Conservation Laws and Finite Volume Methods

AMath 574
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Randall J. LeVeque
Applied Mathematics
University of Washington

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Outline

Today:

- Review acoustics
- Solving Riemann problem for linear systems
- Coupled acoustics-advection
- Acoustics in heterogeneous medium
- Finite volume methods
- Conservation form
- Godunov's method

Next:

- More about finite volumes

Reading: Chapter 4

Eigenvectors for acoustics

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = u_0 - c_0, \qquad \lambda^2 = u_0 + c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

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Note: Eigenvectors are independent of u_0.

Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

In a simple 1-wave (propagating at speed $\lambda^1 = -c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is $-Z_0$ times the velocity variation.

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Similarly, in a simple 2-wave ($\lambda^2 = c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

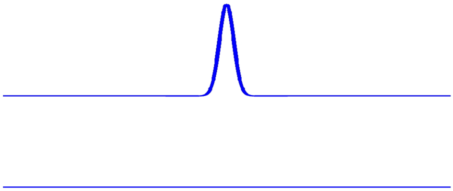
The pressure variation is Z_0 times the velocity variation.

Acoustic waves

$$q(x,0) = \begin{bmatrix} \mathring{p}(x) \\ 0 \end{bmatrix} = -\frac{\mathring{p}(x)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\mathring{p}(x)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

$$= w^1(x,0)r^1 + w^2(x,0)r^2$$

$$= \begin{bmatrix} \mathring{p}(x)/2 \\ -\mathring{p}(x)/(2Z_0) \end{bmatrix} + \begin{bmatrix} \mathring{p}(x)/2 \\ \mathring{p}(x)/(2Z_0) \end{bmatrix}.$$



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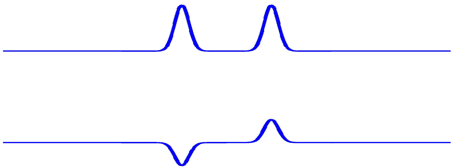

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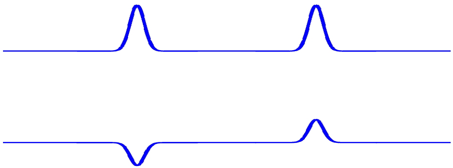

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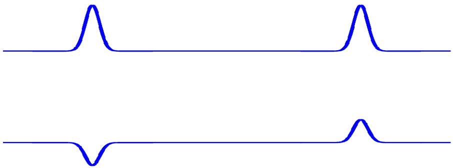

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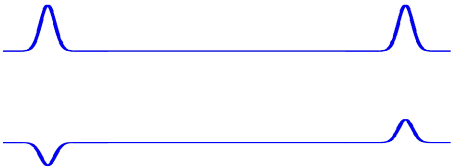
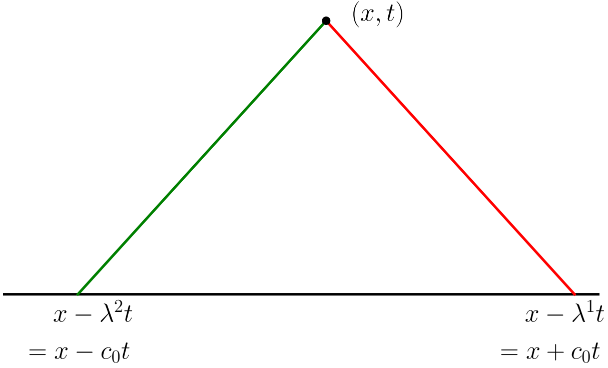


Solution by tracing back on characteristics

The general solution for acoustics:

$$\begin{aligned} q(x,t) &= w^1(x-\lambda^1 t, 0)r^1 + w^2(x-\lambda^2 t, 0)r^2 \\ &= w^1(x+c_0 t, 0)r^1 + w^2(x-c_0 t, 0)r^2 \end{aligned}$$



