Conservation Laws and Finite Volume Methods

AMath 574

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Outline

Today:

- CFL condition
- Numerical examples using Clawpack
- Numerical dissipation of upwind
- Lax-Wendroff method (second order)
- Numerical dispersion, modified equations

Next:

- High resolution methods

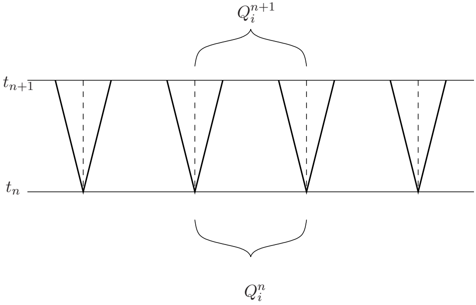
Reading: Chapters 5 and 6

Godunov's method

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.



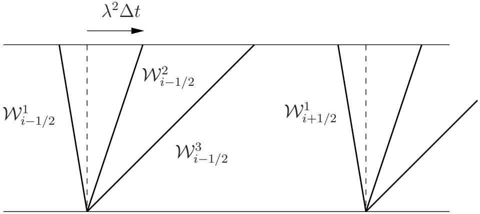
$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\Downarrow}(Q_{i-1}, Q_i) \quad \text{for } t > t_n.$$

$$F_{i-1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n))\,dt = f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n)).$$

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of

waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

Matrix splitting for upwind method

For $q_t + Aq_x = 0$, the upwind method (Godunov) is:

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n + \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m} (\lambda^p)^+ \alpha^p_{i-1/2} r^p + \sum_{p=1}^{m} (\lambda^p)^- \alpha^p_{i+1/2} r^p\right] \\
&= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2}\right] \\
&= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+ (Q_i^n - Q_{i-1}^n) + A^- (Q_{i+1}^n - Q_i^n)\right]
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&= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+ (Q_i^n - Q_{i-1}^n) + A^- (Q_{i+1}^n - Q_i^n)\right]
\end{aligned}
$$

Natural generalization of upwind to a system.

If all eigenvalues are positive, then $A^+ = A$ and $A^- = 0$,

If all eigenvalues are negative, then $A^+ = 0$ and $A^- = A$.

The CFL Condition

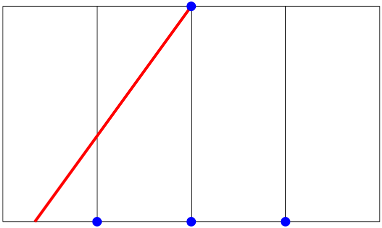
For the method to be stable, the numerical domain of dependence must include the true domain of dependence.

For advection, the solution is constant along characteristics,

$$q(x,t) = q(x - ut, 0)$$

For a 3-point method, CFL condition requires $\left|\frac{u\Delta t}{\Delta x}\right| \leq 1.$

If this is violated:
True solution is determined by data at a point $x - ut$ that is ignored by the numerical method, even as the grid is refined.



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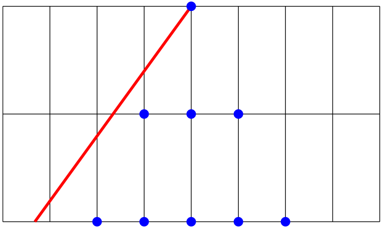

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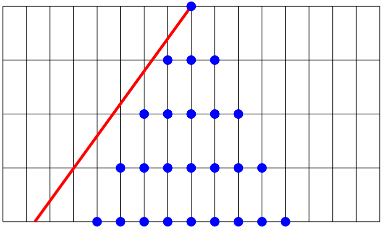

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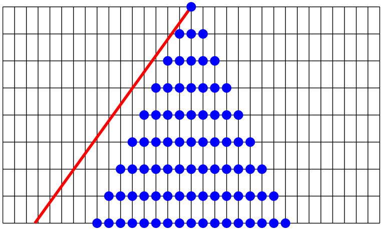
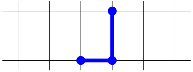
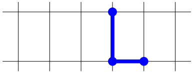


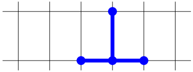
Stencil CFL Condition



$$0 \le \frac{\lambda_p \Delta t}{\Delta x} \le 1, \quad \forall p$$



$$-1 \le \frac{\lambda_p \Delta t}{\Delta x} \le 0, \quad \forall p$$



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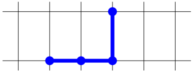
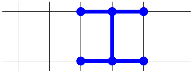


$$0 \le \frac{\lambda_p \Delta t}{\Delta x} \le 2, \quad \forall p$$



$$-\infty < \frac{\lambda_p \Delta t}{\Delta x} < \infty, \quad \forall p$$

Numerical Experiments

Experiment with the code in

`$CLAW/apps/advection/1d/example1`

Make the following changes in `setrun.py`:

- Upwind method (`clawdata.order = 1`)
- Finer grid (`clawdata.mx = 100`)
- Periodic boundary conditions
  ```
  clawdata.mthbc_xlower = 2
  clawdata.mthbc_xupper = 2
  ```
- Narrower pulse (`beta = 300` or `3000`)
- Courant number greater than 1.
  ```
  clawdata.cfl_desired = 1.1
  clawdata.cfl_max = 1.1
  ```

Upwind for a linear system

The one-sided method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} A(Q_i^n - Q_{i-1}^n)$$

is stable only if $0 \le \Delta t \lambda^p / \Delta x \le 1$ for all p.

