Conservation Laws and Finite Volume Methods

AMath 574
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Notes:

Outline

Today:

- Gas dynamics
- Linearization of gas dynamics
- Linear acoustics
- Diagonalization of linear systems
- Meaning of eigenvectors
- Characteristic solution for acoustics

Next:

- Riemann problem for acoustics
- Finite volume methods

Reading: Chapter 3 and start Chapter 4

Notes:

Compressible gas dynamics

In one space dimension (e.g. in a pipe).
$\rho(x,t) =$ density, $u(x,t) =$ velocity,
$p(x,t) =$ pressure, $\rho(x,t)u(x,t) =$ momentum.

Conservation of:

mass:	ρ	flux:	ρu
momentum:	ρu	flux:	$(\rho u)u + p$
(energy)			

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

(Later: p may also depend on internal energy / temperature)

Notes:

Compressible gas dynamics

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Momentum flux:

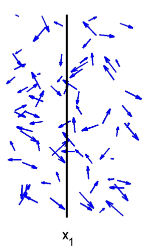
$\rho u^2 = (\rho u) u =$ advective flux

p term in flux?

- $-p_x =$ force in Newton's second law,
- as momentum flux: microscopic motion of gas molecules.

Notes:

Momentum flux arising from pressure



Note that:

- molecules with positive x-velocity crossing x_1 to right increase the momentum in $[x_1, x_2]$
- molecules with negative x-velocity crossing x_1 to left also increase the momentum in $[x_1, x_2]$

Hence momentum flux increases with pressure $p(x_1, t)$ even if macroscopic (average) velocity is zero.

Notes:

Compressible gas dynamics

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

Same as shallow water if $P(\rho) = \frac{1}{2} g \rho^2$ (with $\rho \equiv h$).

Isothermal: $P(\rho) = a^2 \rho$ (since T proportional to p/ρ).

Isentropic: $P(\rho) = \hat{\kappa} \rho^\gamma$ ($\gamma \approx 1.4$ for air)

Jacobian matrix:

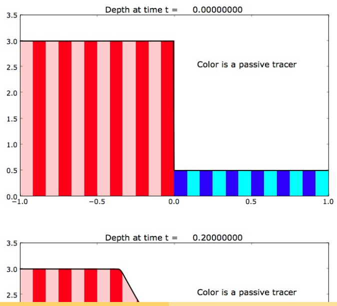
$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

Notes:

The Riemann problem

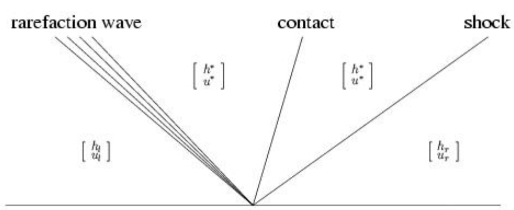
Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



Notes:

Riemann solution for the SW equations in x-t plane



Similarity solution:
Solution is constant on any ray: $q(x,t) = Q(x/t)$

Riemann solution can be calculated for many problems.
Linear: Eigenvector decomposition. Nonlinear: more difficult.

In practice "approximate Riemann solvers" used numerically.

Notes:

Compressible gas dynamics

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

Sound speed: $c = \sqrt{P'(\rho)}$ varies with ρ.

System is hyperbolic if $P'(\rho) > 0$.

Notes:

Linearization of gas dynamics

Suppose $\rho(x,t) \approx \rho_0$ and $u(x,t) \approx u_0$.

Model small perturbations to this steady state (sound waves).

$$\begin{bmatrix} \rho(x,t) \\ (\rho u)(x,t) \end{bmatrix} = \begin{bmatrix} \rho_0 \\ \rho_0 u_0 \end{bmatrix} + \begin{bmatrix} \widetilde{\rho}(x,t) \\ (\widetilde{\rho u})(x,t) \end{bmatrix}$$

or $q(x,t) = q_0 + \tilde{q}(x,t)$ where $\|\tilde{q}(x,t)\| = \epsilon$ is small.

Then nonlinear equation $q_t + f(q)_x = 0$ leads to

$$\begin{aligned} \tilde{q}_t &= q_t \\ &= -f(q)_x \\ &= -f'(q)q_x \\ &= -f'(q_0 + \tilde{q})\tilde{q}_x \\ &= -f'(q_0)\tilde{q}_x + \mathcal{O}(\epsilon^2). \end{aligned}$$

Linearization: $\tilde{q}_t + A\tilde{q}_x = 0$ where $A = f'(q_0)$.

