

Conservation Laws and Finite Volume Methods

AMath 574
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Notes:

Outline

Today:

- Review acoustics
- Solving Riemann problem for linear systems
- Coupled acoustics-advection
- Acoustics in heterogeneous medium
- Finite volume methods
- Conservation form
- Godunov's method

Next:

- More about finite volumes

Reading: Chapter 4

Notes:

Eigenvectors for acoustics

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \quad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $A r^p = \lambda^p r^p$, where

$$\lambda^1 = u_0 - c_0, \quad \lambda^2 = u_0 + c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

Note: Eigenvectors are independent of u_0 .

Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} = \text{impedance}$.

Notes:

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \quad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

In a simple 1-wave (propagating at speed $\lambda^1 = -c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is $-Z_0$ times the velocity variation.

Similarly, in a simple 2-wave ($\lambda^2 = c_0$),

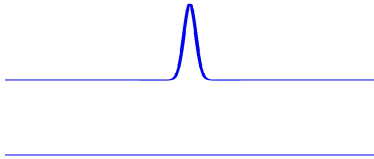
$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is Z_0 times the velocity variation.

Notes:

Acoustic waves

$$\begin{aligned} q(x, 0) = \begin{bmatrix} \overset{\circ}{p}(x) \\ 0 \end{bmatrix} &= -\frac{\overset{\circ}{p}(x)}{2Z_0} \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x)}{2Z_0} \begin{bmatrix} Z_0 \\ 1 \end{bmatrix} \\ &= w^1(x, 0)r^1 + w^2(x, 0)r^2 \\ &= \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ -\overset{\circ}{p}(x)/(2Z_0) \end{bmatrix} + \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ \overset{\circ}{p}(x)/(2Z_0) \end{bmatrix}. \end{aligned}$$

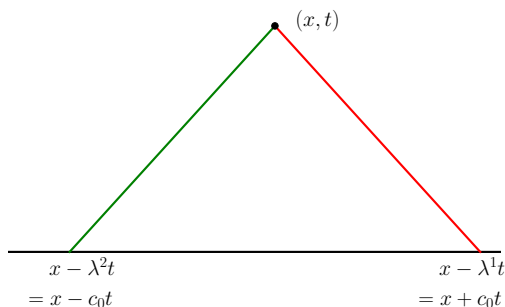


Notes:

Solution by tracing back on characteristics

The general solution for acoustics:

$$\begin{aligned} q(x, t) &= w^1(x - \lambda^1 t, 0)r^1 + w^2(x - \lambda^2 t, 0)r^2 \\ &= w^1(x + c_0 t, 0)r^1 + w^2(x - c_0 t, 0)r^2 \end{aligned}$$

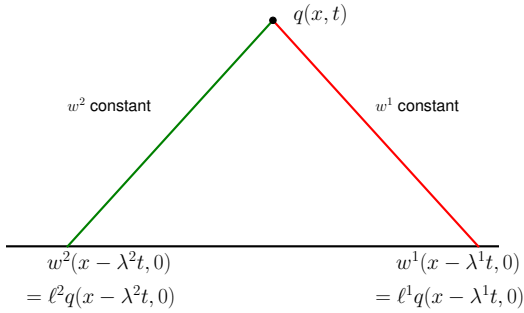


Notes:

Solution by tracing back on characteristics

The general solution for acoustics:

$$q(x, t) = w^1(x - \lambda^1 t, 0)r^1 + w^2(x - \lambda^2 t, 0)r^2$$



Notes:

Riemann Problem

Special initial data:

$$q(x, 0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphragm



Pressure:

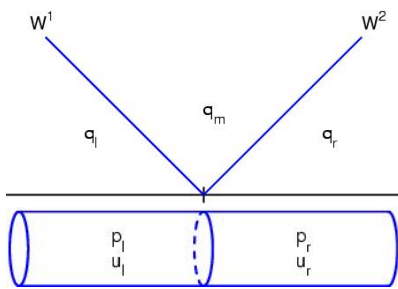


Acoustic waves propagate with speeds $\pm c$.

Notes:

Riemann Problem for acoustics

Waves propagating in $x-t$ space:

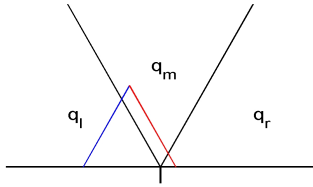


Left-going wave $\mathcal{W}^1 = q_m - q_l$ and right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A .