Upwind method based on sign of each λ^p:

Let $\lambda^+ = \max(\lambda, 0),\ \lambda^- = \min(\lambda, 0),$
$\Lambda^+ = \text{diag}((\lambda^p)^+),\ \ \Lambda^- = \text{diag}((\lambda^p)^-),$
$A^+ = R\Lambda^+ R^{-1},\ \ A^- = R\Lambda^- R^{-1}$

Then

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} A^+(Q_i^n - Q_{i-1}^n) - \frac{\Delta t}{\Delta x} A^-(Q_{i+1}^n - Q_i^n).$$

Symmetric methods

Centered in space, forward in time:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}A\right)(Q_i^n - Q_{i-1}^n) - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}A\right)(Q_{i+1}^n - Q_i^n) \\ &= Q_i^n - \frac{\Delta t}{2\Delta x}A(Q_{i+1}^n - Q_{i-1}^n) \end{aligned}$$

Centered approximation to q_x, but unstable for any fixed $\Delta t/\Delta x$.

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&= Q_i^n - \frac{\Delta t}{2\Delta x}A(Q_{i+1}^n - Q_{i-1}^n)
\end{aligned}
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Centered approximation to q_x, but unstable for any fixed $\Delta t/\Delta x$.

Lax-Friedrichs:

$$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{\Delta t}{2\Delta x}A(Q_{i+1}^n - Q_{i-1}^n)$$

This is stable if $\left|\frac{\lambda^p \Delta t}{\Delta x}\right| \leq 1$ for all p.

Numerical dissipation

Lax-Friedrichs:

$$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{\Delta t}{2\Delta x}A(Q_{i+1}^n - Q_{i-1}^n)$$

This can be rewritten as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

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The unstable method with the addition of artificial viscosity,

Approximates $q_t + Aq_x = \epsilon q_{xx}$ (modified equation)

with $\epsilon = \frac{\Delta x^2}{2\Delta t} = \mathcal{O}(\Delta x)$ if $\Delta t/\Delta x$ is fixed as $\Delta x \to 0$.

Modified Equations

The upwind method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} u(Q_i^n - Q_{i-1}^n).$$

gives a first-order accurate approximation to $q_t + uq_x = 0$.

But it gives a second-order approximation to

$$q_t + uq_x = \frac{u\Delta x}{2}\left(1 - \frac{u\Delta t}{\Delta x}\right) q_{xx}.$$

This is an advection-diffusion equation.

Indicates that the numerical solution will diffuse.

Note: coefficient of diffusive term is $O(\Delta x)$.

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Indicates that the numerical solution will diffuse.

Note: coefficient of diffusive term is $O(\Delta x)$.

Note: No diffusion if $\frac{u\Delta t}{\Delta x} = 1$ $\quad (Q_i^{n+1} = Q_{i-1}^n$ exactly).

Lax-Wendroff

Second-order accuracy?

Taylor series:

$$q(x,t+\Delta t) = q(x,t) + \Delta t q_t(x,t) + \frac{1}{2}\Delta t^2 q_{tt}(x,t) + \cdots$$

From $q_t = -Aq_x$ we find $q_{tt} = A^2 q_{xx}$.

$$q(x,t+\Delta t) = q(x,t) - \Delta t A q_x(x,t) + \frac{1}{2}\Delta t^2 A^2 q_{xx}(x,t) + \cdots$$

Replace q_x and q_{xx} by centered differences:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2 (Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

Modified Equation for Lax-Wendroff

The Lax-Wendroff method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2 (Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

gives a second-order accurate approximation to $q_t + uq_x = 0$.

But it gives a third-order approximation to

$$q_t + uq_x = -\frac{uh^2}{6}\left(1 - \left(\frac{u\Delta t}{\Delta x}\right)^2\right) q_{xxx}.$$

This has a dispersive term with $O(\Delta x^2)$ coefficient.

Indicates that the numerical solution will become oscillatory.

Dispersion relation

Consider a single Fourier mode:

$$q(x,0) = e^{i\xi x} \implies q(x,t) = e^{i(\xi x - \omega t)}$$

Determine $\omega(\xi)$ based on the PDE.
This is the dispersion relation.

$$q_t = -i\omega q, \quad q_x = i\xi q, \quad q_{xx} = -\xi^2 q, \quad q_{xxx} = -i\xi^3 q, \ldots$$

$$q_t + uq_x = 0 \implies \omega(\xi) = u\xi, \qquad q(x,t) = e^{i\xi(x-ut)}$$

(translates at speed u for all ξ)

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(translates at speed u for all ξ)

$$q_t + uq_x = \epsilon q_{xx} \implies \qquad q(x,t) = e^{-\epsilon\xi^2 t} e^{i\xi(x-ut)}$$

(decays)

Dispersion relation

Consider a single Fourier mode:

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Determine $\omega(\xi)$ based on the PDE.
This is the dispersion relation.

$$q_t = -i\omega q, \quad q_x = i\xi q, \quad q_{xx} = -\xi^2 q, \quad q_{xxx} = -i\xi^3 q, \ldots$$

$$q_t + uq_x = 0 \implies \omega(\xi) = u\xi, \qquad q(x,t) = e^{i\xi(x-ut)}$$

(translates at speed u for all ξ)

$$q_t + uq_x = \epsilon q_{xx} \implies \qquad q(x,t) = e^{-\epsilon\xi^2 t} e^{i\xi(x-ut)}$$ (decays)

$$q_t + uq_x = \beta q_{xxx} \implies \quad q(x,t) = e^{i\xi(x-(u+\beta\xi^2)t)}$$

(translates at speed $u + \beta\xi^2$ that depends on wave number!)