Notes:

Linearization of gas dynamics

Linearization: $\tilde{q}_t + A\tilde{q}_x = 0$ where $A = f'(q_0)$.

$$A = f'(q_0) = \begin{bmatrix} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{bmatrix}.$$

This can be written out as (2.47):

$$\begin{aligned} \tilde{\rho}_t + (\widetilde{\rho u})_x &= 0 \\ (\widetilde{\rho u})_t + (-u_0^2 + P'(\rho_0))\tilde{\rho}_x + 2u_0(\widetilde{\rho u})_x &= 0. \end{aligned}$$

Rewrite in terms of p and u perturbations (Exer. 2.1):

$$\begin{aligned} \tilde{p}_t + u_0\tilde{p}_x + K_0\tilde{u}_x &= 0, \\ \rho_0\tilde{u}_t + \tilde{p}_x + \rho_0 u_0\tilde{u}_x &= 0, \end{aligned}$$

where $K_0 = \rho_0 P'(\rho_0)$ is the bulk modulus.

Notes:

Linearization of gas dynamics

$$\begin{aligned} \tilde{p}_t + u_0\tilde{p}_x + K_0\tilde{u}_x &= 0, \\ \rho_0\tilde{u}_t + \tilde{p}_x + \rho_0 u_0\tilde{u}_x &= 0, \end{aligned}$$

gives the system $q_t + Aq_x = 0$ (Drop tildes)

$$q(x,t) = \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix}, \qquad A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$

Eigenvalues: $\lambda = u_0 \pm c_0$

where $c_0 = \sqrt{K_0/\rho_0} = \sqrt{P'(\rho_0)}$ is the linearized sound speed.

Usually $u_0 = 0$ for linear acoustics. Then $\lambda^1 = -c_0, \ \lambda^2 = +c_0$.

Notes:

Example: Linear acoustics in a 1d tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity} \end{array}$$

Equations:

$$\begin{aligned} p_t + \kappa u_x &= 0 \qquad & \kappa &= \text{bulk modulus} \\ \rho u_t + p_x &= 0 \qquad & \rho &= \text{density} \end{aligned}$$

or

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Eigenvalues: $\lambda = \pm c$, where $c = \sqrt{\kappa/\rho}$ = sound speed

Second order form: Can combine equations to obtain

$$p_{tt} = c^2 p_{xx}$$

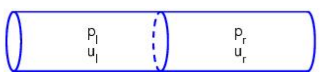
Notes:

Riemann Problem

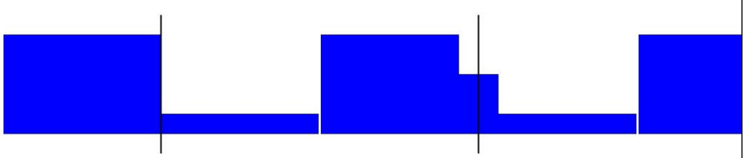
Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram



Pressure:

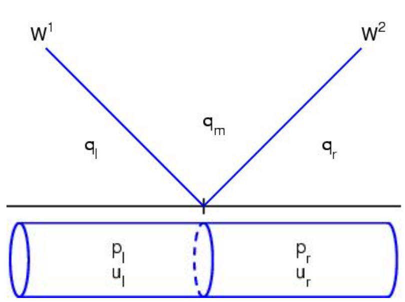


Acoustic waves propagate with speeds $\pm c$.

Notes:

Riemann Problem for acoustics

Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

Notes:

Eigenvectors for acoustics

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = u_0 - c_0, \qquad \lambda^2 = u_0 + c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

Note: Eigenvectors are independent of u_0.

Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0}$ = impedance.

Notes:

Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Suppose hyperbolic:

- Real eigenvalues $\lambda^1 \le \lambda^2 \le \cdots \le \lambda^m$,
- Linearly independent eigenvalues $r^1,\ r^2,\ \ldots,\ r^m$.

Let $R = [r^1 | r^2 | \cdots | r^m]$ $\quad m \times m$ matrix of eigenvectors.

Then $Ar^p = \lambda^p r^p$ means that $AR = R\Lambda$ where

$$\Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix} \equiv \text{diag}(\lambda^1, \lambda^2, \ldots, \lambda^m).$$

$AR = R\Lambda \implies A = R\Lambda R^{-1}$ and $R^{-1}AR = \Lambda$.
Similarity transformation with R diagonalizes A.

Notes:

Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Multiply system by R^{-1}:

$$R^{-1}q_t(x,t) + R^{-1}Aq_x(x,t) = 0.$$

Introduce $RR^{-1} = I$:

$$R^{-1}q_t(x,t) + R^{-1}ARR^{-1}q_x(x,t) = 0.$$

Use $R^{-1}AR = \Lambda$ and define $w(x,t) = R^{-1}q(x,t)$:

$$w_t(x,t) + \Lambda w_x(x,t) = 0.$$ Since R is constant!