Notes:

Riemann Problem for acoustics

In $x-t$ plane:



$$q(x, t) = w^1(x + ct, 0)r^1 + w^2(x - ct, 0)r^2$$

Decompose q_l and q_r into eigenvectors:

$$q_l = w_l^1 r^1 + w_l^2 r^2$$

$$q_r = w_r^1 r^1 + w_r^2 r^2$$

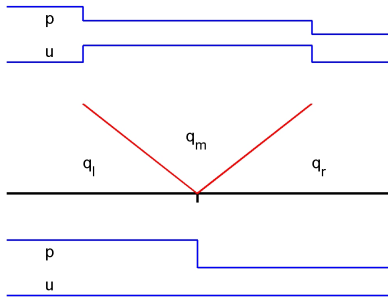
Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

Notes:

Riemann Problem for acoustics

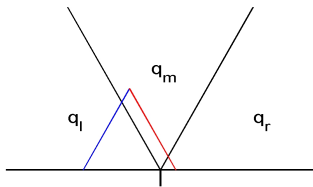
Waves propagating in $x-t$ space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A .

Notes:

Riemann Problem for acoustics



$$q_l = w_l^1 r^1 + w_l^2 r^2$$

$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

So the waves \mathcal{W}^1 and \mathcal{W}^2 are eigenvectors of A :

$$\mathcal{W}^1 = q_m - q_l = (w_r^1 - w_l^1)r^1$$

$$\mathcal{W}^2 = q_r - q_m = (w_r^2 - w_l^2)r^2.$$

Notes:

Riemann solution for a linear system

Linear hyperbolic system: $q_t + Aq_x = 0$ with $A = R\Lambda R^{-1}$.
 General Riemann problem data $q_l, q_r \in \mathbb{R}^m$.

Decompose jump in q into eigenvectors:

$$q_r - q_l = \sum_{p=1}^m \alpha^p r^p$$

Note: the vector α of eigen-coefficients is

$$\alpha = R^{-1}(q_r - q_l) = R^{-1}q_r - R^{-1}q_l = w_r - w_l.$$

Riemann solution consists of m waves $\mathcal{W}^p \in \mathbb{R}^m$:

$$\mathcal{W}^p = \alpha^p r^p, \quad \text{propagating with speed } s^p = \lambda^p.$$

Notes:

Coupled advection–acoustics

Flow in pipe with constant background velocity \bar{u} .

$\phi(x, t)$ = concentration of advected tracer

$u(x, t), p(x, t)$ = acoustic velocity / pressure perturbation

Equations include advection at velocity \bar{u} :

$$\begin{aligned} p_t + \bar{u}p_x + Ku_x &= 0 \\ u_t + (1/\rho)p_x + \bar{u}u_x &= 0 \\ \phi_t + \bar{u}\phi_x &= 0 \end{aligned}$$

This is a linear system $q_t + Aq_x = 0$ with

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \quad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

Notes:

Coupled advection–acoustics

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \quad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

eigenvalues: $\lambda^1 = u - c, \quad \lambda^2 = u, \quad \lambda^3 = u + c,$

eigenvectors: $r^1 = \begin{bmatrix} -Z \\ 1 \\ 0 \end{bmatrix}, \quad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^3 = \begin{bmatrix} Z \\ 1 \\ 0 \end{bmatrix},$

where $c = \sqrt{\kappa/\rho}$, $Z = \rho c = \sqrt{\rho\kappa}$.

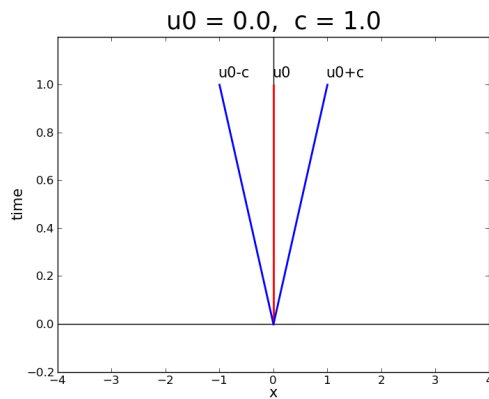
$$R = \begin{bmatrix} -Z & 0 & Z \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad R^{-1} = \frac{1}{2Z} \begin{bmatrix} -1 & Z & 0 \\ 0 & 0 & 1 \\ 1 & Z & 0 \end{bmatrix}.$$

Notes:

