AMath 574 February 28, 2011

Today:

- Another example nonlinear system: Burgers' + Advection
- Shallow water Riemann solution

Next Monday:

- Finite volume methods
- Approximate Riemann solvers

Reading: Chapter 15

Burgers' + advection

Another example of a nonlinear system:

$$q = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u+1)v \end{bmatrix}.$$

This is simply Burgers' equation

$$u_t + \frac{1}{2}(u^2)_x = 0$$

coupled to conservative advection

$$v_t + ((u+1)v)_x = 0$$

But... Advection velocity $u + 1$ comes from solution of Burgers' equation.

Burgers' + advection

Solving $u_t + \frac{1}{2}(u^2)_x = 0$ gives rarefaction wave (if $u_l < u_r$)

or shock wave with speed $s^1 = \frac{1}{2}(u_l + u_r)$ (if $u_l > u_r$).

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Advection equation can be rewritten as

$$v_t + (u+1)v_x = -u_x v$$

and characteristic theory shows that

$$\frac{d}{dt}v(X(t),t) = -u_x(X(t),t)v(X(t),t)$$

along the curve $X'(t) = u(X(t),t) + 1$.

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In regions where u is constant:

Characteristics are straight lines,

$u_x = 0 \implies v$ is constant.

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If u has a shock, then source term in v has form of delta function.

If delta moves a different speed than advection velocity, this leads to a jump in v at the shock location.

Resonant case: If shock moves at same speed as advection velocity then delta function is stationary relative to advecting v and we expect solution to blow up!

Burgers' + advection

Reconsider as nonlinear system:

$$q = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u+1)v \end{bmatrix}.$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} u & 0 \\ v & u+1 \end{bmatrix}.$$

Always hyperbolic since $u \neq u + 1$.

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$$\lambda^1 = u, \quad r^1 = \begin{bmatrix} 1 \\ -v \end{bmatrix}, \qquad \nabla\lambda^1 \cdot r^1 \equiv 1, \quad \text{genuinely nonlinear}$$

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$\lambda^2 = u+1, \quad r^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \nabla\lambda^2 \cdot r^2 \equiv 0,$ linearly degenerate

Burgers' + advection: 2-waves

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Integral curves:

$$\begin{aligned} \tilde{u}'(\xi) = 0 \qquad &\implies \tilde{u}(\xi) = u_* \\ \tilde{v}'(\xi) = v(\xi) \;&\implies \tilde{v}(\xi) = v_* e^{\xi} \end{aligned}$$

Integral curves are vertical lines.

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Integral curves are vertical lines.

These lines are also contours of λ^2 (linearly degenerate!)

We'll see later these are also the Hugoniot loci for 2-waves.

Burgers' + advection: 1-waves

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Integral curves:

$$\begin{aligned} \tilde{u}'(\xi) &= 1 &&\Longrightarrow \tilde{u}(\xi) = u_* + \xi \\ \tilde{v}'(\xi) &= -v(\xi) &&\Longrightarrow \tilde{v}(\xi) = v_* e^{-\xi} \end{aligned}$$

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Integral curves:

$$\begin{aligned} \tilde{u}'(\xi) = 1 \quad &\Longrightarrow \tilde{u}(\xi) = u_* + \xi \quad \Longrightarrow \xi = \tilde{u} - u_* \\ \tilde{v}'(\xi) = -v(\xi) \quad &\Longrightarrow \tilde{v}(\xi) = v_* e^{-\xi} \end{aligned}$$

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Burgers' + advection: Hugoniot loci

$$q = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u+1)v \end{bmatrix}.$$

States q and q_* must satisfy Rankine-Hugoniot jump condition:

$$f(q) - f(q_*) = s(q - q_*)$$

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One solution:

$u = u_*$ (and jump in v arbitrary) $\implies$ vertical lines

These are Hugoniot loci for 2-waves.

2-waves are discontinuities in v alone, speed $s = u_* + 1$
(determined from second equation of R-H conditions).

Burgers' + advection: Hugoniot loci

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Second solution:

$$s = s^1 = \frac{1}{2}(u + u_*) \implies$$ shock waves in Burgers' equation

Burgers' + advection: Hugoniot loci

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Relation between v and u across shock:

Second equation of R-H relation:

$$(u+1)v - (u_* + 1)v_* = s(v - v_*) \quad = \frac{1}{2}(u + u_*)(v - v_*)$$

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Burgers' + advection: Hugoniot loci

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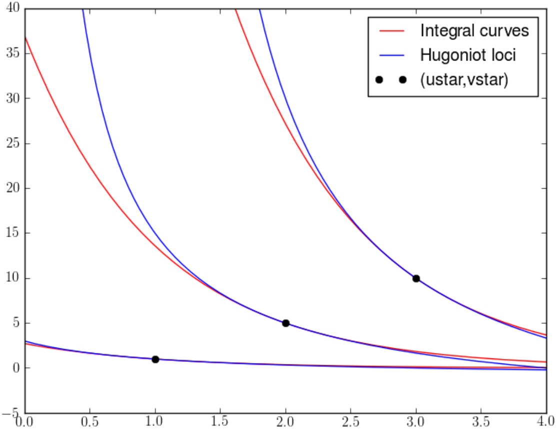
$$\implies \quad v = \left(\frac{1 + \frac{1}{2}(u_* - u)}{1 - \frac{1}{2}(u_* - u)}\right) v_* \quad \approx e^{u_* - u} v_*$$

The Hugoniot locus agrees to $\mathcal{O}(|u_* - u|^3)$ with integral curve.

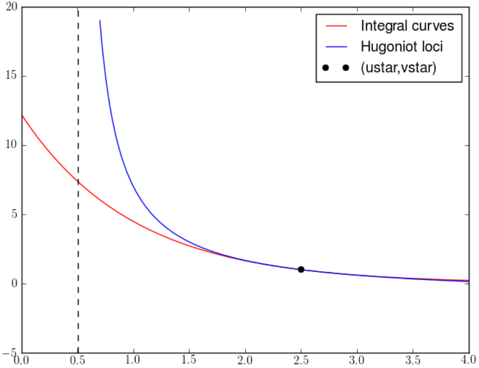
Burgers' + advection: Phase plane



Burgers' + advection: Phase plane

But note that

$$v = \left(\frac{1 + \frac{1}{2}(u_* - u)}{1 - \frac{1}{2}(u_* - u)} \right) v_* \quad \to \infty \quad \text{as } u \to u_* - 2$$



Burgers' + advection: Riemann solution

To be discussed on the board...

See also the description and codes at

`http://www.clawpack.org/links/burgersadv`