This decouples to m independent scalar advection equations:

$$w_t^p(x,t) + \lambda^p w_x^p(x,t) = 0. \qquad p = 1,\ 2,\ \ldots,\ m.$$

Notes:

Solution to Cauchy problem

Suppose $q(x,0) = \overset{\circ}{q}(x)$ for $-\infty < x < \infty$.

From this initial data we can compute data

$$\overset{\circ}{w}(x) \equiv R^{-1}\overset{\circ}{q}(x)$$

The solution to the decoupled equation $w_t^p + \lambda^p w_x^p = 0$ is

$$w^p(x,t) = w^p(x - \lambda^p t, 0) = \overset{\circ}{w}^p(x - \lambda^p t).$$

Putting these together in vector gives $w(x,t)$ and finally

$$q(x,t) = Rw(x,t).$$

We can rewrite this as

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t)\, r^p = \sum_{p=1}^{m} \overset{\circ}{w}^p(x - \lambda^p t)\, r^p$$

Notes:

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Consider a pure 1-wave (simple wave), at speed $\lambda^1 = -c_0$,
If $\overset{\circ}{q}(x) = \bar{q} + \overset{\circ}{w}^1(x) r^1$ then

$$q(x,t) = \bar{q} + \overset{\circ}{w}^1(x - \lambda^1 t) r^1$$

Variation of q, as measured by q_x or $\Delta q = q(x + \Delta x) - q(x)$ is proportional to eigenvector r^1, e.g.

$$q_x(x,t) = \overset{\circ}{w}_x^1(x - \lambda^1 t) r^1$$

Notes:

Physical meaning of eigenvectors

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In a simple 1-wave (propagating at speed $\lambda^1 = -c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is $-Z_0$ times the velocity variation.

Similarly, in a simple 2-wave ($\lambda^2 = c_0$),

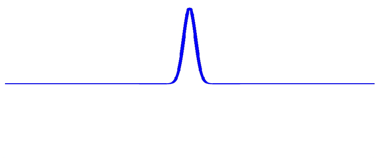
$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is Z_0 times the velocity variation.

Notes:

Acoustic waves

$$
\begin{aligned}
q(x,0) = \begin{bmatrix} \overset{\circ}{p}(x) \\ 0 \end{bmatrix} &= -\frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix} \\
&= w^1(x,0)r^1 + w^2(x,0)r^2 \\
&= \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ -\overset{\circ}{p}(x)/(2Z_0) \end{bmatrix} + \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ \overset{\circ}{p}(x)/(2Z_0) \end{bmatrix}.
\end{aligned}
$$



Notes:

Solution by tracing back on characteristics

The general solution for acoustics:

$$
\begin{aligned}
q(x,t) &= w^1(x-\lambda^1 t,0)r^1 + w^2(x-\lambda^2 t,0)r^2 \\
&= w^1(x+c_0 t,0)r^1 + w^2(x-c_0 t,0)r^2
\end{aligned}
$$

Recall that $w(x,0) = R^{-1}q(x,0)$, i.e.

$$w^1(x,0) = \ell^1 q(x,0), \qquad w^2(x,0) = \ell^2 q(x,0)$$

where ℓ^1 and ℓ^2 are rows of R^{-1}.

$$R^{-1} = \begin{bmatrix} \ell^1 \\ \ell^2 \end{bmatrix}$$

Note: ℓ^1 and ℓ^2 are left-eigenvectors of A:

$$\ell^p A = \lambda^p \ell^p \qquad \text{since } R^{-1}A = \Lambda R^{-1}.$$

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Solution by tracing back on characteristics

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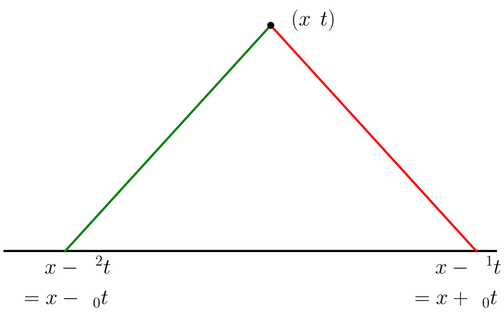

Notes:

Solution by tracing back on characteristics

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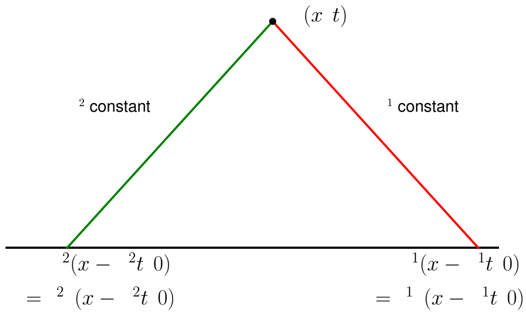

Notes: