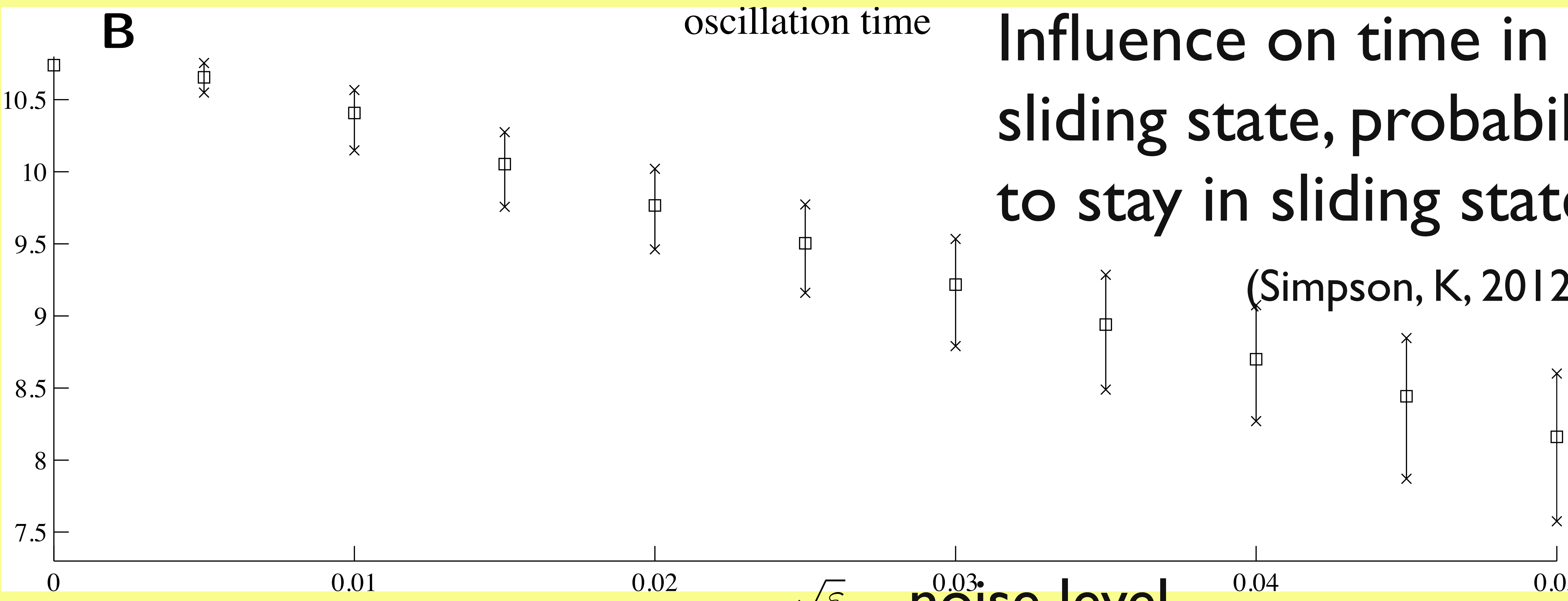
Implications for (near) sliding dynamics

$$\langle y(t)\rangle = y_{\text{slide}}(t) + \frac{(a_L^2 d_R - a_R^2 d_L)(a_L + a_R) - (a_L^2 c_R - a_R^2 c_L)(b_L - b_R)}{2a_L a_R (a_L + a_R)^2}\varepsilon t + o(\varepsilon)$$

$$Var(y(t)) = \varepsilon\kappa t + \frac{(b_L - b_R)^2}{(a_L + a_R)^2}\varepsilon t + O(\varepsilon^2)$$

Standard deviation larger than mean shift from sliding

Large coefficients can shift average oscillation time



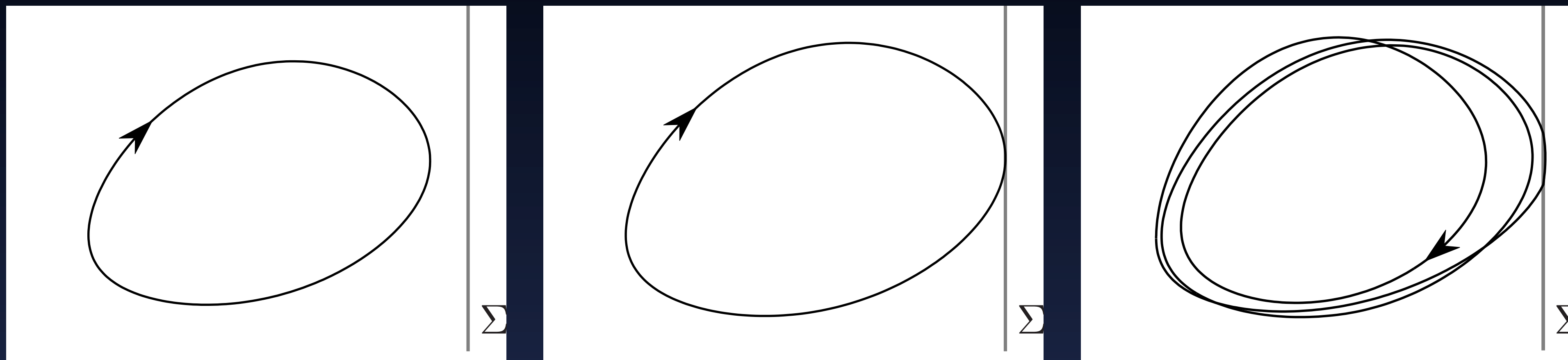
Influence on time in sliding state, probability to stay in sliding state

(Simpson, K, 2012)

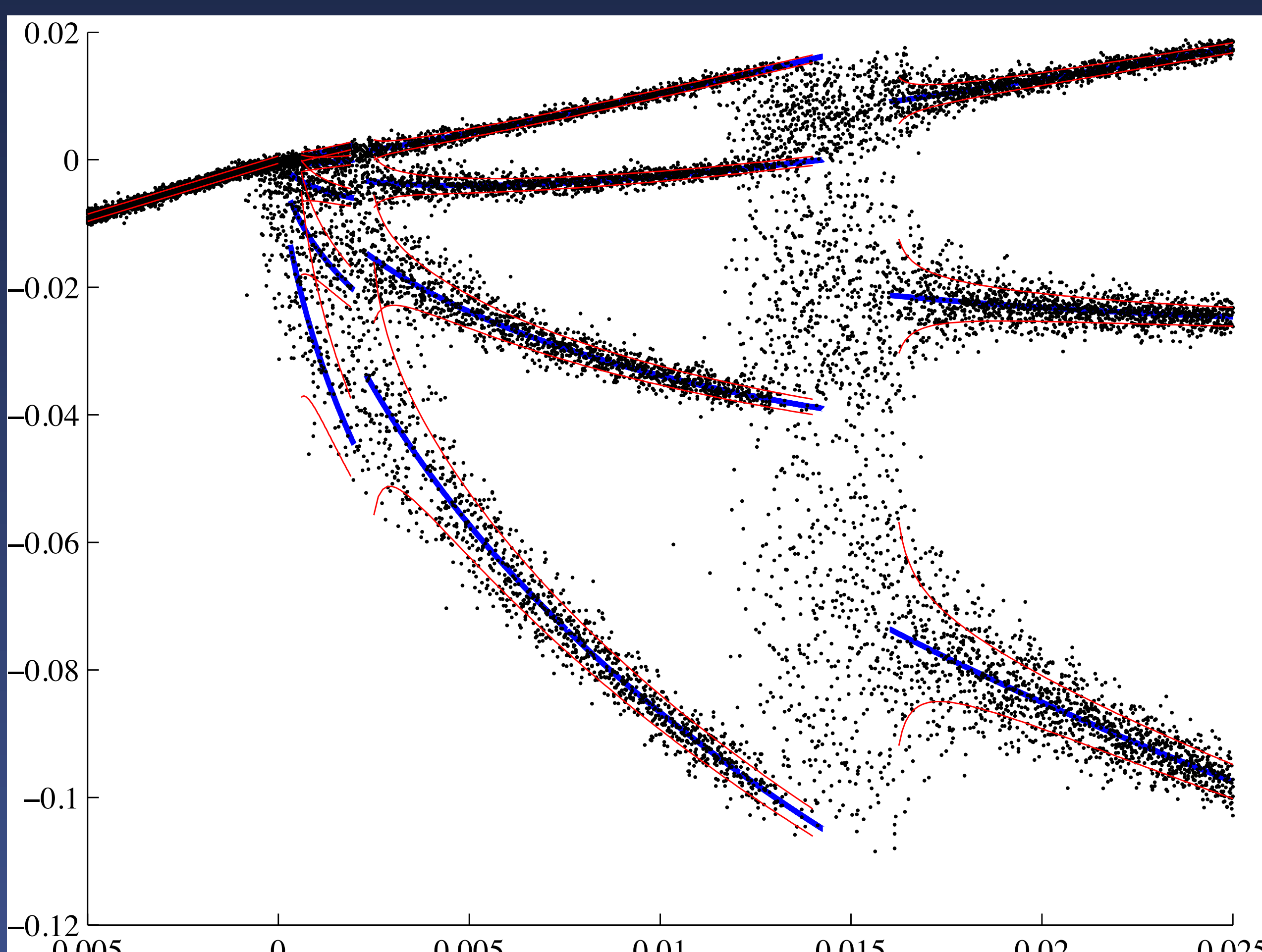
noise level

Noise sensitivity: grazing

Grazing: vibro-impacts, friction, AFM, stick-slip



Poincare map: a discontinuity map captures impact



Grazing normal form:

Nordmark map

Square root behavior

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{cases} A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mu \,, & x \le 0 \\ A \begin{bmatrix} x \\ y - \chi\sqrt{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mu \,, & x \ge 0 \end{cases}$$

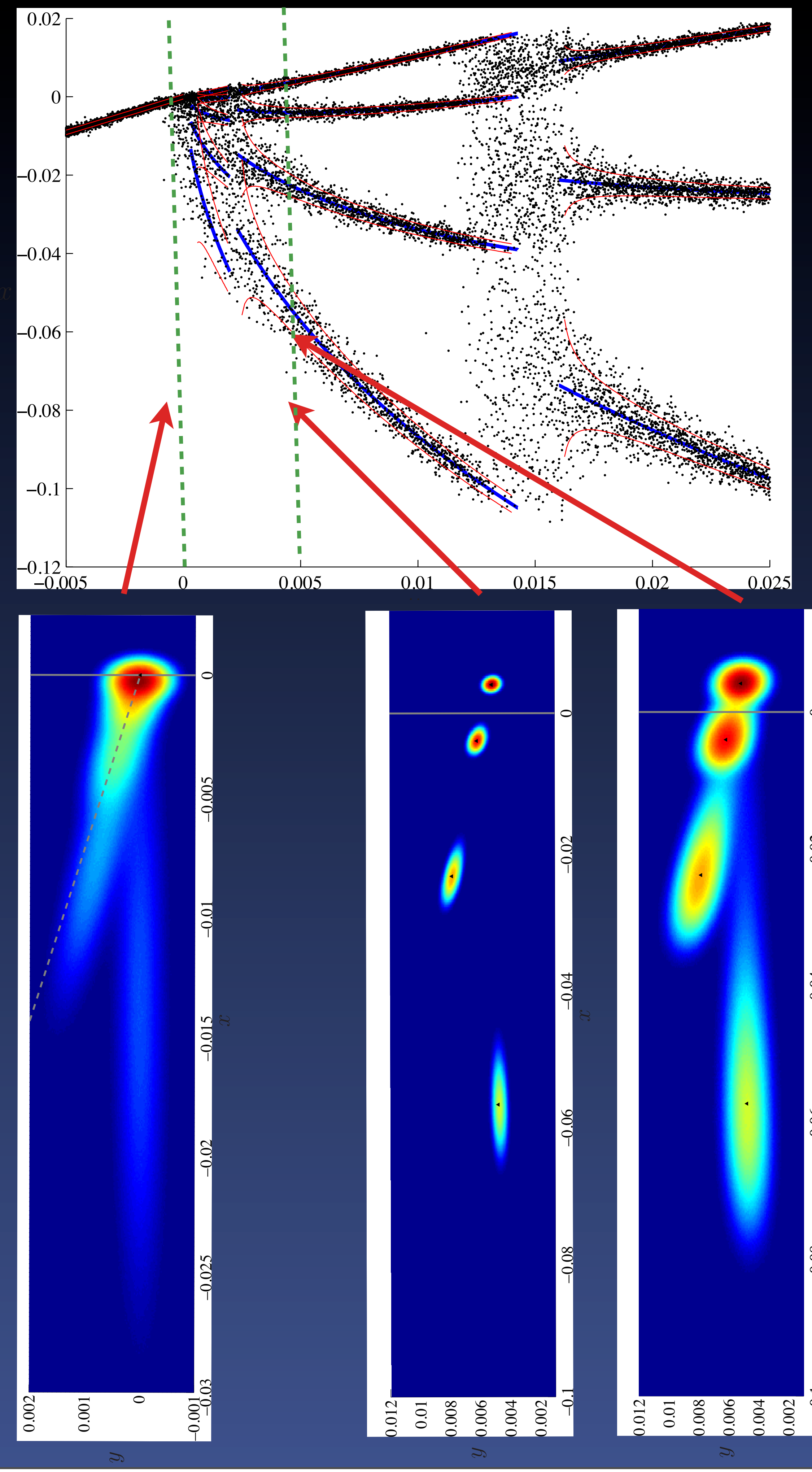
Noise sensitivity: grazing

Stochastic Poincare map derived from cts model (vs. Poincare map + noise)

Gaussian densities: well separated branches

Non- Gaussian: branches overlap, square root "stretching" follows iterates near switching

Simpson, Hogan, K. SIADS, to appear



Different types of stochastic discontinuous dynamics: need a variety of ideas

- Mixed mode oscillations: Semi-analytical iterations of time dependent probability density functions
- Discontinuity induced bifurcations - underlying sources of noise-sensitivity
- Positive occupation times: sliding
- Boundary layers and non-standard scaling limits: sliding transitions
- Grazing: Stochastic Poincare maps

Lots of mathematical and modeling challenges:

- Nonlinear models with delay: complex behaviors
- Piecewise continuous nonlinear systems: recently receiving more attention
- Stochastic modeling for systems with delay and discontinuities: open problems analytically and computationally, new approaches needed
- Robustness of different on/off control strategies
- Recent work: extension of results for continuous cases to discontinuous drift cases with "nice" noise (Mohammed, et al, 2013 in progress)