Chapter 9 Exercises

From: *Finite Difference Methods for Ordinary and Partial Differential Equations*
by R. J. LeVeque, SIAM, 2007. `http://www.amath.washington.edu/~rjl/fdmbook`

Exercise 9.1 *(leapfrog for heat equation)*

Consider the following method for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}).$$

(a) Determine the order of accuracy of this method (in both space and time).

(b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of α (if any) will this method be Lax-Richtmyer stable and hence convergent?

Hint: Consider the MOL interpretation and the stability region of the time-discretization being used.

(c) Is this a useful method?

Exercise 9.2 *(codes for heat equation)*

(a) The m-file `heat_CN.m` solves the heat equation $u_t = \kappa u_{xx}$ using the Crank-Nicolson method. Run this code, and by changing the number of grid points, confirm that it is second-order accurate. (Observe how the error at some fixed time such as $T = 1$ behaves as k and h go to zero with a fixed relation between k and h, such as $k = 4h$.)

You might want to use the function `error_table.m` to print out this table and estimate the order of accuracy, and `error_loglog.m` to produce a log-log plot of the error vs. h. See `bvp_2.m` for an example of how these are used.

(b) Modify `heat_CN.m` to produce a new m-file `heat_trbdf2.m` that implements the TR-BDF2 method on the same problem. Test it to confirm that it is also second order accurate. Explain how you determined the proper boundary conditions in each stage of this Runge-Kutta method.

(c) Modify `heat_CN.m` to produce a new m-file `heat_FE.m` that implements the forward Euler explicit method on the same problem. Test it to confirm that it is $\mathcal{O}(h^2)$ accurate as $h \to 0$ provided when $k = 24h^2$ is used, which is within the stability limit for $\kappa = 0.02$. Note how many more time steps are required than with Crank-Nicolson or TR-BDF2, especially on finer grids.

(d) Test `heat_FE.m` with $k = 26h^2$, for which it should be unstable. Note that the instability does not become apparent until about time 1.6 for the parameter values $\kappa = 0.02$, $m = 39$, $\beta = 150$. Explain why the instability takes several hundred time steps to appear, and why it appears as a sawtooth oscillation.

Hint: What wave numbers ξ are growing exponentially for these parameter values? What is the initial magnitude of the most unstable eigenmode in the given initial data? The expression (16.52) for the Fourier transform of a Gaussian may be useful.

Exercise 9.3 *(heat equation with discontinuous data)*

(a) Modify `heat_CN.m` to solve the heat equation for $-1 \le x \le 1$ with step function initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0. \end{cases} \tag{Ex9.3a}$$

With appropriate Dirichlet boundary conditions, the exact solution is

$$u(x,t) = \frac{1}{2}\,\text{erfc}\left(x/\sqrt{4\kappa t}\right), \tag{Ex9.3b}$$

where erfc is the complementary error function

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\, dz.$$

(i) Test this routine $m = 39$ and $k = 4h$. Note that there is an initial rapid transient decay of the high wave numbers that is not captured well with this size time step.

(ii) How small do you need to take the time step to get reasonable results? For a suitably small time step, explain why you get much better results by using $m = 38$ than $m = 39$. What is the observed order of accuracy as $k \to 0$ when $k = \alpha h$ with α suitably small and m even?

(b) Modify `heat_trbdf2.m` (see Exercise 9.2) to solve the heat equation for $-1 \le x \le 1$ with step function initial data as above. Test this routine using $k = 4h$ and estimate the order of accuracy as $k \to 0$ with m even. Why does the TR-BDF2 method work better than Crank-Nicolson?

Exercise 9.4 *(Jacobi iteration as time stepping)*

Consider the Jacobi iteration (4.4) for the linear system $Au = f$ arising from a centered difference approximation of the boundary value problem $u_{xx}(x) = f(x)$. Show that this iteration can be interpreted as forward Euler time stepping applied to the MOL equations arising from a centered difference discretization of the heat equation $u_t(x,t) = u_{xx}(x,t) - f(x)$ with time step $k = \frac{1}{2}h^2$.

Note that if the boundary conditions are held constant then the solution to this heat equation decays to the steady state solution that solves the boundary value problem. Marching to steady state with an explicit method is one way to solve the boundary value problem, though as we saw in Chapter 4 this is a very inefficient way to compute the steady state.

Exercise 9.5 *(Diffusion and decay)*

Consider the PDE

$$u_t = \kappa u_{xx} - \gamma u, \tag{Ex9.5a}$$

which models a diffusion with decay provided $\kappa > 0$ and $\gamma > 0$. Consider methods of the form

$$U_j^{n+1} = U_j^n + \frac{k}{2h^2}[U_{j-1}^n - 2U_j^n + U_{j+1}^n + U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}] - k\gamma[(1-\theta)U_j^n + \theta U_j^{n+1}] \tag{Ex9.5b}$$

where θ is a parameter. In particular, if $\theta = 1/2$ then the decay term is modeled with the same centered-in-time approach as the diffusion term and the method can be obtained by applying the Trapezoidal method to the MOL formulation of the PDE. If $\theta = 0$ then the decay term is handled explicitly. For more general reaction-diffusion equations it may be advantageous to handle the reaction terms explicitly since these terms are generally nonlinear, so making them implicit would require solving nonlinear systems in each time step (whereas handling the diffusion term implicitly only gives a linear system to solve in each time step).

(a) By computing the local truncation error, show that this method is $\mathcal{O}(k^p + h^2)$ accurate, where $p = 2$ if $\theta = 1/2$ and $p = 1$ otherwise.

(b) Using von Neumann analysis, show that this method is unconditionally stable if $\theta \geq 1/2$.

(c) Show that if $\theta = 0$ then the method is stable provided $k \leq 2/\gamma$, independent of h.