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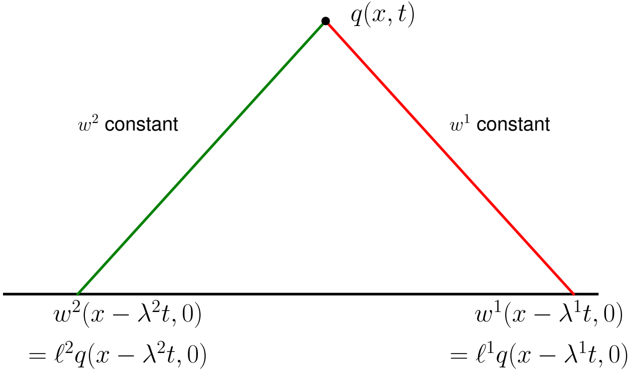


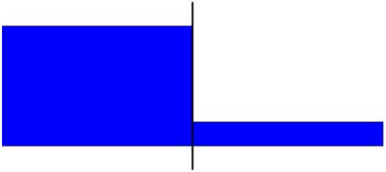
Riemann Problem

Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram

Pressure:



Acoustic waves propagate with speeds $\pm c$.

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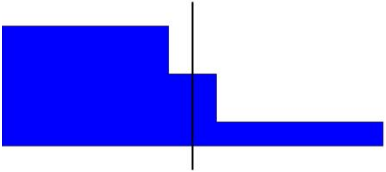

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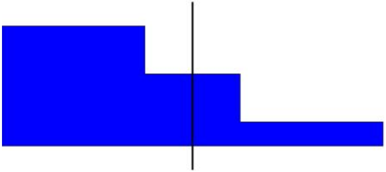

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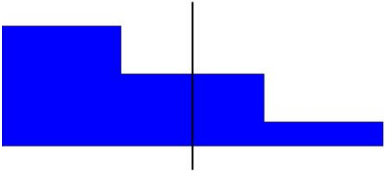

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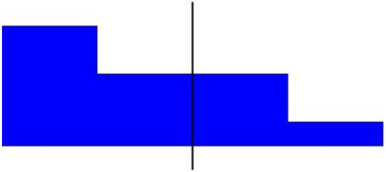

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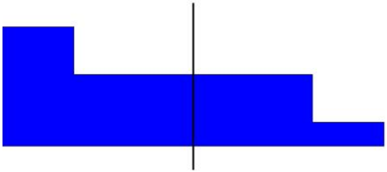
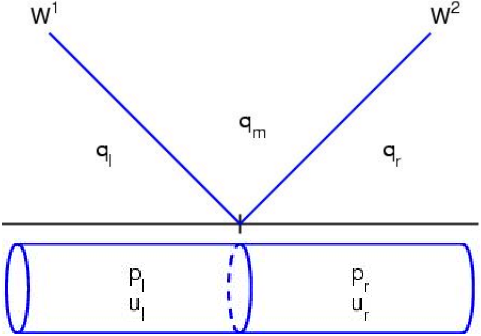


Acoustic waves propagate with speeds $\pm c$.

Riemann Problem for acoustics

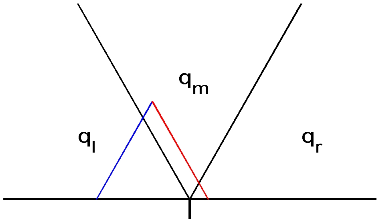
Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

Riemann Problem for acoustics

In x–t plane:



$$q(x,t) = w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2$$

Decompose q_l and q_r into eigenvectors:

$$q_l = w_l^1 r^1 + w_l^2 r^2$$
$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

Riemann Problem for acoustics

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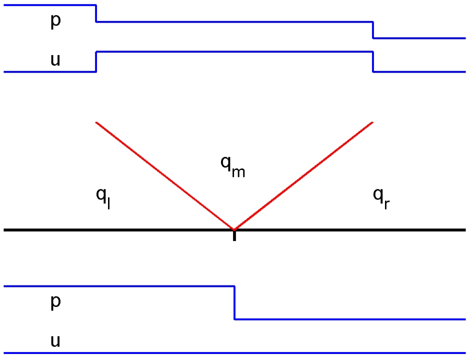