Coupled advection–acoustics

Wave structure of solution in the $x-t$ plane

With no advection:

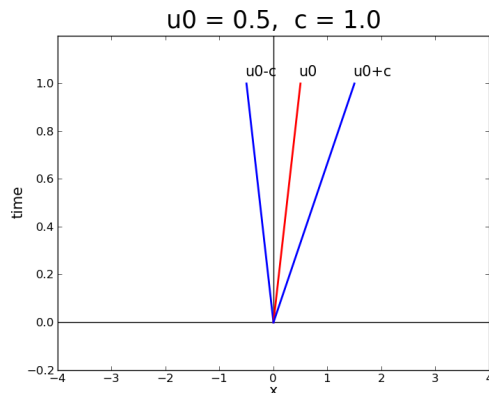


Notes:

Coupled advection–acoustics

Wave structure of solution in the $x-t$ plane

Subsonic case ($|u_0| < c$):

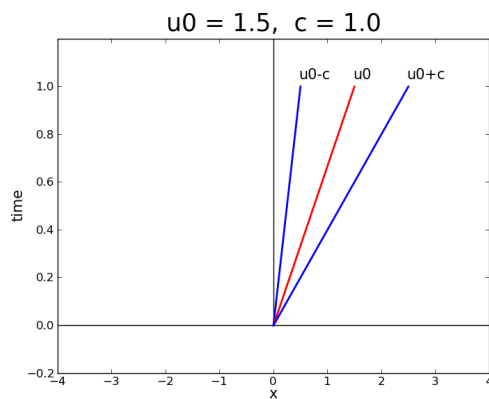


Notes:

Coupled advection–acoustics

Wave structure of solution in the $x-t$ plane

Supersonic case ($|u_0| > c$):



Notes:

Wave propagation in heterogeneous medium

Linear system $q_t + A(x)q_x = 0$. For acoustics:

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho(x) & 0 \end{bmatrix}.$$

eigenvalues: $\lambda^1 = -c(x)$, $\lambda^2 = +c(x)$,

where $c(x) = \sqrt{\kappa(x)/\rho(x)}$ = local speed of sound.

eigenvectors: $r^1(x) = \begin{bmatrix} -Z(x) \\ 1 \end{bmatrix}$, $r^2(x) = \begin{bmatrix} Z(x) \\ 1 \end{bmatrix}$

where $Z(x) = \rho c = \sqrt{\rho\kappa}$ = impedance.

$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \quad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

Cannot diagonalize unless $Z(x)$ is constant.

Notes:

Wave propagation in heterogeneous medium

Multiply system

$$q_t + A(x)q_x = 0$$

by $R^{-1}(x)$ on left to obtain

$$R^{-1}(x)q_t + R^{-1}(x)A(x)R(x)R^{-1}(x)q_x = 0$$

or

$$(R^{-1}(x)q)_t + \Lambda(x) [(R^{-1}(x)q)_x - R_x^{-1}(x)q] = 0$$

Let $w(x, t) = R^{-1}(x)q(x, t)$ (characteristic variable).

There is a coupling term on the right:

$$w_t + \Lambda(x)w_x = \Lambda(x)R_x^{-1}(x)R(x)w$$

\implies reflections (unless $R_x^{-1}(x) \equiv 0$).

Notes:

Wave propagation in heterogeneous medium

Generalized Riemann problem: single jump discontinuity in $q(x, 0)$ and in $K(x)$ and $\rho(x)$.

Decompose jump in q as linear combination of eigenvectors, with

- left-going waves: eigenvectors for material on left,
- right-going waves: eigenvectors for material on right.

$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \quad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

Riemann solution: decompose

$$q_r - q_l = \alpha^1 \begin{bmatrix} -Z_l \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix} = \mathcal{W}^1 + \mathcal{W}^2$$

The waves propagate with speeds $s^1 = -c_l$ and $s^2 = c_r$.

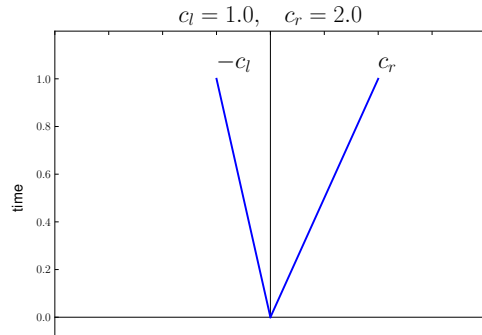
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