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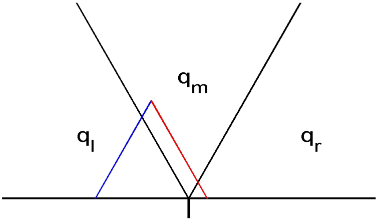


$$q_l = w_l^1 r^1 + w_l^2 r^2$$
$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

So the waves $\mathcal{W}^1$ and $\mathcal{W}^2$ are eigenvectors of A:

$$\mathcal{W}^1 = q_m - q_l = (w_r^1 - w_l^1) r^1$$
$$\mathcal{W}^2 = q_r - q_m = (w_r^2 - w_l^2) r^2.$$

Riemann solution for a linear system

Linear hyperbolic system: $q_t + Aq_x = 0$ with $A = R\Lambda R^{-1}$.
General Riemann problem data $q_l,\ q_r \in \mathbb{R}^m$.

Decompose jump in q into eigenvectors:

$$q_r - q_l = \sum_{p=1}^{m} \alpha^p r^p$$

Note: the vector α of eigen-coefficients is

$$\alpha = R^{-1}(q_r - q_l) = R^{-1}q_r - R^{-1}q_l = w_r - w_l.$$

Riemann solution consists of m waves $\mathcal{W}^p \in \mathbb{R}^m$:

$$\mathcal{W}^p = \alpha^p r^p, \qquad \text{propagating with speed } s^p = \lambda^p.$$

Coupled advection–acoustics

Flow in pipe with constant background velocity $\bar{u}$.
$\phi(x,t) =$ concentration of advected tracer
$u(x,t),\ p(x,t) =$ acoustic velocity / pressure perturbation

Equations include advection at velocity $\bar{u}$:

$$\begin{array}{llllllll} p_t & + & \bar{u}p_x & + & Ku_x & & & = 0 \\ u_t & + & (1/\rho)p_x & + & \bar{u}u_x & & & = 0 \\ \phi_t & & & & & + & \bar{u}\phi_x & = 0 \end{array}$$

This is a linear system $q_t + Aq_x = 0$ with

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \qquad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

Coupled advection–acoustics

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \qquad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

eigenvalues: $\lambda^1 = u - c, \qquad \lambda^2 = u \qquad \lambda^3 = u + c,$

eigenvectors: $r^1 = \begin{bmatrix} -Z \\ 1 \\ 0 \end{bmatrix}, \quad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^3 = \begin{bmatrix} Z \\ 1 \\ 0 \end{bmatrix},$

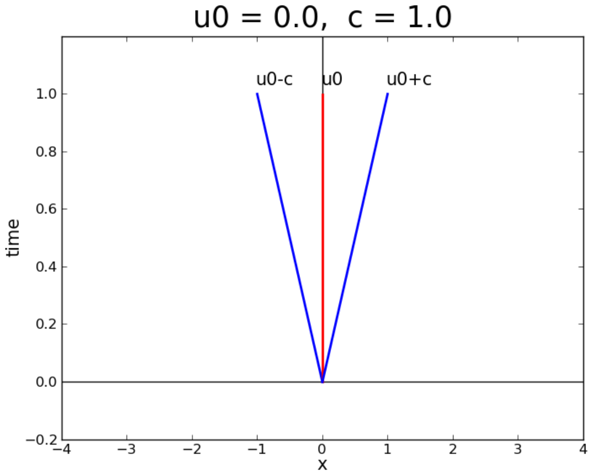
where $c = \sqrt{\kappa/\rho}, \;\; Z = \rho c = \sqrt{\rho\kappa}$.

$$R = \begin{bmatrix} -Z & 0 & Z \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z} \begin{bmatrix} -1 & Z & 0 \\ 0 & 0 & 1 \\ 1 & Z & 0 \end{bmatrix}.$$

Coupled advection–acoustics

Wave structure of solution in the x–t plane

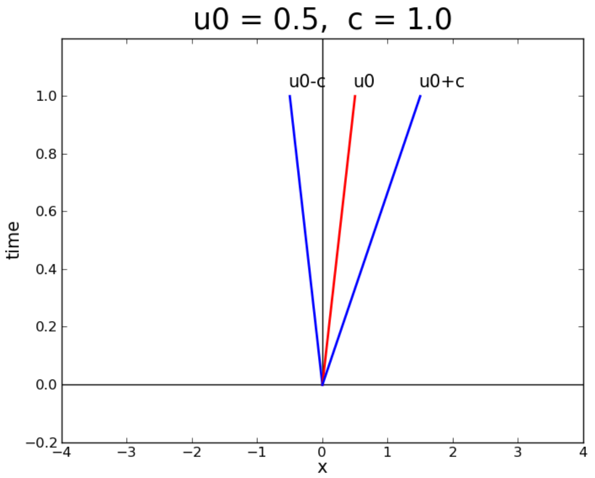
With no advection:



Coupled advection–acoustics

Wave structure of solution in the x–t plane

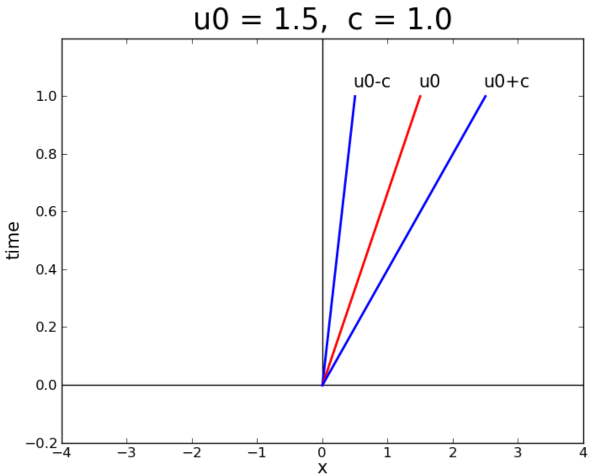
Subsonic case ($|u_0| < c$):



Coupled advection–acoustics

Wave structure of solution in the x–t plane

Supersonic case ($|u_0| > c$):



Wave propagation in heterogeneous medium

Linear system $q_t + A(x)q_x = 0$. For acoustics:

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho(x) & 0 \end{bmatrix}.$$

eigenvalues: $\lambda^1 = -c(x), \qquad \lambda^2 = +c(x),$

where $c(x) = \sqrt{\kappa(x)/\rho(x)}$ = local speed of sound.

eigenvectors: $r^1(x) = \begin{bmatrix} -Z(x) \\ 1 \end{bmatrix}, \qquad r^2(x) = \begin{bmatrix} Z(x) \\ 1 \end{bmatrix}$

where $Z(x) = \rho c = \sqrt{\rho\kappa}$ = impedance.

$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \qquad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

Cannot diagonalize unless $Z(x)$ is constant.

Wave propagation in heterogeneous medium

Multiply system

$$q_t + A(x)q_x = 0$$

by $R^{-1}(x)$ on left to obtain

$$R^{-1}(x)q_t + R^{-1}(x)A(x)R(x)\,R^{-1}(x)q_x = 0$$

or

$$(R^{-1}(x)q)_t + \Lambda(x)\,\left[(R^{-1}(x)q)_x - R_x^{-1}(x)q\right] = 0$$

Let $w(x,t) = R^{-1}(x)q(x,t)$ (characteristic variable).
There is a coupling term on the right:

$$w_t + \Lambda(x)\,w_x = \Lambda(x)R_x^{-1}(x)R(x)w$$

$\Longrightarrow$ reflections (unless $R_x^{-1}(x) \equiv 0$).

Wave propagation in heterogeneous medium

Generalized Riemann problem: single jump discontinuity in $q(x,0)$ and in $K(x)$ and $\rho(x)$.

Decompose jump in q as linear combination of eigenvectors, with

- left-going waves: eigenvectors for material on left,
- right-going waves: eigenvectors for material on right.

$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \qquad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

Riemann solution: decompose

$$q_r - q_l = \alpha^1 \begin{bmatrix} -Z_l \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix} = \mathcal{W}^1 + \mathcal{W}^2$$

The waves propagate with speeds $s^1 = -c_l$ and $s^2 = c_r$.

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